# PHASE TRANSITION IN THE DENSITY OF STATES OF QUANTUM SPIN GLASSES 

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#### Abstract

We prove that the empirical density of states of quantum spin glasses on arbitrary graphs converges to a normal distribution as long as the maximal degree is negligible compared with the total number of edges. This extends the recent results of [6] that were proved for graphs with bounded chromatic number and with symmetric coupling distribution. Furthermore, we generalise the result to arbitrary hypergraphs. We test the optimality of our condition on the maximal degree for $p$-uniform hypergraphs that correspond to $p$-spin glass Hamiltonians acting on $n$ distinguishable spin- $1 / 2$ particles. At the critical threshold $p=n^{1 / 2}$ we find a sharp classical-quantum phase transition between the normal distribution and the Wigner semicircle law. The former is characteristic to classical systems with commuting variables, while the latter is a signature of noncommutative random matrix theory.


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## 1. Introduction

The distribution of the energy levels for classical spin glasses converges to the normal distribution in the thermodynamic limit by the central limit theorem. On the other hand, the Hamiltonian of the quantum spin glasses can be considered as a random Hermitian matrix and thus the Wigner semicircle law might be expected. In fact, the mean field quantum spin glass on the full hypergraph with Gaussian coupling constants is equivalent to the Gaussian unitary ensemble (GUE). It turns out that, despite the inherent noncommutativity, the density of states for a large class of quantum spin glasses still follows the normal law. For quantum spin glasses on graphs with bounded chromatic number and with symmetrically distributed coupling constants this has recently been shown by Keating, Linden and Wells [6]. In fact, their result extends to bounded deterministic couplings in case of spin chains, see [7].

In the first part of this paper, we generalise their result in several directions by considering general graphs and even hypergraphs. Moreover, we relax the symmetry condition on the couplings. We find that the central limit theorem holds for a quantum spin glass on an arbitrary hypergraph, provided that the maximal degree of any vertex (the number of edges adjacent to it) is much smaller than the total number of edges. This condition guarantees that the noncommutative effects, related to edges sharing a common

[^0]vertex, are subleading: most degrees of freedom are still commutative. Thus the system is essential classical as far as the density of states is concerned. We also present an example (star graph) where the degree of a distinguished vertex is comparable with the total number of edges. The density of states is explicitly computable and it is neither Gaussian nor the semicircle law.

In the second part of the paper we investigate the transition from the classical regime dominated by commuting variables to the quantum regime where noncommutativity determines the leading behaviour. This transition is particularly transparent for the quantum $p$-spin model, i.e. a quantum spin glass on a $p$-uniform hypergraph. The case $p=2$ corresponds to the quantum version of the standard SherringtonKirkpatrick model and its density of states follows the normal law. The other extreme case, $p=n$, is the GUE model with the semicircle law. We prove a sharp phase transition at $p \sim \sqrt{n}$; for $p \ll \sqrt{n}$ we have the normal law, while for $p \gg \sqrt{n}$ we get the semicircle law. For $p=\lambda \sqrt{n}$ with a fixed $\lambda \in(0, \infty)$, we establish a new family of densities of states, parametrised by $\lambda$, that naturally interpolates between the normal distribution and the semicircle law. We emphasise that the regime $p \sim n^{\alpha}, \alpha \in(1 / 2,1)$, is still far from the mean field regime in the sense of random matrices: we have only $3^{p}\binom{n}{p} \ll 2^{n}$ independent random variables parametrising an operator acting an $N=2^{n}$ dimensional Hilbert space. In contrast, Wigner random matrices of dimension $N \times N$ have $N^{2}$ independent degrees of freedom. The $p$-spin model thus corresponds to a very sparse random matrix, still it follows the Wigner semicircle law if $p \gg \sqrt{n}$. Our result gives a rigorous proof of the transition between the Gaussian and the semicircle density of states that has been numerically observed in [4] for $k$-body interactions as $k$ approaches the total number of particles.

We mention that this phase transition is apparently present only for the density of states; as far as the local eigenvalue statistics is concerned all these models seem to belong to the random matrix (GUE) universality class. The numerical tests presented in [6] deal with the one-dimensional quantum chain, one of the sparsest model, and still demonstrate a very strong agreement with the GUE gap distribution. Certainly the same is expected for spin glasses on denser graphs. Quantum spin glasses are one of the simplest interacting many-body disordered quantum models. Therefore, this remarkable feature is yet another manifestation of Wigner's vision on the ubiquity of the random matrix gap statistics for essentially any disordered quantum system. For more details on the physical motivation and related works we refer to [6].

Our approach is different from that of [6]; we use the very robust moment method. In particular, this allows us to consider arbitrary coupling distributions without much effort and to identify new limiting laws in the $p \sim \sqrt{n}$ transition regime.

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## 2. Model and main results

Given a sequence of undirected graphs $\Gamma_{n}$ on the vertex sets $\{1, \ldots, n\}$, we are considering Hermitian random matrices $H_{n}^{\left(\Gamma_{n}\right)}$ defined by

$$
\begin{equation*}
H_{n}^{\left(\Gamma_{n}\right)}:=\frac{1}{\sqrt{9 e\left(\Gamma_{n}\right)}} \sum_{(i j) \in \Gamma_{n}} \sum_{a, b=1}^{3} \alpha_{a, b,(i j)} \sigma_{i}^{(a)} \sigma_{j}^{(b)} \tag{1}
\end{equation*}
$$

where $e\left(\Gamma_{n}\right)$ denotes the number of edges in $\Gamma_{n}$. The normalisation factor of $\left(9 e\left(\Gamma_{n}\right)\right)^{-1 / 2}$ corresponds to the $9 e\left(\Gamma_{n}\right)$ terms under the sum and is chosen to keep the spectrum of order 1 . As a convention, the edge connecting $i<j$ is called ( $i j$ ) and since the vertex set of the graphs is canonical, we shall, with a slight abuse of notation, identify the graph with its collection of edges. The coefficients $\alpha_{a, b,(i j)}$ are assumed to be independent random variables with zero mean and unit variance. The Pauli matrices acting on the $j$-th qubit are denoted by $\sigma_{j}^{(a)}:=1_{2}^{\otimes(j-1)} \otimes \sigma^{(a)} \otimes 1_{2}^{\otimes(n-j)}$ where $\sigma^{(a)}$ are the standard spin-1/2 Pauli matrices

$$
\sigma^{(1)}=\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right) \quad \sigma^{(2)}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right) \quad \sigma^{(3)}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

and $1_{2}=\sigma^{(0)}$ is the $2 \times 2$ identity matrix. For definiteness we will work with spin- $1 / 2$ systems, but all our results hold for spin- $s$ models with any fixed $s$, see Remark 11 .

We are interested in the eigenvalue density of the operators $H_{n}^{\left(\Gamma_{n}\right)}$ in the limit $n \rightarrow \infty$. The expected eigenvalue density of $H_{n}^{\left(\Gamma_{n}\right)}$ is given by

$$
\mu_{n}:=\frac{1}{2^{n}} \mathbf{E} \sum_{j=1}^{2^{n}} \delta_{\lambda_{j}},
$$

where $\lambda_{j}$ are the eigenvalues of $H_{n}^{\left(\Gamma_{n}\right)}$. The result in [6] shows that under the assumption that the random variables are bounded, symmetric about 0 and that the graphs have a uniformly bounded chromatic number, $\mu_{n}$ converges weakly to a standard normal distribution as $n \rightarrow \infty$. Theorem 1 generalises this result by removing the symmetry condition and also allowing sequences of graphs for which the maximal vertex degree $d_{\max }(n)$ grows slower than the number of edges $e\left(\Gamma_{n}\right)$. Since graph sequences with uniformly bounded chromatic numbers have a uniformly bounded maximal degree, our degree condition is implied by the condition from [6] on the chromatic number, but it is much more general, and in some sense optimal.

Theorem 1. Let $\Gamma_{n}$ be a sequence of graphs on the vertex sets $\{1, \ldots, n\}$ such that $\lim _{n \rightarrow \infty} \frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}=0$ and let

$$
\left\{\alpha_{a, b,(i j)} \mid 1 \leq a, b \leq 3,(i j) \in \Gamma_{n}\right\}
$$

be a tight collection of independent (not necessarily identically distributed) random variables with zero mean and unit variance. Then the expected density of states of the Hamiltonian $H_{n}^{\left(\Gamma_{n}\right)}$ defined in (1) converges weakly to a standard normal distribution.

Remark 2. The empirical eigenvalue distribution $\nu_{n}:=2^{-n} \sum_{j=1}^{2^{n}} \delta_{\lambda_{j}}$ is concentrated around its expectation $\mu_{n}=\mathbf{E} \nu_{n}$ and therefore the convergence in expectation, as proved in Theorem 11, also implies that $\nu_{n}$ converges weakly in probability to a standard normal distribution. This strengthening of Theorem 1 can be proved with a standard extension of the moment method to estimating the variance following the proof of [1. Lemma 2.1.7]. Since noncommutative features play no role in this argument, we omit the details. The same remark also applies to our subsequent Theorems 3 and 8 ,

Theorem 1 addresses both the model of nearest neighbour interactions in a 1-dimensional closed chain (where the labelling is cyclic in the sense $\sigma_{n+1}^{(a)}=\sigma_{1}^{(a)}$ )

$$
H_{n}:=\frac{1}{\sqrt{9 n}} \sum_{j=1}^{n} \sum_{a, b=1}^{3} \alpha_{a, b, j} \sigma_{j}^{(a)} \sigma_{j+1}^{(b)},
$$

as well as the mean field model realised by the complete graph

$$
H_{n}^{(\text {comp })}:=\frac{1}{\sqrt{9 n(n-1) / 2}} \sum_{1 \leq i<j \leq n} \sum_{a, b=1}^{3} \alpha_{a, b,(i j)} \sigma_{i}^{(a)} \sigma_{j}^{(b)} .
$$

It also applies to all $d_{n}$-regular graphs in between, i.e. those where every vertex has the same degree $d_{n} \geq 1$ (here $\left(d_{n}\right)_{n \in \mathbb{N}}$ is an arbitrary sequence of parameters). Indeed, these graphs satisfy $n d_{n}=2 e\left(\Gamma_{n}\right)$ and therefore

$$
\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}=\frac{d_{n}}{e\left(\Gamma_{n}\right)}=\frac{2}{n} \rightarrow 0
$$

as $n \rightarrow \infty$.
We can generalise Theorem 1 to hypergraphs allowing not only quadratic but also higher order spin interactions. The main condition is that the maximal hyperedge degree, i.e. the maximal number of hyperedges intersecting any fixed hyperedge, should be negligible compared with the total number of hyperedges. The precise formulation will be given in Theorem 8 here we present only a prominent example of this generalisation, the quantum $p$-spin glasses. For any $p \geq 1$, the Hamiltonian of a quantum $p$-spin glass is given by

$$
H_{n}^{(p-\text { glass })}:=3^{-p / 2}\binom{n}{p}^{-1 / 2} \sum_{1 \leq i_{1}<\cdots<i_{p} \leq n} \sum_{a_{1}, \ldots, a_{p}=1}^{3} \alpha_{a_{1}, \ldots, a_{p},\left(i_{1} \ldots i_{p}\right)} \sigma_{i_{1}}^{\left(a_{1}\right)} \ldots \sigma_{i_{p}}^{\left(a_{p}\right)}
$$

The following theorem shows that the limiting density of states is Gaussian if $p$ is fixed or it is $n$-dependent, $p=p_{n}$, but grows slower than $\sqrt{n}$ i.e. $\lim _{n \rightarrow \infty} \frac{p_{n}}{\sqrt{n}}=0$. On the other hand, if $p_{n}$ grows faster than $\sqrt{n}$ i.e. $\lim _{n \rightarrow \infty} \frac{\sqrt{n}}{p_{n}}=0$, then the density of states is given by the semicircle law. We shall use the notations $a_{n} \ll b_{n}$ and $a_{n} \gg b_{n}$ meaning that $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=0$ or $\lim _{n \rightarrow \infty} \frac{b_{n}}{a_{n}}=0$, respectively.

Theorem 3. Let $1 \leq p_{n} \leq n$ be any sequence in $n$ and assume that the independent random variables $\alpha_{a_{1}, \ldots, a_{p_{n}},\left(i_{1} \ldots i_{p_{n}}\right)}$ have zero mean, unit variance and form a tight family of random variables. Then the expected density of states of the Hamiltonians $H_{n}^{\left(p_{n}-g l a s s\right)}$ converges weakly to
(i) a standard normal distribution if $p_{n} \ll \sqrt{n}$,
(ii) a semicircle distribution with density function $\rho(x)=\frac{1}{2 \pi} \sqrt{4-x^{2}} \chi_{[-2,2]}(x)$ if $p_{n} \gg \sqrt{n}$,
(iii) a distribution with the compactly supported density function

$$
\rho_{\lambda}(x)= \begin{cases}v\left(x \mid e^{-4 \lambda / 3}\right) & \text { if } x \in\left[-\frac{2}{\sqrt{1-e^{-4 \lambda / 3}}}, \frac{2}{\sqrt{1-e^{-4 \lambda / 3}}}\right],  \tag{2}\\ 0 & \text { else }\end{cases}
$$

where

$$
v(x \mid q):=\frac{\sqrt{1-q}}{\pi \sqrt{1-(1-q) x^{2} / 4}} \prod_{k=0}^{\infty}\left[\frac{1-q^{2 k+2}}{1-q^{2 k+1}}\left(1-\frac{x^{2}(1-q) q^{k}}{\left(1+q^{k}\right)^{2}}\right)\right]
$$

$$
\text { if } \lim _{n \rightarrow \infty} \frac{p_{n}}{\sqrt{n}}=\lambda \text {. }
$$

## 3. Moment Method

To show that the expected density of states converges weakly to some distribution $\mu$ it often suffices to show that the moments

$$
m_{k, n}:=\int_{\mathbb{R}} x^{n} \mathrm{~d} \mu_{n}(x)=\mathbf{E} \frac{1}{2^{n}} \operatorname{Tr}\left(H_{n}^{\left(\Gamma_{n}\right)}\right)^{k}=\frac{1}{2^{n}} \operatorname{Tr} \mathbf{E}\left(H_{n}^{\left(\Gamma_{n}\right)}\right)^{k}
$$

of $\mu_{n}$ converge pointwise to the moments $m_{k}:=\int_{\mathbb{R}} x^{k} \mathrm{~d} \mu(x)$ of $\mu$ as $n \rightarrow \infty$. A sufficient condition for the uniqueness of the limiting distribution is given by Carleman's condition (see [2]): A probability distribution $\mu$ is uniquely determined by its moments $m_{k}$ if $\sum_{k=1}^{\infty} m_{2 k}^{-1 / 2 k}=\infty$.

To keep the terms simple we introduce the notations

$$
\sigma_{J}:=\sigma_{i}^{\left(a_{1}\right)} \sigma_{j}^{\left(a_{2}\right)}, \quad \alpha_{J}:=\alpha_{a_{1}, a_{2},(i j)}
$$

for tuples $J=(\mathbf{a},(i j))=\left(a_{1}, a_{2},(i j)\right)$ and denote the index sets by

$$
I_{n}:=\{1,2,3\}^{2} \times \Gamma_{n}=\left\{(\mathbf{a},(i j))=\left(a_{1}, a_{2},(i j)\right) \mid \mathbf{a}=\left(a_{1}, a_{2}\right) \in\{1,2,3\}^{2},(i j) \in \Gamma_{n}\right\}
$$

In order to compute the $k$-th moment $m_{k}$ we have to evaluate the sum

$$
\begin{align*}
m_{k, n} & =2^{-n} \operatorname{Tr} \mathbf{E}\left(H_{n}^{\left(\Gamma_{n}\right)}\right)^{k}=2^{-n} \operatorname{Tr} \mathbf{E}\left(\frac{1}{\sqrt{9 e\left(\Gamma_{n}\right)}} \sum_{J \in I_{n}} \sigma_{J} \alpha_{J}\right)^{k} \\
& =\left(9 e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{J_{1}, \ldots, J_{k} \in I_{n}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}} \tag{3}
\end{align*}
$$

in the limit $n \rightarrow \infty$. We split the sum in (3) into three disjoint parts

$$
\begin{equation*}
\sum_{J_{1}, \ldots, J_{k} \in I_{n}}=\sum_{D_{n, k}}+\sum_{A_{n, k}}+\sum_{B_{n, k}} \tag{4}
\end{equation*}
$$

for a partition $I_{n}^{k}=A_{n, k} \cup B_{n, k} \cup D_{n, k}$ into three subsets defined below.
Given any set $X$ and any integer $k \geq 1$ we define $P_{2}\left(X^{k}\right)$ to be those tuples $\left(x_{1}, \ldots, x_{k}\right) \in X^{k}$ for which all entries $x_{1}, \ldots, x_{k}$ appear exactly twice. Firstly, we split $I_{n}^{k}$ into the disjoint sets $P_{2}\left(I_{n}^{k}\right)$ and its complement $D_{n, k}:=I_{n}^{k} \backslash P_{2}\left(I_{n}^{k}\right)$ and then further split $P_{2}\left(I_{n}^{k}\right)$ into

$$
A_{n, k}:=\left\{\left(\left(\mathbf{a}_{1}, e_{1}\right), \ldots,\left(\mathbf{a}_{k}, e_{k}\right)\right) \in P_{2}\left(I_{n}^{k}\right) \mid e_{i} \cap e_{j}=\emptyset \text { if } e_{i} \neq e_{j}\right\}
$$

the family of $k$-tuples with all edges non-intersecting, and its complement $B_{n, k}:=P_{2}\left(I_{n}^{k}\right) \backslash A_{n, k}$. The condition $e_{i} \cap e_{j}=\emptyset$ (meaning that the edges have no vertex in common) assures that the matrices $\sigma_{\left(\mathbf{a}_{i}, e_{i}\right)}$ and $\sigma_{\left(\mathbf{a}_{j}, e_{j}\right)}$ commute. The reasons for these two splits are of entirely different nature. As Lemma 4 below shows the sum over $D_{n, k}$ is negligible under fairly general circumstances due to combinatorics without using any properties of the traces but their boundedness. The second split of the remaining $P_{2}\left(I_{n}^{k}\right)$ into $A_{n, k} \cup B_{n, k}$ is important since for $\left(J_{1}, \ldots, J_{k}\right) \in A_{n, k}$ the $\sigma_{J_{i}}$ corresponding to different $J_{i}$ commute and the can be reordered in such a way that only squares of Pauli matrices remain and the normalised trace is 1. That means that all relevant quantum effects due to (potential) non-commutativities are isolated in the index set $B_{n, k}$. The system is essentially classical if the contribution of the index set $B_{n, k}$ to the rhs. of eq. (3) can be neglected. If this is the case, the asymptotic eigenvalue distribution equals the asymptotic energy histogram of the corresponding classical model where the spin matrices are replaced by commuting spins $s_{i} \in\{-1,1\}$ or $s_{i} \in S^{2}$.

Lemma 4. Let $\left(X_{n}\right)_{n \in \mathbb{N}}$ be a growing sequence of index sets and let $\left\{\alpha_{x} \mid n \in \mathbb{N}, x \in X_{n}\right\}$ be a family of independent random variables with zero mean and unit variance and uniformly bounded moments $\left|\mathbf{E} \alpha_{x}^{k}\right| \leq$ $C_{k}<\infty$. Then we have the bound

$$
\begin{aligned}
\left|X_{n}\right|^{-k / 2} & \sum_{\left(x_{1}, \ldots, x_{k}\right) \in X_{n}^{k} \backslash P_{2}\left(X_{n}^{k}\right)}\left|\mathbf{E} \alpha_{x_{1}} \ldots \alpha_{x_{k}}\right| \\
& \leq \begin{cases}0 & \text { if } k<3 \text { is odd, } \\
\frac{k!!}{\left|X_{n}\right|^{1 / 2}} \frac{(k-1) C_{3}}{3}+\mathcal{O}\left(\left|X_{n}\right|^{-3 / 2}\right) & \text { if } k \geq 3 \text { is odd, } \\
0 & \text { if } k<4 \text { is even, } \\
\frac{(k-1)!!}{\left|X_{n}\right|}\left(\frac{k(k-2)(k-4) C_{3}^{2}}{18}+\frac{k(k-2) C_{4}}{4}\right)+\mathcal{O}\left(\left|X_{n}\right|^{-2}\right) & \text { if } k \geq 4 \text { is even }\end{cases}
\end{aligned}
$$

as $n \rightarrow \infty$ while $k$ is fixed.
Proof. First note that in the case that some $x_{i}$ only appears once, by independence and zero mean hypothesis these terms vanish identically. Since the case that all $x_{i}$ 's appear exactly twice is excluded from the index set in the sum above, we only have to consider those terms for which there are strictly less than $\frac{k}{2}$ distinct $x_{i}$ 's. There are only $\mathcal{O}\left(\left|X_{n}\right|^{m}\right)$ (as $\left.n \rightarrow \infty\right)$ terms with $m<\frac{k}{2}$ distinct $x_{i}$ 's, so we find, after summation, that the total contribution of these terms vanish as $\mathcal{O}\left(\left|X_{n}\right|^{m-k / 2}\right)$ as $n \rightarrow \infty$. Let us try to find the coefficient of the highest order term in $n$. In the case that $k$ is odd the term with the highest order comes from $m=\frac{k-1}{2}$ i.e. vanishes for $k<3$. The terms with $\frac{k-1}{2}$ distinct $x_{i}$ 's such that all $x_{i}$ appear at least twice are those for which some $x_{i}$ appears three times and the rest only two times. There are $\binom{n}{k / 2-1 / 2}$ ways of choosing the $x_{i}$ 's to appear, then there are $\frac{k-1}{2}$ ways of choosing the $x_{i}$ that appears three times and there are $\frac{k!}{3 \cdot 2^{k / 2-3 / 2}}$ ways of assigning those pre-described values to the tuples $\left(x_{1}, \ldots, x_{k}\right)$. In total we find for the number of terms contributing to the leading order

$$
\binom{\left|X_{n}\right|}{k / 2-1 / 2} \frac{k-1}{2} \frac{k!}{3 \cdot 2^{k / 2-3 / 2}}=\frac{\left|X_{n}\right|^{k / 2-1 / 2} k!}{3(k-3)!!}+\mathcal{O}\left(\left|X_{n}\right|^{k / 2-3 / 2}\right),
$$

each having a modulus bounded by $C_{3}$. The proof for even $k$ is analogous.
The estimate in Lemma 4 used only the scaling properties of the expectations and applies to the computation of $m_{k, n}$ since the normalised traces have uniformly bounded modulus. By using more specifics of the tracial part we can improve the error estimate from Lemma 4 significantly:

Lemma 5. Assume that the random variables $\alpha_{J}$ are independent, have zero mean, unit variance and uniformly bounded moments $\left|\mathbf{E} \alpha_{J}^{k}\right| \leq C_{k}<\infty$ for all $n \in \mathbb{N}, J \in I_{n}$. Then we have the bound

$$
\begin{aligned}
& \mid\left(9 e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in D_{n, k}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}} \mid \\
& \quad \leq \begin{array}{ll}
0 & \text { if } k<9 \text { is odd } \\
\frac{k!!}{e\left(\Gamma_{n}\right)^{3 / 2}} \frac{C_{3}^{3}(k-1)(k-3)(k-5)(k-7)}{3^{5}}+\mathcal{O}\left(e\left(\Gamma_{n}\right)^{-5 / 2}\right) & \text { if } k \geq 9 \text { is odd } \\
0 & \text { if } k<4 \text { is even } \\
\frac{(k-1)!!}{e\left(\Gamma_{n}\right)} \frac{k(k-2) C_{4}}{36}+\mathcal{O}\left(e\left(\Gamma_{n}\right)^{-2}\right) & \text { if } k \geq 4 \text { is even }
\end{array}
\end{aligned}
$$

as $n \rightarrow \infty$ while $k$ is fixed.

Before going into the proof of Lemma 5 we state some properties of the traces of Pauli matrices we shall need. A proof of this technical Lemma is given in the appendix.

Lemma 6 (Traces of products of Pauli matrices). Given $a_{1}, \ldots, a_{k} \in\{1,2,3\}$ the normalised traces of products of Pauli matrices $\sigma\left(a_{1}, \ldots, a_{k}\right):=\frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{1}\right)} \ldots \sigma^{\left(a_{k}\right)}$ satisfy:
(i) $\sigma\left(a_{1}, \ldots, a_{k}\right) \in\{0,1,-1, i,-i\}$;
(ii) More generally for all $1 \leq j_{1}, \ldots, j_{k} \leq n$

$$
\frac{1}{2^{n}} \operatorname{Tr} \sigma_{j_{1}}^{\left(a_{1}\right)} \ldots \sigma_{j_{k}}^{\left(a_{k}\right)} \in\{0,1,-1, i,-i\}
$$

(iii) $\sigma\left(a_{1}, \ldots, a_{k}\right)$ is non-zero if and only if the parities of the numbers of 1 's, 2 's and 3 's among the $a_{1}, \ldots, a_{k}$ coincide.
(iv) If $k$ is even we have the recursion relation

$$
\sigma\left(a_{1}, \ldots, a_{k}\right)=\sum_{j=2}^{k} \delta_{a_{1} a_{j}}(-1)^{j} \sigma\left(a_{2}, \ldots, \widehat{a_{j}}, \ldots, a_{k}\right)
$$

where $\widehat{a_{j}}$ means that the $j$-th entry is omitted;
Proof of Lemma 5. For odd $k$ first note that up to a factor of $\pm 1$ we can reorder the $\sigma_{J_{i}}$ 's in the expression $2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}}$ since Pauli matrices either commute or anti-commute. Since for any $J_{i}$ the we have $\sigma_{J_{i}}^{2}=1_{2^{n}}$ the normalised trace reduces to $\pm 2^{-n} \operatorname{Tr} \sigma_{J_{i_{1}}} \ldots \sigma_{J_{i_{l}}}$ with the $J_{i_{1}}, \ldots, J_{i_{l}}$ being all exactly those distinct $J_{1}, \ldots, J_{k}$ that appear an odd number of times. By Lemma 6(iii) in each component of the tensor product we get a zero trace if there are either one or two different Pauli matrices acting on it. Hence the normalised trace is zero if there are one or two distinct $J_{i}$ appearing an odd number of times. Since $k$ is odd we therefore see that the highest order contribution comes from the term where three distinct $J_{i}$ appear three times and the rest appear two times. Thus for $k<9$ we see that $m_{k, n}$ is identically zero. For $k \geq 9$ we first choose the $\left(\frac{k-9}{2}+3\right)=\frac{k-3}{2}$ distinct $J_{i}$ to appear and then those three to appear three times. By counting the number of ways of assigning those $J_{i}$ to our tuples we therefore find the factor

$$
\binom{9 e\left(\Gamma_{n}\right)}{k / 2-3 / 2}\binom{k / 2-3 / 2}{3} \frac{k!}{3^{3} \cdot 2^{k / 2-3 / 2}}=\frac{\left(9 e\left(\Gamma_{n}\right)\right)^{k / 2-3 / 2}}{(k / 2-3 / 2)!}\binom{k / 2-3 / 2}{3} \frac{k!}{3^{3} \cdot 2^{k / 2-3 / 2}}+\mathcal{O}\left(\left(e\left(\Gamma_{n}\right)\right)^{k / 2-5 / 2}\right)
$$

as $n \rightarrow \infty$ from which after dividing by $\left(9 e\left(\Gamma_{n}\right)\right)^{k / 2}$ the claimed asymptotics follow. The claims for even $k$ immediately follow from Lemma 4 using that the term with $C_{3}^{2}$ vanishes by the above argument (since there are two distinct $J_{i}$ appearing an odd number of times).

In particular this already shows that in the limit $n \rightarrow \infty$ all odd moments vanish. The situation with the sums over $A_{n, k}$ and $B_{n, k}$ in (41) is a little bit more delicate. For a large class of graph sequences the sum over $B_{n, k}$ is also negligible and the only contribution comes from $A_{n, k}$ where all normalised traces are equal to 1 and the system is essentially classical. In this case the non-commutativity is actually only a small perturbation and consequently we see the same result as in the classical central limit theorem rather than a random matrix semicircle law. We are now ready to give a proof of Theorem [1 including explicit estimates regarding the rate of convergence of the moments.

Theorem $\mathbb{1}^{1}$ (Detailed version). Denote the maximal vertex degree in the graph $\Gamma_{n}$ by $d_{\max }(n)$. Let $\Gamma_{n}$ be a sequence of graphs on the vertex sets $\{1, \ldots, n\}$ such that $\lim _{n \rightarrow \infty} \frac{d_{\text {max }}(n)}{e\left(\Gamma_{n}\right)}=0$ and let

$$
\left\{\alpha_{a, b,(i j)} \mid 1 \leq a, b \leq 3,(i j) \in \Gamma_{n}\right\}
$$

be a tight collection of independent (not necessarily identically distributed) random variables with zero mean and unit variance. Then the Hamiltonian defined by

$$
H_{n}^{\left(\Gamma_{n}\right)}:=\frac{1}{\sqrt{9 e\left(\Gamma_{n}\right)}} \sum_{(i j) \in \Gamma_{n}} \sum_{a, b=1}^{3} \alpha_{a, b,(i j)} \sigma_{i}^{(a)} \sigma_{j}^{(b)}
$$

(where as a convention the edge between $i<j$ is denoted by $(i j)$ ) has an expected density of states which converges weakly to a standard normal distribution. The convergence rate of the moments is of order $e\left(\Gamma_{n}\right)^{-3 / 2}$ for odd moments and $\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}$ for even moments. Moreover $m_{k, n} \equiv m_{k}$ for $k \in\{0,1,2,3,5,7\}$.

Proof. First note that we can, without loss of generality, assume that the random variables are uniformly bounded and therefore have also uniformly bounded moments. This follows from a standard reduction step relying on the Hoffman-Wielandt inequality that allows us to approximate the density of states $\mu_{n}$ by the density of states of a Hamiltonian with truncated random variables. For details the reader is referred to the proof of [1, Theorem 2.1.21] which can be adapted to our model by the tightness assumption on the random variables. This reduction step is also valid in the proofs of Theorems 3, 8 and Proposition 7, where we shall assume it without further explanation.

The treatment of the sum from eq. (3) is performed in three steps according to the split from eq. (4). Lemma 5 dealt with the $D_{n, k}$-part of the sum. We now consider the part of the sum over the index set $B_{n, k}$. From the condition $\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)} \rightarrow 0$ as $n \rightarrow \infty$ it follows that the number $d_{j, n}$ of choosing $j$ non intersecting edges from the graph $\Gamma_{n}$ asymptotically behaves as $e\left(\Gamma_{n}\right)^{j} / j$ ! i.e. $\lim _{n \rightarrow \infty} \frac{d_{j, n}}{e\left(\Gamma_{n}\right)^{j} j!}=1$ for all fixed $j$. Indeed, there are $e\left(\Gamma_{n}\right)$ choices for the first edge (il). For the next edge we can pick all edges except those including $i$ and $l$ i.e. there are at least $e\left(\Gamma_{n}\right)-2 d_{\max }(n)$ choices for the second edge. Continuing we find the bound

$$
\begin{align*}
\frac{e\left(\Gamma_{n}\right)^{j}}{j!} & \geq d_{j, n} \geq \frac{1}{j!} e\left(\Gamma_{n}\right)\left(e\left(\Gamma_{n}\right)-2 d_{\max }(n)\right)\left(\left(e\left(\Gamma_{n}\right)-4 d_{\max }(n)\right)\right) \ldots\left(e\left(\Gamma_{n}\right)-2(j-1) d_{\max }(n)\right) \\
& =\frac{e\left(\Gamma_{n}\right)^{j}}{j!}\left(1-\frac{j(j-1) d_{\max }(n)}{e\left(\Gamma_{n}\right)}+\mathcal{O}\left(\left(\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)^{2}\right)\right) \tag{5}
\end{align*}
$$

as $n \rightarrow \infty$. Dividing by $e\left(\Gamma_{n}\right)^{j} / j$ ! then proves that $d_{j, n}$ asymptotically behaves as $e\left(\Gamma_{n}\right)^{j} / j$ !.
The estimate from eq. (5) also shows that the number of choosing $j$ edges that have at least one intersection is, to leading order, at most given by

$$
\frac{e\left(\Gamma_{n}\right)^{j-1} d_{\max }(n)}{(j-2)!}+\mathcal{O}\left(\left(\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)^{2} e\left(\Gamma_{n}\right)^{j}\right)
$$

Since there are $\binom{k}{2}\binom{k-2}{2} \ldots\binom{2}{2}=\frac{k!}{2^{k / 2}}$ ways of assigning $\frac{k}{2}$ chosen edges to $e_{1}, \ldots, e_{k}$ such that each appears twice, the index set $B_{n, k}$ therefore contains at most

$$
9^{k / 2} \frac{k!}{2^{k / 2}} \frac{d_{\max }(n)}{(k / 2-2)!} e\left(\Gamma_{n}\right)^{k / 2-1}+\mathcal{O}\left(\left(\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)^{2} e\left(\Gamma_{n}\right)^{k / 2}\right)
$$

elements as $n \rightarrow \infty$. Using that the modulus of the normalised traces is at most 1 (see Lemma 6) and that the expectations are all equal to 1 due to unit variance and independence, we therefore found the bound

$$
\begin{align*}
& \left.\frac{1}{\left(9 e\left(\Gamma_{n}\right)\right)^{k / 2}} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in B_{n, k}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}} \right\rvert\, \\
& \quad \leq(k-1)!!\frac{k(k-2)}{4} \frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}+\mathcal{O}\left(\left(\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)^{2}\right) \tag{6}
\end{align*}
$$

as $n \rightarrow \infty$ while $k$ is fixed.
For the summation over $A_{n, k}$ in (4), we first note that all terms under the sum are equal to 1 . Indeed, the expectations are again 1 by independence and unit variance. For the traces we find that since all distinct $J_{i}$ act on distinct qubits, in all components there is either an identity matrix or a product of two identical Pauli matrices i.e. again identity matrices. Again similarly to eq. (5) we can estimate $\left|A_{n, k}\right|$ to get

$$
\begin{equation*}
(k-1)!!\geq\left(9 e\left(\Gamma_{n}\right)\right)^{-k / 2}\left|A_{n, k}\right| \geq(k-1)!!\left(1-\frac{k(k-2)}{4} \frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)+\mathcal{O}\left(\left(\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)^{2}\right) \tag{7}
\end{equation*}
$$

as $n \rightarrow \infty$ while $k$ is fixed.
The moments $m_{k}$ of a standard normal distribution are given by $(k-1)!$ ! for even $k$ and 0 for odd $k$ and satisfy Carleman's continuity condition. Using the bound in Lemma 5 together with eqs. (6) and (7) we arrive at

$$
\begin{aligned}
\left|m_{k, n}-(k-1)!!\right| \leq & \left|\begin{array}{l}
\left(9 e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in D_{n, k}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}}
\end{array}\right| \\
& \left.+\mid 9 e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in B_{n, k}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}} \mid \\
& +\left|\left(9 e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in A_{n, k}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}}-(k-1)!!\right| \\
\leq & (k-1)!!k(k-2)\left(\frac{C_{4}}{36 e\left(\Gamma_{n}\right)}+\frac{d_{\max }(n)}{2 e\left(\Gamma_{n}\right)}\right)+\mathcal{O}\left(\left(\frac{d_{\max }(n)}{e\left(\Gamma_{n}\right)}\right)^{2}\right)
\end{aligned}
$$

for (fixed) even $k \geq 9$ as $n \rightarrow \infty$, whereas

$$
\begin{aligned}
\left|m_{k, n}-0\right| & =\left|\left(9 e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in D_{n, k}} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}}\right| \\
& \leq \frac{k!!}{e\left(\Gamma_{n}\right)^{3 / 2}} \frac{C_{3}^{3}(k-1)(k-3)(k-5)(k-7)}{3^{5}}+\mathcal{O}\left(e\left(\Gamma_{n}\right)^{-5 / 2}\right)
\end{aligned}
$$

for (fixed) odd $k$ as $n \rightarrow \infty$. This shows the convergence of each moment, thus the weak convergence. The claim that the moments $m_{k, n}$ agree identically with $m_{k}$ for $k \in\{1,2,3,5,7\}$ follows immediately from Lemma 5


Figure 1: Empirical density of states for sample size 1000 including the limiting density

As the following example shows, the assumption on the growth of the maximal degree is necessary. Let

$$
H_{n}^{(\text {star })}:=\frac{1}{\sqrt{9(n-1)}} \sum_{j=2}^{n} \sum_{a, b=1}^{3} \alpha_{a, b, j} \sigma_{1}^{(a)} \sigma_{j}^{(b)}
$$

be the Hamiltonian corresponding to the star graph in which, say, the vertex 1 is connected to all other vertices while there are no edges between the rest. This model shows a significantly different limiting behaviour (a proof is given in the Appendix, see also Figure (1):

Proposition 7. Suppose that the random variables $\left\{\alpha_{a, b, j} \mid n \in \mathbb{N}, 1 \leq a, b \leq 3,2 \leq j \leq n\right\}$ are independent, have zero mean, unit variance and form a tight family of random variables. The expected density of states of $H_{n}^{(s t a r)}$ then converges weakly to a distribution with density

$$
\rho(x)=3 \sqrt{\frac{3}{2 \pi}} x^{2} e^{-3 x^{2} / 2}
$$

as $n \rightarrow \infty$.

## 4. Hypergraphs

A hypergraph is a generalised graph in which any hyperedge can contain a variable number of vertices. Formally a hypergraph on a vertex set $V$ is any subset of $\mathcal{P}(V) \backslash \emptyset$. A hyperedge $e$ containing the (distinct) vertices $i_{1}<\cdots<i_{l}$ will be denoted by $\left(i_{1} \ldots i_{l}\right)$. We shall use the notation $|e|:=l$ for the number of vertices in a given hyperedge $e=\left(i_{1} \ldots i_{l}\right)$. Just as in the traditional graph, the degree of a vertex is
defined to be the number of hyperedges containing the given vertex. The total number of hyperedges is again denoted by $e\left(\Gamma_{n}\right)$. For a given hypergraph $\Gamma_{n}$ on the vertex set $\{1, \ldots, n\}$ we introduce the notations

$$
\alpha_{J}:=\alpha_{(\mathbf{a}, e)}:=\alpha_{a_{1}, \ldots, a_{l},\left(i_{1} \ldots i_{l}\right)}, \quad \sigma_{J}:=\sigma_{(\mathbf{a}, e)}:=\sigma_{i_{1}}^{\left(a_{1}\right)} \ldots \sigma_{i_{l}}^{\left(a_{l}\right)}
$$

and $|J|=|e|=l$ for

$$
J=(\mathbf{a}, e)=\left(a_{1}, \ldots, a_{l},\left(i_{1} \ldots i_{l}\right)\right) \in I_{n}:=\left\{(\mathbf{a}, e) \mid e \in \Gamma_{n}, \mathbf{a} \in\{1,2,3\}^{|e|}\right\}
$$

The generalised Hamiltonian corresponding to the hypergraph $\Gamma_{n}$ is defined to be

$$
\begin{equation*}
H_{n}^{\left(\Gamma_{n}\right)}:=\frac{1}{\sqrt{e\left(\Gamma_{n}\right)}} \sum_{e \in \Gamma_{n}} \frac{1}{3^{|e| / 2}} \sum_{\mathbf{a} \in\{1,2,3\}|e|} \alpha_{(\mathbf{a}, e)} \sigma_{(\mathbf{a}, e)} \tag{8}
\end{equation*}
$$

We again want to study the moments

$$
\begin{equation*}
m_{k, n}=2^{-n} \operatorname{Tr} \mathbf{E}\left(H_{n}^{\left(\Gamma_{n}\right)}\right)^{k}=\left(e\left(\Gamma_{n}\right)\right)^{-k / 2} \sum_{J_{1}, \ldots, J_{k} \in I_{n}} 3^{-\left(\left|J_{1}\right|+\cdots+\left|J_{k}\right|\right) / 2} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} \mathbf{E} \alpha_{J_{1}} \ldots \alpha_{J_{k}} \tag{9}
\end{equation*}
$$

in the limit $n \rightarrow \infty$. Lemma 4 again applies and immediately shows that we can restrict our attention to those summands where the $J_{1}, \ldots, J_{k}$ appear in pairs of two. If the hyperedges of the $\frac{k}{2}$ distinct $J_{i}$ 's are disjoint we can reorder the $\sigma_{J_{i}}$ 's freely and therefore get a normalised trace of 1 . As in the proof of Theorem [1 we establish a sufficient criterion on the sequence of graphs such that among all families of $\frac{k}{2}$ edges the proportion of those that have mutually disjoint edges approaches 1 .

As for conventional graphs, the line graph $L\left(\Gamma_{n}\right)$ of a hypergraph $\Gamma_{n}$ is graph whose vertices are the hyperedges $\left\{e_{1}, \ldots, e_{M}\right\}$ of $\Gamma_{n}$. Two vertices of $L\left(\Gamma_{n}\right)$ (i.e. hyperedges of $\left.\Gamma_{n}\right) e_{1}, e_{2}$ are adjacent (connected by an edge in the line graph) if and only if $e_{1}$ and $e_{2}$ are non-disjoint and so the edges of $L\left(\Gamma_{n}\right)$ are given by

$$
\left\{\left(e_{i} e_{j}\right) \mid 1 \leq i, j \leq M, e_{i} \cap e_{j} \neq \emptyset\right\}
$$

Given some fixed hyperedge $e_{1} \in \Gamma_{n}$ there are at least $e\left(\Gamma_{n}\right)-d_{\max }\left(L\left(\Gamma_{n}\right)\right)$ hyperedges $e_{2}$ disjoint from $e_{1}$, where the maximal hyperedge degree $d_{\max }^{(e)}(n):=d_{\max }\left(L\left(\Gamma_{n}\right)\right)$ is the maximal vertex degree of the line graph. Continuing we find for the number $d_{j, n}$ of choices of $j$ disjoint hyperedges from $\Gamma_{n}$ the bound

$$
\begin{align*}
\frac{e\left(\Gamma_{n}\right)^{j}}{j!} & \geq d_{j, n} \geq \frac{1}{j!} e\left(\Gamma_{n}\right)\left(e\left(\Gamma_{n}\right)-d_{\max }^{(e)}(n)\right) \ldots\left(e\left(\Gamma_{n}\right)-(j-1) d_{\max }^{(e)}(n)\right) \\
& =\frac{e\left(\Gamma_{n}\right)^{j}}{j!}\left(1-\frac{j(j-1)}{2} \frac{d_{\max }^{(e)}(n)}{e\left(\Gamma_{n}\right)}+\mathcal{O}\left(\left(\frac{d_{\max }^{(e)}(n)}{e\left(\Gamma_{n}\right)}\right)^{2}\right)\right) \tag{10}
\end{align*}
$$

as $n \rightarrow \infty$ while $j$ is fixed if $\lim _{n \rightarrow \infty} \frac{d_{\max }^{(e)}(n)}{e\left(\Gamma_{n}\right)}=0$. Following the proof of Theorem 1 we therefore proved its generalisation for hypergraphs:
Theorem 8. Let $\Gamma_{n}$ be a sequence of graphs on the vertex sets $\{1, \ldots, n\}$ such that $\lim _{n \rightarrow \infty} \frac{d_{m a x}^{(e)}(n)}{e\left(\Gamma_{n}\right)}=0$ and let

$$
\left\{\alpha_{(\mathbf{a}, e)} \mid e \in \Gamma_{n}, \mathbf{a} \in\{1,2,3\}^{|e|}\right\}
$$

be a tight collection of independent (not necessarily identically distributed) random variables with zero mean and unit variance. Then the Hamiltonian defined in (8) has a density of states which converges weakly to a standard normal distribution.

For 2-uniform hypergraphs (meaning that all edges connect 2 vertices) the statement of this Theorem is equivalent to Theorem 1. More generally the theorem also covers a sequence of $p_{n}$-uniform graphs $\Gamma_{n}$ corresponding to the $p_{n}$-spin glasses. An interesting special case is the sequence of complete $p_{n}$-uniform hypergraphs in which the hyperedges connect any $p_{n}$ distinct vertices. The corresponding Hamiltonians are given by

$$
H_{n}^{\left(p_{n}-\text { glass }\right)}:=3^{-p_{n} / 2}\binom{n}{p_{n}}^{-1 / 2} \sum_{1 \leq i_{1}<\cdots<i_{p_{n}} \leq n} \sum_{a_{1}, \ldots, a_{p_{n}}=1}^{3} \alpha_{a_{1}, \ldots, a_{p_{n}},\left(i_{1} \ldots i_{p_{n}}\right)} \sigma_{i_{1}}^{\left(a_{1}\right)} \ldots \sigma_{i_{p_{n}}}^{\left(a_{p_{n}}\right)}
$$

In this case the degree of any hyperedge is

$$
\operatorname{deg}\left(i_{1} \ldots i_{p_{n}}\right)=\binom{n}{p_{n}}-\binom{n-p_{n}}{p_{n}}
$$

while the total number of hyperedges is given by $e\left(\Gamma_{n}\right)=\binom{n}{p_{n}}$. Since

$$
\lim _{n \rightarrow \infty} \frac{\binom{n-p_{n}}{p_{n}}}{\binom{n}{p_{n}}}= \begin{cases}1 & \text { if } p_{n} \ll \sqrt{n}, \\ 0 & \text { if } p_{n} \gg \sqrt{n}, \\ e^{-\alpha^{2}} & \text { if } \lim _{n \rightarrow \infty} \frac{p_{n}}{\sqrt{n}}=\alpha \in(0, \infty)\end{cases}
$$

(see Lemma 9, a proof is given in the appendix) this $p_{n}$-spin glass model fulfils the condition of Theorem 8 if and only if $p_{n}$ grows slower than $\sqrt{n}$.

We now turn to the question whether for $p_{n} \gg \sqrt{n}$, the expected density of states of $H_{n}^{\left(p_{n}-\text { glass }\right)}$ indeed exhibits a different limiting behaviour. As Theorem 3 shows, this is indeed the case and for $p_{n}$ growing faster than $\sqrt{n}$ the density of states approaches a semicircle distribution. This also shows that the condition about the maximal edge degree in Theorem 8 is in a certain sense optimal. We start with a combinatorial lemma, whose proof is given in the appendix.

Lemma 9 (Asymptotics of intersections of growing sets). Let $a_{n}, b_{n}$ and $c_{n}$ be three sequences taking values in $\{1, \ldots, n\}$.
(i) Given any subsets $A_{n} \subset\{1, \ldots, n\}$ with $a_{n}$ elements, the proportion of $B_{n} \subset\{1, \ldots, n\}$ with $b_{n}$ elements that have a non-empty intersection with $A_{n}$ goes to one if and only if $a_{n} b_{n}$ grows faster than n. More precisely it holds that

$$
\lim _{n \rightarrow \infty} \frac{\binom{n}{b_{n}}-\binom{n-a_{n}}{b_{n}}}{\binom{n}{b_{n}}}= \begin{cases}1 & \text { if } a_{n} b_{n} \gg n, \\ 0 & \text { if } a_{n} b_{n} \ll n, \\ 1-e^{-\lambda} & \text { if } \lim _{n \rightarrow \infty} \frac{a_{n} b_{n}}{n}=\lambda \in(0, \infty)\end{cases}
$$

(ii) Given any subsets $A_{n} \subset\{1, \ldots, n\}$ with $a_{n}$ elements, the proportion of $B_{n} \subset\{1, \ldots, n\}$ with $b_{n}$ elements that share at least $c_{n}$ elements with $A_{n}$ goes to 1 , i.e.

$$
\lim _{n \rightarrow \infty} \frac{\left|\left\{B_{n} \subset\{1, \ldots, n\}| | B_{n}\left|=b_{n},\left|A_{n} \cap B_{n}\right| \geq c_{n}\right\} \mid\right.\right.}{\binom{n}{b_{n}}}=1,
$$

provided $a_{n} b_{n} \gg n$ and $c_{n} \ll \frac{a_{n} b_{n}}{n}$.


Figure 2: Empirical density of states of some $p$-spin glass Hamiltonian with $p \approx n^{0.42}$ and $p \approx n^{0.56}$

Proof of Theorem [3. As already mentioned the first claim is a immediate consequence of Theorem 8 and the estimate from Lemma 9

Now assume that $p_{n}$ grows faster than $\sqrt{n}$. As before, we compute the moments and due to Lemma 4 again know that the odd moments vanish and for even moments we only have to consider those tuples of

$$
J_{i} \in I_{n}:=\left\{\left(\mathbf{a},\left(i_{1} \ldots i_{p_{n}}\right)\right) \mid \mathbf{a} \in\{1,2,3\}^{p_{n}}, 1 \leq i_{1}<\cdots<i_{p_{n}} \leq n\right\}
$$

which come in pairs of two. Using the already established short hand notation for the $\sigma_{J_{i}}$ we have, for even $k$,

$$
m_{k, n} \approx 3^{-k p_{n} / 2}\binom{n}{p_{n}}^{-k / 2} \sum_{\left(J_{1}, \ldots, J_{k}\right) \in P_{2}\left(I_{n}^{k}\right)} 2^{-n} \operatorname{Tr} \sigma_{J_{1}} \ldots \sigma_{J_{k}} .
$$

(here $\approx$ means is equal in the limit $n \rightarrow \infty$ ). We now rephrase condition $\left(J_{1} \ldots J_{k}\right) \in P_{2}\left(I_{n}^{k}\right)$. The tuples can be thought of being constructed by first drawing $\frac{k}{2}$ distinct $J_{i}$ from $I_{n}$ and then assigning those $\frac{k}{2} J_{i}$ 's to the tuples in a way that each $J_{i}$ appears twice. By defining the family of (labelled) pair-partitions of the set $\{1, \ldots, k\}$ into $\frac{k}{2}$ labelled subsets with 2 elements each;

$$
S_{k}:=\left\{\pi:\{1, \ldots, k\} \rightarrow\{1, \ldots, k / 2\}| | \pi^{-1}(\{j\}) \mid=2 \text { for all } 1 \leq j \leq k / 2\right\},
$$

the sum then reads

$$
\begin{equation*}
m_{k, n} \approx 3^{-k p_{n} / 2}\binom{n}{p_{n}}^{-k / 2} \sum_{\pi \in S_{k}} \sum_{\left\{J_{1}, \ldots, J_{k / 2}\right\} \subset I_{n}}^{\star} 2^{-n} \operatorname{Tr} \sigma_{J_{\pi(1)}} \ldots \sigma_{J_{\pi(k)}}, \tag{11}
\end{equation*}
$$

where $\sum^{\star}$ indicates that the elements $J_{1}, \ldots, J_{k / 2}$ are distinct.
At this point it is useful to introduce the notion of non-crossing pair-partitions which often appear in random matrix theory. An element $\pi \in S_{k}$ shall be called crossing if there exists $1 \leq a<b<c<d \leq k$ such that $\pi(a)=\pi(c)$ and $\pi(b)=\pi(d)$, otherwise it is called non-crossing; the corresponding subsets of $S_{k}$ are denoted by $S_{k}^{(c)}$ and $S_{k}^{(n c)}$. These notions emerge in this context since by Lemma 10 (a proof of which is given in the appendix) for a non-crossing $\pi \in S_{k}^{(n c)}$ the matrices $\sigma_{J_{\pi(j)}}$ in the trace in (11) can be reordered such that all appear as squares and therefore the normalised traces are all 1 independent of the $J_{l}$ 's.

Lemma 10 (Product of Pauli matrices ordered in pair-partitions). Let $k$ be even, $\pi \in S_{k}$ and define

$$
I_{n}:=\left\{(\mathbf{a}, e) \mid e \in \Gamma_{n}, \mathbf{a} \in\{1,2,3\}^{|e|}\right\}
$$

for some hypergraph $\Gamma_{n}$.
(i) If $\pi$ is non-crossing, then $\frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}=1$ for all $1 \leq a_{1}, \ldots, a_{k / 2} \leq 3$.
(ii) If $\pi$ is crossing, there exist $1 \leq a_{1}, \ldots a_{k / 2} \leq 3$ such that $\frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)} \neq 1$.
(iii) If $\pi$ is non-crossing, then $2^{-n} \operatorname{Tr} \sigma_{J_{\pi(1)}} \ldots \sigma_{J_{\pi(k)}}=1$ for all $J_{1}, \ldots, J_{k / 2} \in I_{n}$.

For the sum over the non-crossing pair-partitions $S_{k}^{(n c)}$ we thus find a contribution of

$$
3^{-k p_{n} / 2}\binom{n}{p_{n}}^{-k / 2} \sum_{\pi \in S_{k}^{(n c)}} \sum_{\left\{J_{1}, \ldots, J_{k / 2}\right\} \subset I_{n}}^{\star} 1=\frac{\left|S_{k}^{(n c)}\right|\binom{\left|I_{n}\right|}{k / 2}}{\binom{n}{p_{n}}^{k / 2} 3^{k p_{n} / 2}} \approx \frac{\left|S_{k}^{(n c)}\right|}{(k / 2)!}=\frac{k!}{(k / 2)!(k / 2+1)!},
$$

where it was used that the number of non-crossing pair-partitions into unlabelled subsets are given by the Catalan numbers (see e.g. [1, Proposition 2.1.11]). It remains to show that the sum over the crossing pair-partitions gives no contribution. Since the total number of partitions is finite it suffices to show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{J_{1}, \ldots, J_{k / 2} \in I_{n}} 2^{-n} \operatorname{Tr} \sigma_{J_{\pi(1)}} \ldots \sigma_{J_{\pi(k)}}=0 \tag{12}
\end{equation*}
$$

for each crossing $\pi \in S_{k}$. Notice that this summation is normalised, i.e. the combinatorial prefactor is exactly the number of terms in the sum.

Since $\pi$ is assumed to be crossing there are $1 \leq a<b<c<d \leq k$ such that $r:=\pi(a)=\pi(c)$ and $s:=\pi(b)=\pi(d)$. From Lemma 9 it follows that there exists a sequence $q_{n} \gg 1$ such that the proportion of pairs of subsets of $\{1, \ldots, n\}$, with $p_{n}$ elements each, that share at least $q_{n}$ elements approaches 1 as $n \rightarrow \infty$. Applied to our normalised sum in (12), this means that we can restrict our attention to those terms for which $e_{r}$ and $e_{s}$ have at least $q_{n}$ vertices in common. In this way we arrive at

$$
\begin{align*}
& \frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{\left(\mathbf{a}_{1}, e_{1}\right), \ldots,\left(\mathbf{a}_{k / 2}, e_{k / 2}\right) \in I_{n}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)}\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)}\right)} \\
\approx & \frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{\substack{e_{1}, \ldots, e_{k / 2} \in \Gamma_{n} \\
\mid e_{r} \cap e_{s} \geq q_{n}}} \sum_{\mathbf{a}_{1}, \ldots, \mathbf{a}_{k / 2} \in\{1,2,3\}^{p_{n}}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)}\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)}\right)} \tag{13}
\end{align*}
$$

for some $q_{n} \gg 1$ only depending on $p_{n}$.

For a hyperedge $e$, the hyperedge consisting of the first $l$ vertices of $e$ (with respect to the natural ordering) will be denoted by by $e_{1: l}$. We introduce the shorthand notation

$$
g:=\left(e_{r} \cap e_{s}\right)_{1: q_{n}}
$$

(we recall that hyperedges are subsets of the vertex set, thus set theoretical operations, such as $\in, \cup, \cap, \backslash$, are meaningful for them). With this notation we can factorise the inner sum from eq. (13) to get

$$
\begin{aligned}
& \sum_{\mathbf{a}_{1}, \ldots, \mathbf{a}_{k / 2} \in\{1,2,3\}^{p_{n}}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)}\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)}\right)}= \\
&\left(\sum_{\left.\sum_{\mathbf{a}_{1} \in\{1,2,3\}}\right\}^{\left|e_{1} \cap g\right|}} \ldots \sum_{\mathbf{a}_{k / 2} \in\{1,2,3\}^{\left|e_{k / 2} \cap g\right|}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)} \cap g\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)} \cap g\right)}\right) \\
& \times\left(\sum_{\sum_{\mathbf{a}_{1} \in\{1,2,3\}}} \ldots \sum_{\mathbf{a}_{k / 2} \in\{1,2,3\}^{\left|e_{1} \backslash g\right|}{ }^{\left|e_{k / 2} \backslash g\right|}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)} \backslash g\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)} \backslash g\right)}\right)
\end{aligned}
$$

where the second factor is bounded by $3^{\sum_{l=1}^{k / 2}\left|e_{l} \backslash g\right|}$. We then further factorise the first factor to obtain

$$
\prod_{j \in g} \sum_{\substack{a_{l} \in\{1,2,2\} \text { if } j \in e_{\imath} \cap g \\ a_{l}=0 \text { else }}} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)} .
$$

For any fixed $j \in g$, the number

$$
m_{j}:=\left|\left\{1 \leq l \leq k / 2 \mid j \in e_{l} \cap g\right\}\right|
$$

of l's such that $e_{l} \cap g$ contains the vertex $j$, is always between $2 \leq m_{j} \leq k / 2$ since at least $r$ and $s$ satisfy this condition. By ignoring the $a_{l}=0$ factors and writing $m$ for $m_{j}$, we see that we can rewrite

$$
\sum_{\substack{a_{l} \in\{1,2,3\} \text { if } \\ a_{l}=0 \text { else }}} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}=\sum_{a_{1}, \ldots, a_{m}=1}^{3} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\tilde{\pi}(1)}\right)} \ldots \sigma^{\left(a_{\tilde{\pi}(2 m)}\right)}
$$

for some crossing $\tilde{\pi} \in S_{2 m}$. According to part (ii) of Lemma 10, some (but not all) terms in this sum are equal to -1 and therefore there exists a (possibly) $\tilde{\pi}$ and $m$-dependent constant $C(\tilde{\pi}, m)<1$ such that

$$
\left|\sum_{a_{1}, \ldots, a_{m}=1}^{3} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\tilde{\pi}(1)}\right)} \ldots \sigma^{\left(a_{\tilde{\pi}(2 m)}\right)}\right| \leq C(\tilde{\pi}, m) \cdot 3^{m} .
$$

By setting the

$$
C:=\max _{2 \leq m \leq k / 2} \max _{\tilde{\pi} \in S_{2 m}^{c(c)}} C(\tilde{\pi}, m)<1
$$

to be the maximum of those constants, and recalling that $|g|=q_{n}$, we arrive at

$$
\left|\prod_{j \in g} \sum_{\substack{a_{l} \in\{1,2,3\} \text { if } \\ a_{l}=0 \text { else }}} \frac{1}{j \in e_{l} \cap g} 2 \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}\right| \leq \prod_{j \in g}\left[C \cdot 3^{\left|\left\{1 \leq l \leq k / 2 \mid j \in e_{l} \cap g\right\}\right|}\right]=C^{q_{n}} \cdot 3^{\sum_{l=1}^{k / 2}\left|e e_{l} \cap g\right|} .
$$

After plugging in our estimates into eq. (13) we finally find

$$
\begin{align*}
& \left\lvert\, \frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{\left(\mathbf{a}_{1}, e_{1}\right), \ldots,\left(\mathbf{a}_{k / 2}, e_{k / 2}\right) \in I_{n}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)}\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)}\right) \mid}\right. \\
& \quad \leq \frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{\substack{e_{1}, \ldots, e_{k / 2} \in \Gamma_{n} \\
\left|e_{r} \cap e_{s}\right| \geq q_{n}}} C^{q_{n}} 3^{\sum_{l=1}^{k / 2}\left(\left|e_{l} \cap g\right|+\left|e_{l} \backslash g\right|\right)}+\mathcal{O}(1) \\
& \quad=\frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{e_{1}, \ldots, e_{k / 2} \in \Gamma_{n}} C^{q_{n}} 3^{k p_{n} / 2}+\mathcal{O}(1)=\frac{C^{q_{n}}}{(k / 2)!}+\mathcal{O}(1)=\mathcal{O}(1) \tag{14}
\end{align*}
$$

as $n \rightarrow \infty$, proving that the contribution of any crossing partition vanishes.
We have now proved that the $k$-th moment of the limiting distribution is given by $\frac{k!}{(k / 2)!(k / 2+1)!}$ for even $k$ and 0 for odd $k$. A direct computation shows that these are the moments of the semicircular distribution with density function

$$
\rho(x)=\frac{1}{2 \pi} \sqrt{4-x^{2}} \chi_{[-2,2]}(x)
$$

which furthermore satisfy Carleman's continuity condition.
We now turn to part (iii) of Theorem 3 i.e. the case where $\lim _{n \rightarrow \infty} \frac{p_{n}}{\sqrt{n}}=\lambda \in(0, \infty)$. By Lemma 44 the odd moments vanish also in this case. For even $k$ an explicit formula for the $k$-th moment can be derived as follows. For a given partition $\pi \in S_{k}$ we define the number of crossings $\kappa(\pi)$ to be the number of subsets $\{r, s\} \subset\{1, \ldots, k / 2\}$ such that for some $1 \leq a<b<c<d \leq k$ we have that $\pi(a)=\pi(c)=r$ and $\pi(b)=\pi(d)=s$. We claim that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{3^{-k p_{n} / 2}}{(k / 2)!}\binom{n}{p_{n}}^{-k / 2} \sum_{\left(\mathbf{a}_{1}, e_{1}\right), \ldots,\left(\mathbf{a}_{k / 2}, e_{k / 2}\right) \in I_{n}} 2^{-n} \operatorname{Tr} \sigma_{\left(\mathbf{a}_{\pi(1)}, e_{\pi(1)}\right)} \ldots \sigma_{\left(\mathbf{a}_{\pi(k)}, e_{\pi(k)}\right)}=\frac{\left(e^{-4 \lambda / 3}\right)^{\kappa(\pi)}}{(k / 2)!} \tag{15}
\end{equation*}
$$

holds for all partitions $\pi$. If $\left\{r_{1}, s_{2}\right\}, \ldots,\left\{r_{\kappa(\pi)}, s_{\kappa(\pi)}\right\}$ are the crossings of $\pi$, by Lemma 9 the numbers of vertices in the intersections $e_{r_{1}} \cap e_{s_{1}}, \ldots, e_{r_{\kappa(\pi)}} \cap e_{s_{\kappa(\pi)}}$ are approximately independently Poisson- $\lambda$ distributed. It furthermore follows from Lemma 9 that in the limit we can restrict our attention to those edges where the sets $e_{r_{1}} \cap e_{s_{1}}, \ldots, e_{r_{k(\pi)}} \cap e_{s_{k(\pi)}}$ are mutually disjoint. Since the normalised trace of the Hamiltonian acting on a qubit within such a twofold crossing is given by

$$
3^{-2} \sum_{a, b=1}^{3} \frac{1}{2} \operatorname{Tr} \sigma^{(a)} \sigma^{(b)} \sigma^{(a)} \sigma^{(b)}=-\frac{1}{3}
$$

whereas the normalised trace is 1 for those qubits not involved in any crossings we find that the lhs. of eq. (15) can be asymptotically rewritten as

$$
\frac{1}{(k / 2)!} \sum_{m_{1}=0}^{\infty} \cdots \sum_{m_{\kappa(\pi)}=0}^{\infty} \frac{\lambda^{m_{1}+\cdots+m_{\kappa(\pi)}}}{m_{1}!\ldots m_{\kappa(\pi)!}} e^{-\kappa(\pi) \lambda}(-1 / 3)^{m_{1}+\cdots+m_{\kappa(\pi)}}=\frac{\left(e^{-4 \lambda / 3}\right)^{\kappa(\pi)}}{(k / 2)!},
$$

just as claimed. The $k$-th limiting moment, i.e. the normalised trace of $H_{n}^{\left(p_{n}-\text { glass }\right)}$ in the limit $n \rightarrow \infty$, is thus given by

$$
m_{k}(\lambda):=\frac{1}{(k / 2)!} \sum_{\pi \in S_{k}}\left(e^{-4 \lambda / 3}\right)^{\kappa(\pi)}=\sum_{\pi \in \tilde{S}_{k}}\left(e^{-4 \lambda / 3}\right)^{\kappa(\pi)}
$$

where $\tilde{S}_{k}$ denotes the set of unlabelled partitions.
These moments uniquely correspond to the distribution given in eq. (2), as known from the theory of the q-Hermite polynomials, see [5, eqs. (3.2) and (3.8)]. For the convenience of the reader we collect some further properties of this distribution in Proposition [12,

Remark 11. The proof of the Theorem 3 also works for general spin-s systems (instead of spin- $1 / 2$ ) with small changes. Mainly, part (ii) from Lemma 10 has to be replaced by a corresponding Lemma for spin-s which can be proved along the lines of the original proof. This replacement (possibly) changes the value of the $C$-constant from eq. (14) which is irrelevant for the result since $C<1$ is sufficient for the convergence against zero. In part (iii) the proof also applies to general spin- $s$ systems, except that $e^{-4 \lambda / 3}$ has to be replaced by $e^{-4 s \lambda /(2 s+1)}$. Theorems 1 and 8 also carry over to spin- $s$ since for the important bounds only degree properties of the graph and no specifics of the spin- $1 / 2$ system were used.

Proposition 12. Suppose that $\lim _{n \rightarrow \infty} \frac{p_{n}}{\sqrt{n}}=\lambda \in(0, \infty)$, then $m_{k, n}(\lambda)$, the normalised trace of the $k$-th power of $H_{n}^{\left(p_{n}\right)-\text { glass }}$, in the limit $n \rightarrow \infty$ takes the form

$$
m_{k}(\lambda):=\lim _{n \rightarrow \infty} m_{k, n}(\lambda)=0
$$

if $k$ is odd and

$$
\begin{align*}
m_{k}(\lambda) & :=\lim _{n \rightarrow \infty} m_{k, n}(\lambda)=\frac{1}{\left(1-e^{-4 \lambda / 3}\right)^{k / 2}} \sum_{j=-k / 2}^{k / 2}(-1)^{j} e^{-2 \lambda \cdot j(j-1) / 3}\binom{k}{k / 2+j} \\
& =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-x^{2} / 2}\left(\frac{2 \sinh ^{2}(i x \sqrt{\lambda / 3}+\lambda / 3)}{e^{-2 \lambda / 3} \sinh (-2 \lambda / 3)}\right)^{k / 2} \mathrm{~d} x \tag{16}
\end{align*}
$$

if $k$ is even. For any fixed even $k$ it furthermore holds that $m_{k}(\lambda)$ is monotonically decreasing in $\lambda$ and satisfies

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0} m_{k}(\lambda)=(k-1)!!\quad \text { and } \quad \lim _{\lambda \rightarrow \infty} m_{k}(\lambda)=\frac{k!}{(k / 2)!(k / 2+1)!} \tag{17}
\end{equation*}
$$

in agreement with the statements of Theorem 3. The corresponding limiting probability distribution $\mu_{\lambda}$ has the compactly supported density function given in eq. (2) which converges pointwise to the semicircular density function when $\lambda \rightarrow \infty$ and to the density function of the normal distribution when $\lambda \rightarrow 0$. Furthermore for any fixed $\lambda$ the density function $\rho_{\lambda}$ has square root singularities in $\pm 2 / \sqrt{1-e^{-4 \lambda / 3}}$.

Proof. Recall from the proof of Theorem 3 that the moments are given by

$$
m_{k}(\lambda)=\sum_{\pi \in \tilde{S}_{k}}\left(e^{-4 \lambda / 3}\right)^{\kappa(\pi)}
$$

for even $k$ and $m_{k}(\lambda)=0$ for odd $k$. According to an exact formula by Touchard and Riordan and its integral representation (see [3, eqs. (5) and (7) on page 197]) the even moments are given by the formulas in eq. (16). The monotone decrease follows by computing the derivative in $\lambda$ and the claimed limits in (17) are also direct computations. The claimed limiting behaviour of the density function as $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ also follows from the discussion in Section 2 of [5]. The $k=0$ term from eq. (2) is responsible for the square root singularity near the edges $\pm 2 / \sqrt{1-e^{-4 \lambda / 3}}$.

## Appendix: Proofs of some technical lemmas

## Proof of Lemma 6.

(i),(ii) Trivial calculations.
(iii) Two Pauli matrices $\sigma^{(a)}, \sigma^{(b)}$ anti-commute if $a \neq b$. We can therefore up to a factor of $\pm 1$ reorder the arguments of $\sigma\left(a_{1}, \ldots, a_{k}\right)$ in such a way that $a_{1}=\cdots=a_{n_{1}}=0, a_{n_{1}+1}=\cdots=a_{n_{1}+n_{2}}=1$, $a_{n_{1}+n_{2}+1}=\cdots=a_{n_{1}+n_{2}+n_{3}}=2, a_{n_{1}+n_{2}+n_{3}+1}=\cdots=a_{k}=3$ where $n_{0}+n_{1}+n_{2}+n_{3}=k$ denote the numbers of 0 's, 1 's, 2 's and 3 's. Using that the square of any Pauli matrix is the identity we then find

$$
\sigma\left(a_{1}, \ldots, a_{k}\right)= \pm \sigma\left(\pi_{n_{1}}, 2 \pi_{n_{2}}, 3 \pi_{n_{3}}\right)
$$

where $\pi_{n}$ is the parity function i.e. $\pi_{n}=0$ if $n$ is even and $\pi_{n}=1$ if $n$ is odd.
(iv) Using the anti-commutation relation $\sigma^{(a)} \sigma^{(b)}=2 \delta_{a b} 1_{2}-\sigma^{(a)} \sigma^{(b)}$ we compute inductively

$$
\begin{aligned}
\sigma\left(a_{1}, a_{2}, \ldots, a_{k}\right) & =2 \delta_{a_{1}, a_{2}} \sigma\left(a_{3}, \ldots, a_{k}\right)-\sigma\left(a_{2}, a_{1}, a_{3}, \ldots, a_{k}\right) \\
& =2 \delta_{a_{1} a_{2}} \sigma\left(a_{3}, \ldots, a_{k}\right)-2 \delta_{a_{1} a_{3}} \sigma\left(a_{2}, a_{4}, \ldots, a_{k}\right)+\sigma\left(a_{2}, a_{3}, a_{1}, a_{4}, \ldots, a_{k}\right) \\
& =\cdots=2 \sum_{j=2}^{k}(-1)^{j} \delta_{a_{1} a_{j}} \sigma\left(a_{2}, \ldots, \widehat{a_{j}}, \ldots, a_{k}\right)-\sigma\left(a_{2}, \ldots, a_{k}, a_{1}\right)
\end{aligned}
$$

from which the claim follows by the cyclicity of the trace.
Proof of Proposition 7. We again start to compute the moments using the short hand notation $\sigma_{J_{i}}:=$ $\sigma_{1}^{\left(a_{i}\right)} \sigma_{j_{i}}^{\left(b_{i}\right)}$ for $J=\left(a_{i}, b_{i}, j_{i}\right)$ and find by Lemma 4 that

$$
m_{k, n} \approx(9(n-1))^{-k / 2} \sum_{\left\{J_{1}, \ldots, J_{k / 2}\right\} \subset I_{n}}^{\star} \sum_{\pi \in S_{k}} 2^{-n} \operatorname{Tr} \sigma_{J_{\pi(1)}} \ldots \sigma_{J_{\pi(k)}} .
$$

In the limit $n \rightarrow \infty$ the part of the sum where two different $J_{i}$ have the same $j_{i}$-coordinate can be neglected and we find

$$
m_{k, n} \approx(9(n-1))^{-k / 2} \sum_{\left\{j_{1}, \ldots, j_{k / 2}\right\} \subset\{2, \ldots, n\}} \sum_{a_{1}, \ldots, a_{k / 2}=1}^{3} \sum_{b_{1}, \ldots, b_{k / 2}=1}^{3} \sum_{\pi \in S_{k}} 2^{-1} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}
$$

where it was used that in all but the first component the matrices commute (since the $j_{i}$ are mutually distinct), the square of any Pauli matrix is the identity and that the trace of the tensor product is the product of the traces. After performing the sums over the $j_{i}$ 's and $b_{i}$ 's (and using that $\binom{n-1}{k / 2}(n-1)^{-k / 2} \approx \frac{1}{(k / 2)!}$ ) we arrive at

$$
m_{k, n} \approx m_{k}:=\frac{3^{-k / 2}}{(k / 2)!} \sum_{a_{1}, \ldots, a_{k / 2}=1}^{3} \sum_{\pi \in S_{k}} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}
$$

We claim that

$$
f(k):=\sum_{a_{1}, \ldots, a_{k / 2}=1}^{3} \sum_{\pi \in S_{k}} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}=\frac{(k+1)!}{2^{k / 2}}
$$

holds for all even $k$. While $k=2$ is trivial, for the induction step we compute using Lemma 6 (iv) and the notation therein

$$
\begin{aligned}
f(k) & =\sum_{a_{1}, \ldots, a_{k / 2}=1}^{3} \sum_{\pi \in S_{k}} \frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)} \\
& =\sum_{j=2}^{k}(-1)^{j} \sum_{a_{1}, \ldots, a_{k / 2}=1}^{3} \sum_{\pi \in S_{k}} \delta_{a_{\pi(1)} a_{\pi(j)}} \sigma\left(a_{\pi(2)}, \ldots, \widehat{a_{\pi(j)}}, \ldots, a_{\pi(k)}\right)
\end{aligned}
$$

and then split the sum into to parts $f_{1}(k)$ and $f_{2}(k)$ where in $f_{1}(k)$ we only consider those $\pi \in S_{k}$ for which $\pi(1)=\pi(j)$ and in $f_{2}(k)$ those $\pi$ for which $\pi(1) \neq \pi(j)$. For $f_{1}(k)$ we can then compute

$$
\begin{aligned}
f_{1}(k) & =3 \sum_{j=2}^{k}(-1)^{j} \sum_{a_{1}, \ldots, \widehat{a_{\pi(1)}}, \ldots, a_{k / 2}=1}^{3} \sum_{\substack{\pi \in S_{k} \\
\pi(1)=\pi(j)}} \sigma\left(a_{\pi(2)}, \ldots, \widehat{a_{\pi(j)}}, \ldots, a_{\pi(k)}\right) \\
& =3 \frac{k}{2} \sum_{j=2}^{n}(-1)^{j} \sum_{a_{1}, \ldots, a_{(k-2) / 2}=1}^{3} \sum_{\pi \in S_{k-2}} \sigma\left(a_{\pi(1)}, \ldots, a_{\pi(k-2)}\right)=3 \frac{k}{2} \sum_{j=2}^{k}(-1)^{j} f(k-2)=3 \frac{k}{2} f(k-2)
\end{aligned}
$$

where in the first step we performed the sum over $a_{\pi(1)}=a_{\pi(j)}$ and in the second step took out a factor of $\frac{k}{2}$ corresponding to the $\frac{k}{2}$ possible values of $\pi(1)=\pi(j)$. Similarly we find that

$$
f_{2}(k)=\frac{k(k-2)}{2} f(k-2)
$$

and by adding the two recursion relations we finally arrive at

$$
f(k)=f_{1}(k)+f_{2}(k)=\frac{k(k+1)}{2} f(k-2)=\frac{k(k+1)}{2} \frac{(k-1)!}{2^{(k-2) / 2}}=\frac{(k+1)!}{2^{k / 2}}
$$

proving the claim. Inserting this into the expression we had for $m_{k}$ then gives $m_{k}=\frac{(k+1)!}{6^{k / 2}(k / 2)!}$ for even $k$ and $m_{k}=0$ for odd $k$.

These moments again satisfy Carleman's continuity condition and therefore uniquely correspond to a limiting distribution whose characteristic function $\phi$ is given by

$$
\phi(t)=\sum_{k=0}^{\infty} \frac{(i t)^{2 k}}{(2 k)!} \frac{(2 k+1)!}{6^{k} k!}=\sum_{k=0}^{\infty}(-1)^{k} \frac{(2 k+1) t^{2 k}}{6^{k} k!}=\left(1-\frac{t^{2}}{3}\right) e^{-t^{2} / 6}
$$

from which by a Fourier transform we find the density

$$
\rho(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \phi(t) e^{-i t x} \mathrm{~d} t=3 \sqrt{\frac{3}{2 \pi}} x^{2} e^{-3 x^{2} / 2} .
$$

Proof of Lemma 9.
(i) We can safely assume that eventually $a_{n}+b_{n} \leq n$ since the assertion is trivial otherwise and then compute

$$
\binom{n-a_{n}}{b_{n}} /\binom{n}{b_{n}}=\frac{\left(n-a_{n}\right)\left(n-a_{n}-1\right) \ldots\left(n-a_{n}-b_{n}+1\right)}{n(n-1) \ldots\left(n-b_{n}+1\right)}=\prod_{k=0}^{b_{n}-1}\left(1-\frac{a_{n}}{n-k}\right)
$$

where all factors are non-negative. We continue with the obvious bounds

$$
\left(1-\frac{a_{n}}{n}\right)^{b_{n}} \leq \prod_{k=0}^{b_{n}-1}\left(1-\frac{a_{n}}{n-k}\right) \leq\left(1-\frac{a_{n}}{n-b_{n}+1}\right)^{b_{n}}
$$

and after applying a logarithm arrive at

$$
\begin{aligned}
\exp \left(-\frac{a_{n} b_{n}}{n}\right) & \approx \exp \left(b_{n} \log \left(1-\frac{a_{n}}{n}\right)\right) \leq\binom{ n-a_{n}}{b_{n}} /\binom{n}{b_{n}} \\
& \leq \exp \left(b_{n} \log \left(1-\frac{a_{n}}{n-b_{n}+1}\right)\right) \approx \exp \left(-\frac{a_{n} b_{n}}{n-b_{n}+1}\right)
\end{aligned}
$$

from which the claim follows immediately.
(ii) For any fixed $k \geq 0$ the proportion of sets $B_{n}$ of size $b_{n}$ that share exactly $k$ elements with $A_{n}$ is given by

$$
\binom{a_{n}}{k}\binom{n-a_{n}}{b_{n}-k} /\binom{n}{b_{n}}
$$

i.e. the number of elements in the intersection is hypergeometrically distributed with parameters ( $n, a_{n}, b_{n}$ ) and therefore has a mean of $\frac{a_{n} b_{n}}{n}$ and a variance of

$$
\frac{a_{n} b_{n}}{n} \frac{n-b_{n}}{n} \frac{n-a_{n}}{n-1}
$$

which shows that for $c_{n}$ growing slower than $\frac{a_{n} b_{n}}{n}$ the proportion of $B_{n}$ 's that share at least $c_{n}$ elements with $A_{n}$ converges to 1 .

## Proof of Lemma 10.

(i) Suppose that $\pi$ is non-crossing. Let $i<j$ be those indices for which $\pi(i)=\pi(j)=1$. Since the partition is non-crossing in the tuple $(\pi(1), \ldots, \pi(k))$ there are either zero or two $l$ indices between $\pi(i)$ and $\pi(j)$ for all $1<l \leq k / 2$. Recall that $\sigma^{\left(a_{1}\right)}$ anti-commutes with $\sigma^{\left(a_{l}\right)}$ if $a_{l} \neq a_{1}$ and commutes otherwise. Hence we can freely permute $\sigma^{\left(a_{\pi(j)}\right)}$ to the left next to $\sigma^{\left(a_{\pi(j)}\right)}$ and then the claim follows inductively since $\left(\sigma^{\left(a_{1}\right)}\right)^{2}=1_{2}$ and we therefore proved the claim assuming the result for $k-2$. For $k=2$ the assertion is trivially true.
(ii) Suppose now that $\pi$ is crossing, i.e. there exist $a<b<c<d$ such that $r:=\pi(a)=\pi(c)$ and $s:=\pi(b)=\pi(d)$. Then by setting $a_{l}=1$ for $l \notin\{r, s\}$ the expression simplifies to

$$
\frac{1}{2} \operatorname{Tr} \sigma^{\left(a_{\pi(1)}\right)} \ldots \sigma^{\left(a_{\pi(k)}\right)}=\sigma\left(a_{\pi(1)}, \ldots, a_{\pi(k)}\right)=\sigma\left(\alpha, a_{r}, \beta, a_{s}, \gamma, a_{r}, \delta, a_{s}, \epsilon\right)
$$

for some $\alpha, \beta, \gamma, \delta, \epsilon \in\{0,1\}$. Using the anti-commutation relations we then find that for $a_{r}, a_{s} \in\{2,3\}$ it holds that

$$
\begin{aligned}
\sigma\left(a_{\pi(1)},\right. & \left.\ldots, a_{\pi(k)}\right)=(-1)^{\gamma} \sigma\left(\alpha, a_{r}, \beta, a_{s}, a_{r}, \gamma, \delta, a_{s}, \epsilon\right)=(-1)^{\gamma+1-\delta_{a_{r}, a_{s}}} \sigma\left(\alpha, a_{r}, \beta, a_{r}, a_{s}, \gamma, \delta, a_{s}, \epsilon\right) \\
& =\cdots=(-1)^{2 \gamma+\beta+\delta+1-\delta_{a_{r}, a_{s}}} \sigma\left(\alpha, a_{r}, a_{r}, \beta, \gamma, a_{s}, a_{s}, \delta, \epsilon\right)=(-1)^{\beta+\delta+1-\delta_{a_{r}, a_{s}}} \sigma(\alpha, \beta, \gamma, \delta, \epsilon) .
\end{aligned}
$$

i.e. the result changes sign depending on whether $a_{r}=2$ and $a_{s}=3$ or $a_{r}=a_{s}=2$ and in particular cannot be equal to 1 for all choices of $a_{1}, \ldots, a_{k / 2}$.
(iii) This is an immediate consequence of applying part (i) to all components of the tensor product separately and using that the trace of a tensor product factorises.

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