

DIRECTIONAL EXTREMAL STATISTICS FOR GINIBRE EIGENVALUES

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ABSTRACT. We consider the eigenvalues of a large dimensional real or complex Ginibre matrix in the region of the complex plane where their real parts reach their maximum value. This maximum follows the Gumbel distribution and that these extreme eigenvalues form a Poisson point process, asymptotically as the dimension tends to infinity. In the complex case these facts have already been established by Bender [10] and in the real case by Akemann and Phillips [2] even for the more general elliptic ensemble with a sophisticated saddle point analysis. The purpose of this note is to give a very short direct proof in the Ginibre case with an effective error term. Moreover, our estimates on the correlation kernel in this regime serve as a key input for accurately locating $\max \Re \text{Spec}(X)$ for any large matrix X with i.i.d. entries in the companion paper [15].

1. INTRODUCTION

The Ginibre matrix ensemble [21] is the simplest and most commonly used prototype of non-Hermitian random matrices. It consists of $n \times n$ matrices X with independent, identically distributed (i.i.d.) Gaussian entries x_{ij} . We use the normalization $\mathbf{E} x_{ij} = 0$, $\mathbf{E} |x_{ij}|^2 = \frac{1}{n}$, i.e. $\sqrt{n}x_{ij}$ is a standard real or complex normal random variable. Correspondingly, we talk about real or complex Ginibre matrices. The empirical density of eigenvalues converges to the uniform distribution on the unit disk in the complex plane, known as *Girko's circular law* and proven in increasing generality even without Gaussian assumption [22, 8, 35], while the spectral radius converges to 1 [20, 9, 11, 12] with an explicit speed of convergence [7]. For the Gaussian case, the eigenvalues form a determinantal (or Pfaffian) point process with an explicit correlation kernel $K_n(z, w)$ (see (7) and (50) later). This kernel was computed by Ginibre in the complex case [21] and later by Borodin and Sinclair for the more complicated real case [13, 14] based upon earlier works on Pfaffian formulas [19, 32] (some special cases have been solved earlier [26, 17, 18, 33, 24] and see also [3] for a comprehensive summary of all known related kernels). While the eigenvalue distribution is rotationally symmetric in the complex case, the main complication in the real case stems from the fact that the real axis plays a special role, in fact there are many real eigenvalues [18].

The explicit formula for the eigenvalue correlation function allows one, in principle, to compute the distribution of any interesting statistics of the eigenvalues. In reality, these calculations may require very precise asymptotic analysis of certain special functions where the complex and real cases may differ substantially. For example, the distribution of $\rho(X) := \max |\text{Spec}(X)|$, the spectral radius of X (i.e.

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the largest eigenvalue in modulus), can still be easily identified in the complex case by using Kostlan's observation [25] on the moduli of the complex Ginibre eigenvalues. The precise result, stated in this form in [30], asserts that

$$\rho(X) \stackrel{d}{=} 1 + \sqrt{\frac{\alpha_n}{4n}} + \frac{1}{\sqrt{4n\alpha_n}} G_n, \quad \alpha_n := \log n - 2 \log \log n - \log(2\pi), \quad (1)$$

where G_n converges in distribution to a standard *Gumbel random variable*, i.e.

$$\lim_{n \rightarrow \infty} \mathbf{P}(G_n \leq t) = \exp(-e^{-t})$$

for any fixed $t \in \mathbf{R}$. On the other hand, lacking radial symmetry, which is key element of Kostlan's observation, the analogous result for the real Ginibre ensemble required a much more sophisticated analysis by Rider and Sinclair in [31]. They showed that (1) also holds for real case with the same scaling factor α_n , but G_n converges to a slightly rescaled Gumbel law with distribution function $\exp(-\frac{1}{2}e^{-t})$. The additional factor $1/2$ stems from the fact that the spectrum of a real Ginibre matrix is symmetric with respect to the real axis.

In the current paper we investigate a related quantity, the maximum real part of the spectrum of X , where radial symmetry does not help even in the complex case. It turns out that a similar asymptotics holds but with a new scaling factor:

$$\max \Re \text{Spec}(X) \stackrel{d}{=} 1 + \sqrt{\frac{\gamma}{4n}} + \frac{1}{\sqrt{4n\gamma}} G_n, \quad \gamma = \gamma_n := \frac{\log n - 5 \log \log n - \log(2\pi^4)}{2}, \quad (2)$$

with G_n still converging to a Gumbel variable. More precisely:

Theorem 1 (Gumbel distribution). *Let $\sigma_1, \dots, \sigma_n$ denote the eigenvalues of a real ($\beta = 1$) or complex ($\beta = 2$) $n \times n$ Ginibre matrix. Then for any fixed $^1 t \in \mathbf{R}$ it holds that*

$$\mathbf{P}\left(\max_i \Re \sigma_i < 1 + \sqrt{\frac{\gamma}{4n}} + \frac{t}{\sqrt{4\gamma n}}\right) = \exp\left(-\frac{\beta}{2} \exp(-t)\right) + \mathcal{O}\left(\frac{(\log \log n)^2}{\log n}\right) \quad (3)$$

as $n \rightarrow \infty$.

In the complex case (3) as a limit statement was proven by Bender [10] and in the real case by Ake-mann and Phillips in [2] even for the more involved elliptic Ginibre ensemble where the kernel K_n is expressed by a contour integral (later it was extended to the chiral two-matrix model with complex entries [1]). Here we give a short alternative proof that also provides an effective estimate on the speed of convergence.

In Theorem 1 we only considered the eigenvalue with the largest real part for simplicity, however similar result holds for the largest eigenvalue in any chosen direction. More precisely, in the complex case the distribution of $\max_i \Re(e^{i\theta} \sigma_i)$ is independent of $\theta \in \mathbf{R}$ by rotational symmetry. For real Ginibre matrices and for any fixed $\theta \neq 0$ independent of n , $\max_i \Re(e^{i\theta} \sigma_i)$ still satisfies (3) but with $\beta = 2$. Our proof can easily be extended to cover this more general case using that the local eigenvalue correlation functions for real and complex Ginibre matrices practically coincide away from the real axis.

As a motivation we remark that $\max \Re \text{Spec}(X)$ is the basic quantity determining the exponential growth rate of the long time asymptotics of the solution of the linear system of differential equations

$$\frac{d}{dt} \mathbf{u}(t) = X \mathbf{u}(t).$$

Starting from the pioneering work of May [27] (see also the more recent review [6]), this equation is frequently used in phenomenological models to describe the evolution of many interacting agents with random couplings both in theoretical neuroscience [34, 29] and in mathematical ecology [4, 5].

The appearance of the universal Gumbel distribution in (1)–(2) is typical for extreme value statistics of independent random variables as one of the three main cases described in the Fisher-Tippet-Gnedenko theorem. While nearby Ginibre eigenvalues inside the unit disk are strongly correlated, the extreme eigenvalues are essentially independent which heuristically explains the Gumbel law. The key point is that the correlation length of the eigenvalues is of order $n^{-1/2}$, as the scaling of the Ginibre kernel $K_n(z, w)$ indicates, but in the extreme regime the few eigenvalues that may contribute to $\rho(X)$ or $\max \Re \text{Spec}(X)$ are much farther away from each other than $n^{-1/2}$. In fact, the scaling factor $\gamma = \gamma_n$ is chosen in such a way that there are typically finitely many (independent of n) eigenvalues in an elongated box of size $(4\gamma n)^{-1/2} \times i(\gamma n)^{-1/4}$ around $1 + \sqrt{\gamma/4n}$ (see Fig. 1). The height of this box,

¹Our estimates actually give a slightly weaker effective error for any $|t| \ll \sqrt{\log n}$

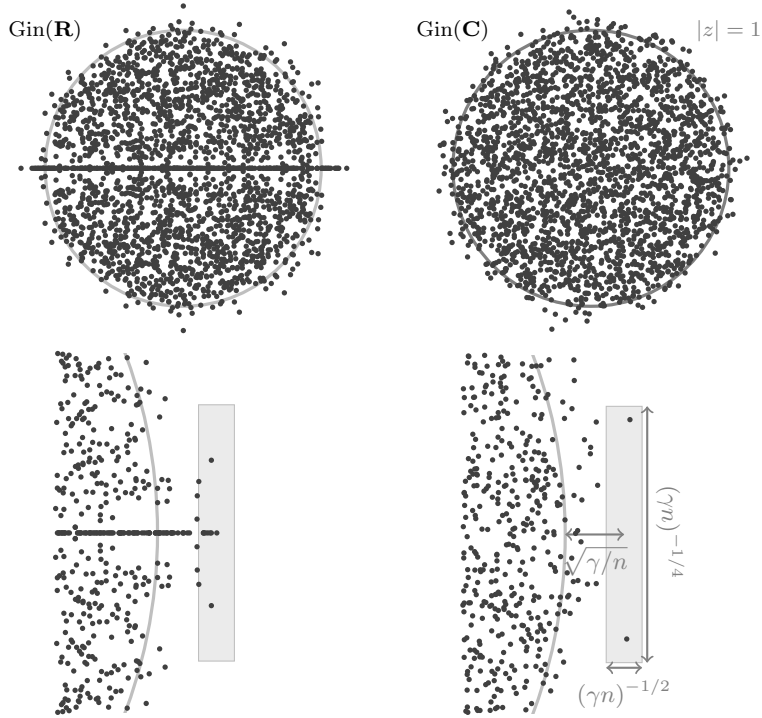


FIGURE 1. The figure shows the eigenvalues of real and complex Ginibre matrices. The eigenvalues for the top figures have been computed for 50 independent Ginibre matrices of size 50×50 , while for the bottom figure 100 independent matrices of size 100×100 have been sampled. Note that the eigenvalues of the real Ginibre matrix are symmetric with respect to the real axis, and that some (in fact $\sim \sqrt{n}$) eigenvalues are on the axis itself. Furthermore, the top left figure misleadingly hints that the rightmost eigenvalue is real. This is a finite n effect (see [31] for a detailed discussion of this so called “Saturn effect”); we actually prove (see (60) to (62)) that in the large n limit the largest real eigenvalue is much smaller than the real part of the rightmost complex eigenvalue.

which is essentially the square root of its width, is determined by the curvature of the boundary of the circular law: above or below this box there are no eigenvalues since their modulus would be too large. Given this heuristic picture, the typical distance between the eigenvalues in the relevant box is of order $n^{-1/4}$ modulo logarithmic factors, so they are well beyond the correlation scale hence independent. As a second result, we also establish this independence rigorously; in fact we show that within this box the eigenvalues form a Poisson point process in the $n \rightarrow \infty$ limit. Again, as a pure limit statement this result has already been proven in [10] for the complex Ginibre ensemble and in [2] for the real case; our contribution is to give an alternative direct proof with an effective error bound.

Theorem 2 (Poisson Point Process). *Let $\sigma_1, \dots, \sigma_n$ denote the eigenvalues of a real or complex $n \times n$ Ginibre matrix. Fix any $t \in \mathbf{R}$ and any function $f: \mathbf{C} \rightarrow [0, \infty)$ supported on $[t, \infty) \times i\mathbf{R}$, which, additionally, is assumed to be symmetric $f(z) = f(\bar{z})$ in the real case². Then we have*

$$\mathbf{E} e^{-\sum_{i=1}^n f(x_i + iy_i)} = \exp\left(-\int_{\mathbf{F}} (1 - e^{-f(x+iy)}) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx\right) + \mathcal{O}\left(\frac{(\log \log n)^2}{\log n}\right), \quad (4)$$

where we introduced the eigenvalue rescaling

$$\sigma_i = 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x_i}{\sqrt{4\gamma n}} + \frac{iy_i}{(\gamma n)^{1/4}} \quad (5)$$

²This restriction is only for convenience, since by spectral symmetry $\overline{\sigma(X)} = \sigma(X)$, any non-symmetric function f can be replaced by its symmetrization $[f(z) + f(\bar{z})]/2$

and we set $\mathbf{F} = \mathbf{H} := \{z \in \mathbf{C} \mid \Im z \geq 0\}$ in the real and $\mathbf{F} = \mathbf{C}$ in the complex case.

Both our main results follow from a precise asymptotics of the rescaled Ginibre kernel $K_n(z, w)$ in the relevant box combined with the idea of the regularized Fredholm determinant also used in [31]. The compact form of K_n in the Ginibre case makes the calculations considerably shorter than the saddle point analysis for its contour integral representation used for the elliptic ensemble in [10, 2]. In particular, we obtain an effective bound on the speed of convergence unlike [10, 2] that rely on dominated convergence. As a byproduct, we also obtain concentration result with an effective error term for the linear statistics (in particular the number) of eigenvalues on a slightly larger box. This result is crucially used in our companion paper [15] in which we accurately identify the size of $\max \Re \text{Spec}(X)$ for matrices with general i.i.d. entries, going well beyond the explicitly solvable models.

We close this introduction with a remark about eigenvectors. For many Hermitian random matrices or operators originating from disordered quantum systems, the general prediction is that Poisson eigenvalue statistics entails localized eigenvectors (while strongly correlated eigenvalue statistics, e.g. Wigner-Dyson, imply delocalized eigenvectors). This is not the case here: all eigenvectors, even those corresponding to extreme eigenvalues in the Poisson regime are fully delocalized [7, Corollary 2.4].

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2. COMPLEX GINIBRE

We recall a few basic facts about the correlation functions. The joint probability density of the eigenvalues of a complex Ginibre matrix is given by [30]

$$\rho_n(\mathbf{z}) = \rho_n(z_1, \dots, z_n) := \frac{n^n}{\pi^n 1! \dots n!} \exp\left(-n \sum_i |z_i|^2\right) \prod_{i < j} (n|z_i - z_j|^2). \quad (6)$$

The product can be written as a product of Vandermonde determinants and we obtain

$$\begin{aligned} \prod_{i < j} (n|z_i - z_j|^2) &= \det \begin{pmatrix} 1 & \sqrt{n}z_1 & \cdots & (\sqrt{n}z_1)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{n}z_n & \cdots & (\sqrt{n}z_n)^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \sqrt{n}z_1 & \cdots & \sqrt{n}z_n \\ \vdots & \ddots & \vdots \\ (\sqrt{n}z_1)^{n-1} & \cdots & (\sqrt{n}z_n)^{n-1} \end{pmatrix} \\ &= 1! \cdots (n-1)! \det \left(K_n(z_i, z_j) \right)_{i,j=1}^n, \quad K_n(z, w) := \sum_{l=0}^{n-1} \frac{(nz\bar{w})^l}{l!}, \end{aligned} \quad (7)$$

so that we conclude

$$\rho_n(\mathbf{z}) = \frac{n^n}{\pi^n n!} e^{-n|\mathbf{z}|^2} \det \left(K_n(z_i, z_j) \right)_{i,j=1}^n, \quad (8)$$

i.e. the eigenvalues form a *determinantal process*. Note that K_n is the kernel of a positive operator of rank n , in particular its off-diagonal terms are estimated by the diagonal ones via Cauchy-Schwarz inequality:

$$|K_n(z, w)|^2 \leq K_n(z, z)K_n(w, w), \quad (9)$$

which also follows directly from the formula for $K_n(z, w)$. In order to integrate out variables we rely on the following well-known identities:

$$\frac{n}{\pi} \int_{\mathbf{C}} e^{-n|z|^2} K_n(z, z) d^2z = n, \quad (10)$$

and for any fixed $w_1, w_2 \in \mathbf{C}$

$$\frac{n}{\pi} \int_{\mathbf{C}} e^{-n|z|^2} K_n(w_1, z)K_n(z, w_2) d^2z = K_n(w_1, w_2). \quad (11)$$

We recall that both claims follow directly from the identity

$$\frac{n}{\pi} \int_{\mathbf{C}} e^{-n|z|^2} (\sqrt{n}z)^a (\sqrt{n}\bar{z})^b d^2z = \delta_{ab} a! \quad (12)$$

for any $a, b \in \mathbf{N}$ and the definition of K_n . As a consequence of these identities, an arbitrary number of variables can be integrated out and we obtain the following standard formula for the correlation functions:

Lemma 3 (k -point correlation function). *For*

$$\rho_n^k(z_1, \dots, z_k) := \int_{\mathbf{C}^{n-k}} \rho_n(\mathbf{z}) d^2 z_{k+1} \cdots d^2 z_n \quad (13)$$

it holds that

$$\rho_n^k(\mathbf{z}) = \frac{n^k (n-k)!}{\pi^n n!} e^{-n|\mathbf{z}|^2} \det \left(K_n(z_i, z_j) \right)_{i,j=1}^k. \quad (14)$$

Consider a function $g: \mathbf{C} \rightarrow [0, 1]$ and evaluate

$$\begin{aligned} \mathbf{E} \prod_{i=1}^n (1 - g(\sigma_i)) &= \int_{\mathbf{C}^n} \rho_n(\mathbf{z}) \prod_{i=1}^n (1 - g(z_i)) d^2 \mathbf{z} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \int_{\mathbf{C}^k} \rho_n^k(\mathbf{z}) \prod_{i=1}^k g(z_i) d^2 \mathbf{z} \\ &= \sum_{k=0}^n \frac{(-1)^k}{k!} \frac{n^k}{\pi^k} \int_{\mathbf{C}^n} e^{-n|\mathbf{z}|^2} \det \left(K_n(z_i, z_j) \right)_{i,j=1}^n \prod_{i=1}^k g(z_i) d^2 \mathbf{z} \\ &= \sum_{k=0}^n \frac{(-1)^k}{k!} \int_{\mathbf{C}^k} \det \left(\sqrt{g(z_i)} \tilde{K}_n(z_i, z_j) \sqrt{g(z_j)} \right)_{i,j=1}^k d^2 \mathbf{z} \\ &= \det(1 - \sqrt{g} \tilde{K}_n \sqrt{g}) \end{aligned} \quad (15)$$

which we recognize as the Fredholm determinant of $1 - \sqrt{g} \tilde{K}_n \sqrt{g}$ (see Definition 4 below, and recall that \tilde{K}_n has rank n), where

$$\tilde{K}_n(z, w) := \frac{n}{\pi} e^{-n(|z|^2 + |w|^2)/2} K_n(z, w) = \frac{n}{\pi} e^{-n(|z|^2 + |w|^2 - 2z\bar{w})/2} \frac{\Gamma(n, n z \bar{w})}{\Gamma(n)}. \quad (16)$$

Here $\Gamma(\cdot, \cdot)$ denotes the incomplete Gamma function defined as

$$\Gamma(s, z) := \int_z^\infty t^{s-1} e^{-t} dt, \quad (17)$$

where $s \in \mathbf{N}$ and the integration contour goes from $z \in \mathbf{C}$ to real infinity.

Definition 4 (Fredholm determinant). *Let (Ω, μ) denote a measure space and let $K(z, w)$ be a kernel on Ω . Then the Fredholm determinant of $1 - K$ is defined as*

$$\det(1 - K) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\Omega^k} \det \left(K(z_i, z_j) \right)_{i,j=1}^k d\mu(z_1) \cdots d\mu(z_k). \quad (18)$$

2.1. Scaling limit for $\max \Re \sigma_i$. We now consider the scaling limit for the part of the complex plane in which the eigenvalue with the largest real part is located c.f. Fig. 1. We will show that the eigenvalue with the largest real part lives on a scale $(4\gamma n)^{-1/2} \times i(\gamma n)^{-1/4}$ around $1 + \sqrt{\gamma/4n}$.

The fact that outside the unit circle the kernel \tilde{K}_n has small Hilbert-Schmidt norm prompts the introduction of the regularised determinant [23, IV.(7.8)]

$$\det_2(1 - K) := \det \left((1 - K) e^K \right) \quad (19)$$

which for finite-rank K allows to write $\det(1 - K) = \det_2(1 - K) \exp(-\text{Tr } K)$. From [23, IV.(7.11)] we thus conclude

$$|\det(1 - K) - \exp(-\text{Tr } K)| \leq \|K\|_2 e^{(\|K\|_2 + 1)^2/2 - \text{Tr } K} \quad (20)$$

where

$$\text{Tr } K = \int_{\Omega} K(x, x) d\mu(x), \quad \|K\|_2^2 = \int_{\Omega^2} |K(x, y)|^2 d\mu(x) d\mu(y). \quad (21)$$

The regularized determinant as a technical tool was used in [31] in a very similar context for the spectral radius of real Ginibre matrices.

Proposition 5. *Let $|t| \leq \sqrt{\log n}/10$ and define the set*

$$A = A(t) := \left\{ z \in \mathbf{C} \mid \Re z \geq 1 + \sqrt{\frac{\gamma}{4n}} + \frac{t}{\sqrt{4\gamma n}} \right\}. \quad (22)$$

Then for $g: \mathbf{C} \rightarrow [0, 1]$ supported on $\text{supp } g \subset A(t)$, and for n large enough so that $\gamma > 0$, it holds that

$$\begin{aligned} \text{Tr } \sqrt{g} \tilde{K}_n \sqrt{g} &= \int_t^\infty \int_{\mathbf{R}} g(z) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx + \mathcal{O}\left(e^{-t} \frac{(\log \log n)^2 + |t|^2}{\log n}\right), \\ z &= 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x}{\sqrt{4\gamma n}} + \frac{iy}{(\gamma n)^{1/4}} \end{aligned} \quad (23)$$

and

$$\|\sqrt{g} \tilde{K}_n \sqrt{g}\|_2 \lesssim e^{-\sqrt{\log n}/32}. \quad (24)$$

The unspecified constants in \lesssim and $\mathcal{O}(\cdot)$ are uniform in n and in $|t| \leq \sqrt{\log n}/10$.

In particular, (15) and (20) combined with Proposition 5 for any fixed t gives

$$\begin{aligned} \mathbf{P}\left(\max_i \Re \sigma_i < 1 + \sqrt{\frac{\gamma}{4n}} + \frac{t}{\sqrt{4\gamma n}}\right) &= \mathbf{P}\left(\sigma_1, \dots, \sigma_n \in A(t)^c\right) \\ &= \det(1 - \chi_{A(t)} \tilde{K}_n \chi_{A(t)}) \xrightarrow{n \rightarrow \infty} e^{-e^{-t}} \end{aligned} \quad (25)$$

with χ_A denoting the characteristic function of the set A , completing the proof of Theorem 1 in the complex case. Moreover, for any function $f: \mathbf{C} \rightarrow [0, \infty)$ supported in $A(t)$ we also have that

$$\begin{aligned} \mathbf{E} \exp\left(-\sum_{i=1}^n f(\sigma_i)\right) &= \det\left(1 - \sqrt{1 - e^{-f}} \tilde{K}_n \sqrt{1 - e^{-f}}\right) \\ &\xrightarrow{n \rightarrow \infty} \exp\left(-\int_t^\infty \int_{\mathbf{R}} (1 - e^{-f(z)}) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx\right) \end{aligned} \quad (26)$$

with z as in (23), proving the complex case of Theorem 2 after change of variables. The error terms in (3) and (4) can easily be obtained from (23)–(24).

Hence the remaining task is to prove Proposition 5 which will be an easy consequence of the following Lemma 6.

Lemma 6. *Rescale the kernel variables as*

$$z = 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x_1}{\sqrt{4\gamma n}} + \frac{iy_1}{(\gamma n)^{1/4}}, \quad w = 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x_2}{\sqrt{4\gamma n}} + \frac{iy_2}{(\gamma n)^{1/4}} \quad (27)$$

with $\mathbf{x} := (x_1, x_2)$, $\mathbf{y} := (y_1, y_2)$ real vectors. In the regime $|\mathbf{x}| + |\mathbf{y}|^2 \leq \sqrt{\log n}/2$ and for $|y_1 - y_2| < n^{1/10} n^{-1/4}$ we have the asymptotics

$$\frac{|\tilde{K}_n(z, w)|^2}{4(\gamma n)^{3/2}} = \frac{\gamma e^{-x_1 - x_2 - y_1^2 - y_2^2}}{\pi(\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2)} \left(1 + \mathcal{O}\left(\frac{\log \log n + |\mathbf{x}|^2 + |\mathbf{y}|^4}{\log n}\right)\right). \quad (28)$$

On the other hand, for $|\mathbf{x}| + |\mathbf{y}|^2 \leq \sqrt{\log n}/2$ and $|y_1 - y_2| \geq Cn^{-1/4}$ for some $C \geq 1$ we have the estimate

$$\frac{|\tilde{K}_n(z, w)|^2}{(\gamma n)^{3/2}} \lesssim \frac{\gamma e^{-x_1 - y_1^2 - x_2 - y_2^2}}{\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2} \left(1 + \mathcal{O}\left(\frac{\sqrt{\gamma}}{C^2} + \frac{|\mathbf{x}|^2 + |\mathbf{y}|^4}{\log n}\right)\right). \quad (29)$$

Finally, for $x_1 + y_1^2 \geq 0$, $x_2 + y_2^2 \geq 0$ we have the uniform bound

$$\frac{|\tilde{K}_n(z, w)|^2}{(\gamma n)^{3/2}} \lesssim |z|^2 |w|^2 e^{-(x_1 + y_1^2)/3} e^{-(x_2 + y_2^2)/3}. \quad (30)$$

Proof of Proposition 5. Set $t_0 := 4(\log \log n + |t|)$ and estimate the trace in (23) as follows

$$\begin{aligned} \text{Tr } \sqrt{g} \tilde{K}_n \sqrt{g} &= \int_{A(t)} g(z) \tilde{K}_n(z, z) d^2 z \\ &= \left(\int_t^{t_0} \int_{y^2 < 2t_0} + \int_t^{t_0} \int_{y^2 \geq 2t_0} + \int_{t_0}^\infty \int_{\mathbf{R}} \right) g(z) \frac{\tilde{K}_n(z, z)}{2(\gamma n)^{3/4}} dy dx \\ &= \int_t^{t_0} \int_{y^2 < t_0} g(z) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx \left(1 + \mathcal{O}\left(\frac{(\log \log n)^2 + |t|^2}{\log n}\right)\right) + \mathcal{O}\left(e^{-|t_0|/4}\right) \\ &= \int_t^\infty \int_{\mathbf{R}} g(z) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx + \mathcal{O}\left(e^{-t} \frac{(\log \log n)^2 + |t|^2}{\log n}\right), \end{aligned} \quad (31)$$

where we used (28) for the first integral and (30) for the remaining two integrals.

For the bound on (23) we estimate

$$\mathrm{Tr}(\sqrt{g}\tilde{K}_n\sqrt{g})^2 \leq \iint_{A(t)} |\tilde{K}_n(z, w)|^2 d^2z d^2w, \quad (32)$$

and after a change of variables from (z, w) to (\mathbf{x}, \mathbf{y}) using (27) we split the integral into two parts. First estimate the part where $|\mathbf{x}| + |\mathbf{y}|^2 > \sqrt{\log n}/2$ and obtain

$$\begin{aligned} & \iint_t^\infty \iint_{\mathbf{R}} \frac{|\tilde{K}_n(z, w)|^2}{4(\gamma n)^{3/2}} \mathbf{1}(|\mathbf{x}| + |\mathbf{y}|^2 > \frac{\sqrt{\log n}}{2}) d\mathbf{y} d\mathbf{x} \\ & \leq \int_t^\infty \int_{\mathbf{R}} \iint_{\mathbf{R}} \frac{|\tilde{K}_n(z, w)|^2}{4(\gamma n)^{3/2}} \mathbf{1}(|x_1| + y_1^2 > \frac{\sqrt{\log n}}{4}) d\mathbf{y} dx_2 dx_1 \\ & = \int_t^\infty \int_{\mathbf{R}} \frac{\tilde{K}_n(z, z)}{2(\gamma n)^{3/4}} \mathbf{1}(|x_1| + y_1^2 > \frac{\sqrt{\log n}}{4}) dy_1 dx_1 \\ & \lesssim \int_t^\infty \int_{\mathbf{R}} e^{-(x+y^2)/4} \mathbf{1}(|x| + y^2 > \frac{\sqrt{\log n}}{4}) dy dx \lesssim e^{-\sqrt{\log n}/16} \end{aligned} \quad (33)$$

due to (11) in the second and (30) in the last step. In the remaining integral we use (28) whenever $|y_1 - y_2| \leq n^{-1/6}$ and (29) otherwise to find

$$\begin{aligned} & \iint_t^\infty \iint_{\mathbf{R}} \frac{|\tilde{K}_n(z, w)|^2}{(\gamma n)^{3/2}} \mathbf{1}(|\mathbf{x}| + |\mathbf{y}|^2 \leq \frac{\sqrt{\log n}}{2}) d\mathbf{x} d\mathbf{y} \\ & \lesssim \iint_t^\infty \iint_{\mathbf{R}} e^{-x_1 - x_2 - y_1^2 - y_2^2} \left(\mathbf{1}(|y_1 - y_2| \leq n^{-1/6}) + \frac{\mathbf{1}(|y_1 - y_2| > n^{-1/6})}{\gamma^{-3/2} n^{1/6}} \right) d\mathbf{y} d\mathbf{x} \\ & \lesssim e^{-2t} n^{-1/6} \gamma^{3/2}, \end{aligned} \quad (34)$$

concluding the proof. \square

Proof of Lemma 6. For (30) by Cauchy-Schwarz it is sufficient to prove

$$\frac{\tilde{K}_n(z, z)}{(\gamma n)^{3/4}} \lesssim |z|^2 e^{-(x+y^2)/3}. \quad (35)$$

For the proof of (35) we recall the asymptotics [31, Lemma 3.2] of the incomplete Γ function

$$\frac{\Gamma(n, nt)}{\Gamma(n)} = \frac{t\mu(t) \operatorname{erfc}(\sqrt{n}\mu(t))}{\sqrt{2}(t-1)} \left(1 + \mathcal{O}(n^{-1/2}) \right), \quad \mu(t) := \sqrt{t - \log(t) - 1}, \quad (36)$$

which holds uniformly in $t > 1$, and note that

$$|z|^2 = 1 + \frac{\sqrt{\gamma} + (x + y^2)/\sqrt{\gamma}}{\sqrt{n}} + \frac{(\gamma + x)^2}{4\gamma n} \geq 1 + \frac{\sqrt{\gamma} + (x + y^2)/\sqrt{\gamma}}{\sqrt{n}}. \quad (37)$$

Then, for (35) we use $\operatorname{erfc}(x) \lesssim e^{-x^2}/x$ to estimate

$$\frac{1}{(\gamma n)^{3/4}} \tilde{K}_n(z, z) \lesssim \frac{n^{1/4}}{\gamma^{5/4}} |z|^2 e^{-n\mu(|z|^2)^2} \leq \frac{n^{1/4}}{\gamma^{5/4}} |z|^2 e^{-\gamma(1-\sqrt{\gamma/n})/2} e^{-(x+y^2)/3} \quad (38)$$

using the elementary bound $t - \log t - 1 \geq \delta(1 - \delta)(t - 1)/2$ for $t \geq 1 + \delta$ and $\delta \in [0, 1)$ implying

$$\mu(|z|^2)^2 = |z|^2 - 2 \log |z| - 1 \geq \frac{\gamma + x + y^2}{2n} \left(1 - \sqrt{\frac{\gamma}{n}} \right) \geq \frac{\gamma}{2n} \left(1 - \sqrt{\frac{\gamma}{n}} \right) + \frac{x + y^2}{3n} \quad (39)$$

due to $\gamma/n \ll 1$ in the last step. Now (35) follows from

$$e^{-\gamma/2} = \exp\left(-\frac{1}{4} \log \frac{n}{2\pi^4(\log n)^5}\right) = \frac{2^{1/4}\pi(\log n)^{5/4}}{n^{1/4}} = \frac{2^{3/2}\pi\gamma^{5/4}}{n^{1/4}} \left(1 + \mathcal{O}\left(\frac{\log \log n}{\log n}\right) \right). \quad (40)$$

For (29) we first note

$$z\bar{w} = 1 + \frac{\sqrt{\gamma} + (x_1 + x_2) + y_1 y_2}{\sqrt{n}} + i \frac{y_1 - y_2}{(\gamma n)^{1/4}} + i \frac{y_1(\gamma + x_2/\gamma) - y_2(\gamma + x_1/\gamma)}{(\gamma n)^{3/4}}, \quad (41)$$

and hence $|1 - z\bar{w}| \gtrsim (|y_1 - y_2|(n/\gamma)^{1/4} + \sqrt{\gamma})/\sqrt{n}$. Now we use the asymptotics [31, Lemma 3.4]

$$\frac{\Gamma(n, nz\bar{w})}{\Gamma(n)} = e^{-nz\bar{w}} \frac{e^n (z\bar{w})^n}{\sqrt{2\pi n}(1 - z\bar{w})} \left(1 + \mathcal{O}\left(\frac{1}{n|1 - z\bar{w}|^2}\right) \right) \quad (42)$$

to estimate

$$\frac{|\tilde{K}_n(z, w)|}{(n\gamma)^{3/4}} \lesssim \frac{n^{1/4} e^{n(1-|z|^2/2-|w|^2/2+\log|z\bar{w}|)}}{\gamma^{3/4}(|y_1 - y_2|(n/\gamma)^{1/4} + \sqrt{\gamma})} \left(1 + \mathcal{O}\left(\frac{\sqrt{\gamma}}{C^2}\right)\right). \quad (43)$$

In the exponent we use

$$\begin{aligned} 1 - |z|^2/2 - |w|^2/2 + \log|z| + \log|w| &= -\frac{(|z|^2 - 1)^2}{4} - \frac{(|w|^2 - 1)^2}{4} + \mathcal{O}\left((\gamma/n)^{3/2}\right) \\ &= -\frac{\gamma + x_1 + y_2^2 + x_2 + y_2^2}{2n} + \mathcal{O}\left(\frac{1 + |\mathbf{x}|^2 + |\mathbf{y}|^4}{n\gamma}\right) \end{aligned} \quad (44)$$

to conclude

$$\frac{|\tilde{K}_n(z, w)|}{(n\gamma)^{3/4}} \lesssim \frac{\sqrt{\gamma} e^{-(x_1 + y_1^2)/2} e^{-(x_2 + y_2^2)/2}}{|y_1 - y_2|(n/\gamma)^{1/4} + \sqrt{\gamma}} \left(1 + \mathcal{O}\left(\frac{\sqrt{\gamma}}{C^2} + \frac{\log \log n + x^2 + y^4}{\log n}\right)\right). \quad (45)$$

It remains to consider (28) where we use [31, Lemma 3.3] in the form

$$\frac{\Gamma(n, n z \bar{w})}{\Gamma(n)} = \frac{z \bar{w} \mu(z \bar{w}) \operatorname{erfc}(\sqrt{n} \mu(z \bar{w}))}{\sqrt{2}(z \bar{w} - 1)} \left(1 + \mathcal{O}\left(\frac{1}{n|1 - z \bar{w}|}\right)\right), \quad \mu(z) := \sqrt{z - \log(z) - 1}. \quad (46)$$

We use the Taylor expansion $\mu(1+z) = z/\sqrt{2} + \mathcal{O}(|z|^2)$ (for small enough $|z|$) and the asymptotics [16, Eq. (7.12.1)] of the error function $\operatorname{erfc}(z) = e^{-z^2}/(\sqrt{\pi}z)(1 + \mathcal{O}(|z|^{-2}))$ for $|\arg z| < 3\pi/4$ to obtain

$$\frac{\Gamma(n, n z \bar{w})}{\Gamma(n)} = \frac{e^{-n(z \bar{w} - 1)^2/2}}{\sqrt{2\pi}\sqrt{n}(z \bar{w} - 1)} \left(1 + \mathcal{O}\left(|z \bar{w} - 1| + n|z \bar{w} - 1|^3 + \frac{1}{n|z \bar{w} - 1|^2}\right)\right) \quad (47)$$

and thereby

$$\begin{aligned} \frac{|\tilde{K}_n(z, w)|^2}{4(n\gamma)^{3/2}} &= \frac{n^{1/2}}{\gamma^{3/2}(2\pi)^3} \frac{e^{-\gamma - x_1 - x_2 - y_1^2 - y_2^2}}{\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2} \left(1 + \mathcal{O}\left(\frac{1 + |\mathbf{x}|^2 + |\mathbf{y}|^4}{\gamma}\right)\right) \\ &= \frac{\gamma e^{-x_1 - x_2 - y_1^2 - y_2^2}}{\pi(\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2)} \left(1 + \mathcal{O}\left(\frac{\log \log n + |\mathbf{x}|^2 + |\mathbf{y}|^4}{\log n}\right)\right). \end{aligned} \quad (48)$$

Here we also used the upper bound on $|y_1 - y_2| \leq n^{1/10} n^{-1/4}$ in order to estimate $\sqrt{\gamma/n} \lesssim |1 - z \bar{w}| \lesssim n^{-1/2}(\sqrt{\gamma} + n^{1/10}/\gamma^{1/4})$. \square

3. REAL GINIBRE

We now consider the real case. The analogue of (15) for test functions $g: \mathbf{C} \rightarrow [0, 1]$ invariant under complex conjugation, $g(\bar{z}) = g(z)$, and vanishing on the real line, $g(x) = 0$, $x \in \mathbf{R}$, is given by [31]

$$\mathbf{E} \prod_{i=1}^n (1 - g(\sigma_i)) = [\det(1 - \sqrt{g} K_n^{\mathbf{C}, \mathbf{C}} \sqrt{g})]^{1/2}, \quad (49)$$

where

$$K_n^{\mathbf{C}, \mathbf{C}}(z, w) := \begin{pmatrix} S_n(z, w) & -i S_n(z, \bar{w}) \\ -i S_n(\bar{z}, w) & S_n(w, z) \end{pmatrix} \quad (50)$$

with

$$\begin{aligned} S_n(z, w) &:= \frac{i n e^{-n(z - \bar{w})^2/2}}{\sqrt{2\pi}} \sqrt{n}(\bar{w} - z) \sqrt{\operatorname{erfc}(\sqrt{2n}|\Im z|) \operatorname{erfc}(\sqrt{2n}|\Im w|)} e^{-nz\bar{w}} K_n(z, w) \\ &= \Phi_n(z, w) \tilde{K}_n(z, w) \\ \Phi_n(z, w) &:= e^{n(|z|^2 + |w|^2 - 2z\bar{w})/2} \frac{i\sqrt{\pi} e^{-n(z - \bar{w})^2/2}}{\sqrt{2}} \sqrt{n}(\bar{w} - z) \sqrt{\operatorname{erfc}(\sqrt{2n}|\Im z|) \operatorname{erfc}(\sqrt{2n}|\Im w|)}. \end{aligned} \quad (51)$$

The analogue to Proposition 5 is the following result.

Proposition 7. Let $|t| \leq \sqrt{\log n}/10$, let $A(t)$ be as in (22) and recall $\gamma = \gamma_n$ from (2). Consider any function $g: \mathbf{C} \rightarrow [0, 1]$ supported on $\text{supp } g \subset A(t)$ that is symmetric in the sense $g(z) = g(\bar{z})$, and let n be large enough such that $\gamma > 0$. Then we have

$$\begin{aligned} \text{Tr } \sqrt{g} K_n^{\mathbf{C}, \mathbf{C}} \sqrt{g} &= 2 \int_t^\infty \int_0^\infty g(z) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx + \mathcal{O}\left(e^{-t} \frac{(\log \log n)^2 + |t|^2}{\log n}\right), \\ z &= 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x}{\sqrt{4\gamma n}} + \frac{iy}{(\gamma n)^{1/4}} \end{aligned} \quad (52)$$

and

$$\|\sqrt{g} K_n^{\mathbf{C}, \mathbf{C}} \sqrt{g}\|_2 \lesssim e^{-\sqrt{\log n}/32}. \quad (53)$$

The unspecified constants in \lesssim and $\mathcal{O}(\cdot)$ are uniform in n and in $|t| \leq \sqrt{\log n}/10$.

Proof. We estimate

$$\Phi_n(z, z) = \sqrt{\pi} e^{2n(\Im z)^2} \sqrt{2n} \Im z \operatorname{erfc}(\sqrt{2n} |\Im z|) = 1 + \mathcal{O}\left(\min\left\{1, \frac{1}{n(\Im z)^2}\right\}\right), \quad (54)$$

where we used the asymptotic $\operatorname{erfc}(x) = e^{-x^2}/(\sqrt{\pi}x)(1 + \mathcal{O}(x^{-2}))$ and the bound $\operatorname{erfc}(x) \leq e^{-x^2}/(\sqrt{\pi}x)$. Thus the tracial computation essentially reduces to the complex case (31) and we obtain

$$\begin{aligned} \text{Tr } \sqrt{g} K_n^{\mathbf{C}, \mathbf{C}}(z, z) \sqrt{g} &= 2 \int_{A(t)_+} g(z) S_n(z, z) d^2 z \\ &= 2 \int_{A(t)_+} g(z) \tilde{K}_n(z, z) \mathbf{1}(\Im z > n^{-5/12}) d^2 z \left(1 + \mathcal{O}(n^{-5/12})\right) \\ &\quad + \mathcal{O}\left(\int_{A(t)_+} \tilde{K}_n(z, z) \mathbf{1}(\Im z \leq n^{-5/12}) d^2 z\right) \\ &= 2 \int_t^\infty \int_0^\infty \frac{e^{-x-y^2}}{\sqrt{\pi}} g(z) dy dx + \mathcal{O}\left(e^{-t} \frac{(\log \log n)^2 + |t|^2}{\log n}\right), \end{aligned} \quad (55)$$

where $A(t)_+ := A(t) \cap \mathbf{H}$, we parametrized z with x, y as in (23), and we used (28) and (30).

For the Hilbert-Schmidt norm we estimate, analogously to (33)

$$\begin{aligned} \|\sqrt{g} K_n^{\mathbf{C}, \mathbf{C}} \sqrt{g}\|_2 &= \iint g(z) g(w) \operatorname{Tr} K_n^{\mathbf{C}, \mathbf{C}}(z, w) K_n^{\mathbf{C}, \mathbf{C}}(w, z) d^2 z d^2 w \\ &\leq \iint_{\Re \geq t} \operatorname{Tr} K_n^{\mathbf{C}, \mathbf{C}}(z, w) K_n^{\mathbf{C}, \mathbf{C}}(w, z) \mathbf{1}(|\mathbf{x}| + |\mathbf{y}|^2 \leq \frac{\sqrt{\log n}}{2}) d^2 z d^2 w \\ &\quad + \int \operatorname{Tr} K_n^{\mathbf{C}, \mathbf{C}}(z, z) \mathbf{1}(|x| + y^2 > \frac{\sqrt{\log n}}{4}) d^2 z \\ &\lesssim \iint_{\Re \geq t} |S_n(z, w)|^2 \mathbf{1}(|\mathbf{x}| + |\mathbf{y}|^2 \leq \frac{\sqrt{\log n}}{2}) d^2 z d^2 w + e^{-\sqrt{\log n}/16}, \end{aligned} \quad (56)$$

where we used that the integrals of $|S_n(z, w)|^2$ and $|S_n(z, \bar{w})|^2$ are equal by symmetry of the integration region, and $\Re \geq t$ indicates the integration region $\{\Re z \geq t\} \cap \{\Re w \geq t\}$. Now we use (28) and (29) together with the elementary bound

$$|\Phi_n(z, w)|^2 \lesssim \frac{n|z - \bar{w}|^2}{(1 \vee \sqrt{n} \Im z)(1 \vee \sqrt{n} \Im w)} \lesssim \frac{(x_1 - x_2)^2 / \sqrt{\gamma} + \sqrt{n}(y_1 + y_2)^2}{(\gamma^{1/4} \vee n^{1/4} y_1)(\gamma^{1/4} \vee n^{1/4} y_2)} \quad (57)$$

to estimate

$$\frac{|S_n(z, w)|^2}{(\gamma n)^{3/2}} \lesssim \frac{\gamma e^{-x_1 - x_2 - y_1^2 - y_2^2}}{\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2} \frac{(x_1 - x_2)^2 / \sqrt{\gamma} + \sqrt{n}(y_1 + y_2)^2}{(\gamma^{1/4} \vee n^{1/4} y_1)(\gamma^{1/4} \vee n^{1/4} y_2)} \quad (58)$$

and conclude, similarly to (34), that

$$\iint_{\Re \geq t} |S_n(z, w)|^2 \mathbf{1}(|\mathbf{x}| + |\mathbf{y}|^2 \leq \frac{\sqrt{\log n}}{2}) d^2 z d^2 w \lesssim e^{-2t} n^{-1/6} \gamma^{3/2}. \quad (59)$$

□

As a consequence of (20) and (49) and Proposition 7 we obtain that for any fixed t it holds that

$$\begin{aligned} \mathbf{P}\left(\max_{i: \sigma_i \notin \mathbf{R}} \Re \sigma_i < 1 + \sqrt{\frac{\gamma}{4n}} + \frac{t}{\sqrt{4\gamma n}}\right) &= \mathbf{P}\left(\sigma_1, \dots, \sigma_n \in \mathbf{R} \cup [\mathbf{C} \setminus (A(t) \cup \mathbf{R})]\right) \\ &= [\det(1 - \chi_{A(t)} K_n^{\mathbf{C}, \mathbf{C}} \chi_{A(t)})]^{1/2} \xrightarrow{n \rightarrow \infty} e^{-e^{-t}/2} \end{aligned} \quad (60)$$

with χ_A denoting the characteristic function of the set A , using that $\int_0^\infty e^{-y^2} dy = \sqrt{\pi}/2$. Moreover, for any symmetric function $f: \mathbf{C} \rightarrow [0, \infty)$ supported in $A(t)$ we also have that

$$\begin{aligned} \mathbf{E} \exp\left(-\sum_{i: \sigma_i \notin \mathbf{R}} f(\sigma_i)\right) &= \det\left(1 - \sqrt{1 - e^{-f}} K_n^{\mathbf{C}, \mathbf{C}} \sqrt{1 - e^{-f}}\right)^{1/2} \\ &\xrightarrow{n \rightarrow \infty} \exp\left(-\int_t^\infty \int_0^\infty (1 - e^{-f(z)}) \frac{e^{-x-y^2}}{\sqrt{\pi}} dy dx\right). \end{aligned} \quad (61)$$

In order to complete the proof of Theorems 1 and 2 it remains to estimate the real eigenvalues. However, the real eigenvalues affect neither of these results since the largest real eigenvalue lives on a smaller scale, $1 + \mathcal{O}(1/\sqrt{n})$, than the largest real part of complex eigenvalues, $1 + \mathcal{O}(\sqrt{\log n/n})$. Indeed, the main result of [28] is that for large t

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\max_{i: \sigma_i \in \mathbf{R}} \sigma_i \leq 1 + \frac{t}{\sqrt{n}}\right) = 1 - \frac{1}{4} \operatorname{erfc}(t) + \mathcal{O}(e^{-2t^2}). \quad (62)$$

Together with (60) and (61) this concludes the proof of Theorems 1 and 2 also in the real case.

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