On Brouwer’s proof of invariance of dimension

Bachelor Thesis

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Abstract

In this thesis we give an elementary proof of the Invariance of Dimension Theorem, i.e. the statement that for distinct positive integers $n$ and $m$, no non-empty open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$. We closely follow Brouwer’s paper [2, p.161-165] and work out his geometric lines of thoughts. Throughout the thesis we try to stay as self-contained as possible, building up the prerequisite material from convex analysis and introduce some useful notions in order to reformalize Brouwer’s arguments.
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The goal of this chapter is to describe a canonical way to divide the \(n\)-dimensional unit cube into nice subsets that do only intersect, if at all, along their boundaries. We introduce the notion of a simplex and explain some of its basic properties. Simplices turn out to be easy to handle; therefore, we use them as building blocks for our subdivision of the \(n\)-dimensional unit cube. Such a subdivision into simplices is called a triangulation.

### 1.1 Basic properties of simplices

This is an introductory section setting up the basic definitions. We need to define the notions of a convex hull, simplex, vertex order and the sign of a simplex. All of these notions are essential. Without them we are not able to prove Brouwer’s Invariance of Dimension Theorem later on.

**Definition 1.1 (Convex hull)** Let \(A = \{x_1, \ldots, x_k\}\) be a finite set of points in \(\mathbb{R}^n\). The set

\[
\text{conv}(A) := \left\{ \sum_{i=1}^{k} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} \quad (1.1)
\]

is called the convex hull of \(A\).

Recall that a set \(C \subset \mathbb{R}^n\) is said to be convex if for all points \(x, y \in C\) and \(\lambda \in [0, 1]\) we have that \(\lambda x + (1 - \lambda)y\) is contained in the set \(C\). To begin, we show that the convex hull of a finite set deserves its name and is actually convex.

**Lemma 1.2** The convex hull of a finite set \(A = \{x_1, \ldots, x_k\}\) of points in \(\mathbb{R}^n\) is convex and compact.

**Proof** Let \(A = \{x_1, \ldots, x_k\}\) denote an arbitrary finite subset of \(\mathbb{R}^n\). At first, we deal with the compactness of the set \(\text{conv}(A)\). We want to find a continuous map \(f\) and a compact set \(B\) such that \(\text{conv}(A)\) is the image of \(B\) under
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f. The existence of such a map \( f \) implies immediately the compactness of \( \text{conv}(\mathcal{A}) \). This is well-known from analysis. We define the set

\[
\Delta^k := \left\{ (\lambda_0, \ldots, \lambda_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} \lambda_i = 1, \forall i \in \{0, 1, \ldots, k\} : \lambda_i \geq 0 \right\}. \quad (1.2)
\]

In order to use \( \Delta^k \) as our set \( \mathcal{B} \) we need to show that \( \Delta^k \) is compact. Let us define the map

\[
g : \mathbb{R}^{k+1} \to \mathbb{R} \\
(\lambda_0, \ldots, \lambda_k) \mapsto \lambda_0 + \cdots + \lambda_k.
\]

The continuity of \( g \) implies the closedness of the set \( g^{-1}(1) \). Given that the half spaces \( H_i^+ = \{(\lambda_0, \ldots, \lambda_k) \in \mathbb{R}^{k+1} : \lambda_i \geq 0 \} \) are closed for every \( i \in \{0, \ldots, k\} \) we obtain that

\[
\Delta^k = g^{-1}(1) \cap \left( \bigcap_{i=0}^{k} H_i^+ \right) \quad (1.3)
\]

as an intersection of closed sets is closed as well. For \( \Delta^k \) is bounded we may conclude that it is compact. We define the map

\[
f : \Delta^k \to \text{conv}(\mathcal{A}) \\
(\lambda_0, \ldots, \lambda_k) \mapsto \lambda_0 x_0 + \cdots + \lambda_k x_k. \quad (1.4)
\]

The continuity of \( f \) is immediate. The equality \( f(\Delta^k) = \text{conv}(\mathcal{A}) \) follows from the definition of \( \text{conv}(\mathcal{A}) \). Thus, \( \text{conv}(\mathcal{A}) \) is compact.

Now we deal with the convexity of \( \text{conv}(\mathcal{A}) \). Note that the set \( \Delta^k \) as an intersection of convex sets, see Equation (1.3), is convex as well. The map \( f \) is the restriction of the affine linear map

\[
\hat{f} : \mathbb{R}^{k+1} \to \mathbb{R}^k \\
(\lambda_0, \ldots, \lambda_k) \mapsto \lambda_0 x_0 + \cdots + \lambda_k x_k
\]

(1.5)

to \( \Delta^k \). We may invoke Lemma A.1 to conclude that \( \text{conv}(\mathcal{A}) \) as the image of a convex set under an affine linear map is convex, as desired.

Sometimes it is useful to define the convex hull of a finite subset \( A \) of \( \mathbb{R}^n \) as

\[
\bigcap_{B \subset \mathbb{R}^n \text{convex, closed}} \bigcap_{A \subset B} B. \quad (1.6)
\]

Using that the set \( \text{conv}(\mathcal{A}) \) is convex and compact, see Lemma 1.2, we see that (1.6) is a subset of \( \text{conv}(\mathcal{A}) \). On the other hand, the Definition 1.1 of
conv(A) and the fact that the set (1.6) is convex and contains A imply the opposite inclusion, i.e. \( \text{conv}(A) \subseteq (1.6) \). Hence the two definitions (1.1) and (1.6) of the convex hull of a finite set in \( \mathbb{R}^n \) are equivalent.

Now we built up all the prerequisite material we need to define the \( k \)-dimensional simplex. We follow loosely [4, p. 1] and establish the following definition of a simplex.

**Definition 1.3 (k-dimensional simplex)** The convex hull of the points \( x_0, \ldots, x_k \) contained in \( \mathbb{R}^n \) is said to be a \( k \)-dimensional simplex, if \( x_1 - x_0, \ldots, x_k - x_0 \) are linearly independent. We refer to \( x_0 \) as the base point. The set \( \{x_0, \ldots, x_k\} \) is called vertex set.

Clearly \( k \) cannot be strictly greater than \( n \). If the dimension of a \( k \)-dimensional simplex \( S \) with vertices in \( \mathbb{R}^n \) does not matter, we simply call \( S \) a simplex.

**Lemma 1.4 (Independence of base point)** Let \( S \) denote a \( k \)-dimensional simplex with vertex set \( \{x_0, \ldots, x_k\} \) consisting of points of \( \mathbb{R}^n \). Then for every \( i \) in \( \{0, \ldots, k\} \) the vectors \( x_0 - x_i, \ldots, x_i - x_i, \ldots, x_k - x_i \) are linearly independent.

**Proof** If \( k \) is equal to zero, there is nothing to prove. We may choose \( x_1 \) as base point. Assume there exist \( \alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_k \in \mathbb{R} \) such that

\[
\alpha_0(x_0 - x_1) + \alpha_2(x_2 - x_1) + \cdots + \alpha_k(x_k - x_1) = 0. \tag{1.7}
\]

By adding zero we may convert Equation (1.7) into

\[
\alpha_2(x_2 - x_0) + \cdots + \alpha_k(x_k - x_0) + (-\alpha_0 + \alpha_2 - \cdots - \alpha_k)(x_1 - x_0) = 0. \tag{1.8}
\]

As \( x_0 \) is the base point of the \( k \)-dimensional simplex \( S \) the coefficients in the sum (1.8) all vanish. So for every \( i \) in \( \{2, \ldots, k\} \) the coefficient \( \alpha_i \) of \( x_i - x_0 \) in Equation (1.8) is equal to zero. We conclude the proof by noticing that \( \alpha_0 \) vanishes, for it is the only non zero constituent of the coefficient of \( x_1 - x_0 \) in Equation (1.8).

To state this in a different manner, Lemma 1.4 implies that the definition of \( S \) is independent of the base point. Using the independence of the base point, i.e. Lemma 1.4, it is immediate that every subset with cardinality \( 1 \leq l \leq k + 1 \) of the vertex set of a \( k \)-dimensional simplex is again the vertex set of an \( (l - 1) \)-dimensional simplex. This \( (l - 1) \)-dimensional simplex is called an \( (l - 1) \)-face of the \( k \)-dimensional simplex. We have already seen in the proof of Lemma 1.2 that there is a natural relationship between points in the set \( \Delta^k \), see Equation (1.11), and the points in the convex hull of a finite set consisting of \( k + 1 \) points. In the next lemma we prove that this correspondence is unique provided that the finite set in question is a \( k \)-dimensional simplex.
Lemma 1.5 (Barycentric coordinates) Let \( S = \text{conv}([x_0, x_1, \ldots, x_k]) \) be a k-dimensional simplex with vertices in \( \mathbb{R}^n \). Then for every point \( x \in S \) there exist unique \( \lambda_i \geq 0 \), where \( i \in \{0, 1, \ldots, k\} \), such that

\[
x = \sum_{i=0}^{k} \lambda_i x_i \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i = 1.
\]

**Proof** Due to the fact that \( x \) is contained in \( S \), existence is guaranteed. Now we deal with uniqueness. Assume we have two representations (1.9) for \( x \). Then,

\[
x = \sum_{i=0}^{k} \lambda_i x_i = \sum_{i=0}^{k} \lambda'_i x_i \quad \iff \quad \sum_{i=1}^{k} (\lambda_i - \lambda'_i)(x_i - x_0) = 0.
\]

In Equation (1.10) we used the fact that the coefficients \( \lambda_i \) and \( \lambda'_i \) sum up respectively to one. Seeing that the points \( x_1 - x_0, \ldots, x_k - x_0 \) are linearly independent, it follows that the coefficients \( \lambda_i \) and \( \lambda'_i \) have to coincide for every \( i \in \{1, \ldots, k\} \). Furthermore, again on account of the fact that the coefficients \( \lambda_i \) and \( \lambda'_i \) sum up respectively to one, the points \( \lambda'_0 \) and \( \lambda_0 \) have to coincide as well, thereby the lemma follows.■

To put it differently, Equation (1.9) implies that for every k-dimensional simplex \( S \) in \( \mathbb{R}^n \) there exists an injective continuous map \( \beta \), which we call barycentric map, from the set \( \Delta^k \) to \( S \). The assignment of the barycentric map \( \beta \) is given by

\[
(\lambda_0, \ldots, \lambda_k) \mapsto \lambda_1(x_1 - x_0) + \cdots + \lambda_k(x_k - x_0) + x_0.
\]

Note that \( \beta \) is affine linear. Since \( \Delta^k \) is compact and \( \mathbb{R}^n \) is a Hausdorff space, the map \( \beta \) is even a homeomorphism between \( \Delta^k \) and \( S \). For every point \( x \) in the simplex \( S \) we call the point \( \beta^{-1}(x) \in \Delta^k \) the barycentric coordinates of \( x \).

The fact that the image of a convex set under an affine linear map is again convex played a crucial role in the proof of Lemma 1.2. It is natural to ask given an affine linear map whether the convex hull of the images of the vertices of a simplex is again a simplex of the same dimension.

Lemma 1.6 Suppose \( S \) is a k-dimensional simplex with vertices \( \{x_0, \ldots, x_k\} \) in \( \mathbb{R}^n \) and \( f: S \rightarrow \mathbb{R}^n \) is the restriction of an injective affine linear map. Then the convex hull of the points \( f(x_0), \ldots, f(x_k) \) is again a k-dimensional simplex.

**Proof** All we have to check is the linear independence of the points \( f(x_1) - f(x_0), \ldots, f(x_k) - f(x_0) \). Let \( A \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \) such that \( f(x) = Ax + b \). The injectivity of \( f \) implies that \( A \) is injective as well. Let \( \alpha_1, \ldots, \alpha_k \in \mathbb{R} \) be given such that

\[
\alpha_1(f(x_1) - f(x_0)) + \cdots + \alpha_k(f(x_k) - f(x_0)) = 0.
\]

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The cancellation of the b’s in all of the terms \( f(x_i) - f(x_0) = Ax_i + b - (Ax_0 + b) = A(x_i - x_0) \) and the linearity of \( A \) make it possible to rewrite Equation (1.12) as follows

\[
A(\alpha_1(x_1 - x_0) + \cdots + \alpha_k(x_k - x_0)) = 0.
\]

Since \( A \) is injective, we deduce that

\[
\alpha_1(x_1 - x_0) + \cdots + \alpha_k(x_k - x_0) = 0.
\]

Now we can conclude the proof by noticing that \( \alpha_1 = \ldots = \alpha_k = 0 \), for the points \( x_1 - x_0, \ldots, x_k - x_0 \) are linearly independent. ■

Since every \( k \)-dimensional simplex is homeomorphic to the set \( \Delta^k \) it deserves a special name.

**Definition 1.7 (Standard simplex)** For every \( k \geq 0 \) we call the set \( \Delta^k \), see Equation (1.11), the \( k \)-dimensional standard simplex.

It would be nice if the \( k \)-dimensional standard simplex is actually a \( k \)-simplex. Indeed, this follows immediately from the previous Lemma 1.6, given that the barycentric map \( \beta \) is the restriction of an injective affine linear map. The following theorem is a standard result from convex analysis.

**Theorem 1.8 (Carathéodory's Theorem)** Let \( A = \{x_0, \ldots, x_k\} \) be a finite set of points in \( \mathbb{R}^n \). Suppose that the convex hull of \( A \) is not a \( k \)-dimensional simplex. Then every point in \( \text{conv}(A) \) lies in an \( r \)-simplex with vertices in \( A \), where \( 0 \leq r < k \).

**Proof** This proof closely follows the proof of Carathéodory’s Theorem as given in [9, p. 155].

To begin, we observe that \( k \) must be greater than or equal to 1, since one point sets are always 0-dimensional simplices. Let \( x \in \text{conv}(A) \) be arbitrary. Then, \( x \) can be written as:

\[
x = \lambda_0 x_0 + \cdots + \lambda_k x_k, \quad \text{where} \quad \sum_{i=0}^{k} \lambda_i = 1, \forall i : \lambda_i \geq 0. \tag{1.13}
\]

Recall that we say that \( x \) can be expressed as a *convex combination* of \( k + 1 \) points in \( A \) if Equation (1.13) holds. The points \( x_1 - x_0, \ldots, x_k - x_0 \) are linearly dependent, since \( \text{conv}(A) \) is not a \( k \)-dimensional simplex. Hence there exist \( \xi_1, \ldots, \xi_k \in \mathbb{R} \), not all equal to zero, such that

\[
\xi_1(x_1 - x_0) + \cdots + \xi_k(x_1 - x_0) = 0. \tag{1.14}
\]

We can multiply Equation (1.14) by \( \xi \in \mathbb{R} \) and add it to Equation (1.13) to obtain:

\[
x = (\lambda_0 - \xi \cdot (\xi_1 + \cdots + \xi_k))x_0 + (\lambda_1 + \xi \cdot \xi_1)x_1 + \cdots + (\lambda_k + \xi \cdot \xi_k)x_k. \tag{1.15}
\]
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Define $\xi_0 = -\xi_1 - \cdots - \xi_k$. Since $\xi$ was arbitrary we can choose $-\xi = \min\left\{ \frac{\lambda_i}{\xi_i} : i \in \{0, \ldots, k\}, \xi_i \neq 0 \right\}$. (1.16)

In fact, since the $\xi_i$’s are not all equal to zero, the minimum in Equation (1.16) is obtained. Hence there exists an $i \in \{0, \ldots, k\}$ such that we can write $-\xi = \frac{\lambda_i}{\xi_i}$. Observe that for all $j \in \{0, \ldots, k\}$ the number $\lambda_j + \xi \cdot \xi_j$ is non-negative and

$$\sum_{j=0}^{k} \lambda_j + \xi \cdot \xi_j = 1.$$ (1.17)

In particular, $\lambda_i + \xi \cdot \xi_i = 0$. We deduce that $x$ can be expressed as a convex combination of $k$ points in $A$. If the convex hull of these points is a $(k - 1)$-dimensional simplex, we are done. Otherwise we repeat the construction until we reach the point that the convex hull of the remaining points is a simplex. At worst, we are left with two points. If they are equal, we conclude that $x$ is contained in a 0-dimensional simplex with vertices in $A$. If the two points are distinct, we see that $x$ is contained in a 1-dimensional simplex with vertices in $A$. Thus, the lemma follows.

Carathéodory’s Theorem plays a crucial role in many proofs of the next chapter and gives furthermore a connection between non collapsed points, see Definition 2.10, and non collapsed sets, see Definition 2.7. This connection is stated in Lemma 2.12.

It turns out to be useful later on to be able to speak about the orientation of a simplex. The first step towards a proper setting for the orientation of a simplex is the following definition.

**Definition 1.9 (Vertex order)** Let $A = \{x_0, \ldots, x_n\}$ be a finite set in $\mathbb{R}^n$. Every $(n + 1)$-tuple in the set $\{(x_{\sigma(0)}, \ldots, x_{\sigma(n)}) : \sigma \in S_{n+1}\}$ is called vertex order of the finite set $A$.

The vertex order of a finite set $A = \{x_0, \ldots, x_n\}$ is uniquely determined by $\sigma \in S_{n+1}$. Brouwer uses the name indikatrix for something almost similar to what we call vertex order. See [1, p. 100] for the proper definition of Brouwer’s notion of an indikatrix.

**Definition 1.10 (Sign of a finite set)** Suppose $A = \{x_0, \ldots, x_n\}$ is a finite set in $\mathbb{R}^n$ with vertex order $(x_{\sigma(0)}, \ldots, x_{\sigma(n)})$. The signum of the determinant of the matrix that has the vectors $x_{\sigma(1)} - x_{\sigma(0)}, \ldots, x_{\sigma(n)} - x_{\sigma(0)}$ (in this order) as columns is called the sign of $A$. We denote this number symbolically by $\text{sgn}(A)$.

The sign of $A$ depends on the vertex order. Suppose we are given a set $B = \text{conv}(\{x_0, \ldots, x_n\})$ and a vertex order for the finite set $A = \{x_0, \ldots, x_n\}$,
then we breach notation and call the number \( \text{sgn}(A) \) the sign of \( B \). Assume \( A \) happens to be an \( n \)-dimensional simplex. As the definition of \( A \) is independent of the base point, see Lemma 1.4, we know that for every \( \sigma \in S_{n+1} \) the points \( x_{\sigma(1)} - x_{\sigma(0)}, \ldots, x_{\sigma(n)} - x_{\sigma(0)} \) are linearly independent and therefore their determinant is never equal to zero. Hence the sign of every \( n \)-dimensional simplex is always either \(-1\) or \(1\).

1.2 Standard triangulation of the \( n \)-dimensional unit cube

In this subsection we construct a standard method to triangulate the \( n \)-dimensional unit cube, see Definition 1.15, into simplices. Furthermore, we even canonically assign to each simplex of this triangulation a vertex order and therefore also a sign. We denote by \((S_n, \circ)\) the group of bijections of the set \( \{1, \ldots, n\} \). Recall that \( S_n \) acts linearly on \( \mathbb{R}^n \) by the law

\[
\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

Symbolically we write \( S_n \curvearrowright \mathbb{R}^n \). This action serves as a tool to address the different simplices of our triangulation. In fact, every element of \( S_n \) corresponds to a simplex of the triangulation. We begin with the set that corresponds to the neutral element of the group \((S_n, \circ)\).

**Lemma 1.11** The set \( S_{1d} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \ldots \leq x_n \leq 1\} \) is an \( n \)-dimensional simplex with vertex set \( V_{1d} = \{(0, \ldots, 0), (0, \ldots, 0, 1), \ldots, (1, \ldots, 1)\} \).

**Proof** To begin, we introduce a handy notation. We denote \((0, \ldots, 0)\) in \( V_{1d}\) by \( v_0 \) and for each \( i \) in the set \( \{1, \ldots, n\} \) the point \((0, \ldots, 0, 1, \ldots, 1)\) in \( V_{1d}\), where the first \( i-1 \) coordinates are zero and all other coordinates are equal to one, by \( v_i \). The points \( v_0, \ldots, v_n \) are clearly contained in \( S_{1d}\). Furthermore,

\[
\det \begin{pmatrix}
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
1 & \ldots & \ldots & 1
\end{pmatrix} = 1. \tag{1.18}
\]

Hence the points \( v_1 - v_0, \ldots, v_n - v_0 \) are linearly independent. Now we are left to show that convex hull of the points \( v_0, \ldots, v_n \) is equal to \( S_{1d}\). Let \( x = (x_1, \ldots, x_n) \in S_{1d} \) be arbitrary. Define \( \lambda_n := x_1 \geq 0 \) and for every \( i \in \{1, \ldots, n-1\} \) define \( \lambda_i := x_{n-i+1} - x_{n-i} \). Note that \( \lambda_i \geq 0 \). Define \( \lambda_0 := 1 - \lambda_n - \cdots - \lambda_1 = 1 - x_n \). Note that \( \lambda_0 \geq 0 \) and \( 1 = \lambda_0 + \cdots + \lambda_n \). It follows that

\[
x = \sum_{i=0}^{n} \lambda_i v_i.
\]
Thus, we have shown that $x \in \text{conv}(v_0, \ldots, v_n)$. Let $x, x' \in S_{id}$ be two distinct points. Elementary estimates show that every point on the line segment $[x, x']$ is contained in $S_{id}$. Therefore $\text{conv}(v_0, \ldots, v_n) \subset S_{id}$, for $S_{id}$ is convex and contains the points $v_0, \ldots, v_n$. The Lemma follows. ■

The fact that $S_{id}$ is a simplex enables us to deduce that for every $\sigma$ contained in $S_n$ the set $\sigma S_{id}$ is again a simplex.

**Corollary 1.12** For each $\sigma \in S_n$ the set $S_{\sigma} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \leq 1 \}$ is an $n$-dimensional simplex with vertex set $V_{id} = \{\sigma v_0, \ldots, \sigma v_n\}$.

**Proof** As in the proof of Lemma 1.11 we have to show that the convex hull $\text{conv}(\sigma v_0, \ldots, \sigma v_n)$ is equal to $S_{\sigma}$. By definition $S_{\sigma} = \sigma S_{id}$. Since the action $S_n \curvearrowright \mathbb{R}^n$ is linear, we may deduce, using the fact that in finite dimensional vector spaces convexity is preserved under linear maps, that the set $\sigma S_{id} = S_{\sigma}$ is convex. Thus, we have that $\text{conv}(\sigma v_0, \ldots, \sigma v_n) \subset S_{\sigma}$. Let $x \in S_{\sigma}$ be arbitrary. The point $y = \sigma^{-1} x$ has a representation as

$$y = \sum_{i=0}^{n} \lambda_i v_i$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$. Hence we can conclude the proof by observing that

$$x = \sum_{i=0}^{n} \lambda_i \sigma v_i$$

Note that every $v_i \in V_{id}$ has a distinct number of “1”s and that this number is preserved under the action of every $\sigma \in S_n$. Hence we obtain for every two distinct $\sigma, \tau \in S_n$ that

$$\sigma v_i = \tau v_j \Rightarrow i = j. \quad (1.19)$$

Equation (1.19) is helpful in the proof of Lemma 1.19.

**Lemma 1.13** The sign of the simplex $S_{\sigma}$ with vertex type $(\sigma v_0, \ldots, \sigma v_n)$ is equal to $\text{sgn}(\sigma)$.

**Proof** Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$. Define the $n \times n$ matrix $P_\sigma := [\delta_{\sigma(i)}]$. It follows immediately that $\sigma x = P_\sigma x$, for every $x \in \mathbb{R}^n$. Thus,

$$\det(\sigma v_1 | \cdots | \sigma v_n) = \det P_\sigma \cdot 1 = \text{sgn}(\sigma),$$

as the determinant is a multiplicative map and Equation (1.18) holds. Our Lemma follows. ■
1.2. Standard triangulation of the n-dimensional unit cube

Note that Lemma 1.13 implies that the linear map $\sigma(\cdot)$ preserves volumes. For later purposes we introduce the following convention.

**Convention 1.14** Let $\sigma$ be contained in the set $S_n$. By convention, we use the vertex order $(\sigma.v_0, \ldots, \sigma.v_n)$ for the simplex $S_\sigma$. Hence we canonically assume that our simplices $S_\sigma$ have signs equal to $\text{sgn}(\sigma)$.

Sometimes it is handy to have a short term expression for the n-dimensional unit cube.

**Definition 1.15 (n-dimensional unit cube)** The set $I^n = [0,1]^n \subset \mathbb{R}^n$ is called the n-dimensional unit cube.

Note that $I^n = \text{conv}([0,1]^n)$. Let $O(n)$ denote the orthogonal group, i.e. the group of distance-preserving endomorphisms of $\mathbb{R}^n$. Recall that for a fixed basis of $\mathbb{R}^n$ there is a bijective correspondence between the set $O(n)$ and the set of orthogonal matrices with entries over $\mathbb{R}$; therefore, we may work from now on with the set of orthogonal matrices, which we call $O(n)$ again for convenience. The group $O(n)$ acts linearly on $\mathbb{R}^n$ by matrix multiplication, i.e. by the law

$$A.x = Ax,$$

where $x$ is contained in $\mathbb{R}^n$. The action $O(n) \curvearrowright \mathbb{R}^n$ gives rise to a homomorphism

$$\varphi: O(n) \to \text{Aut}(\mathbb{R}^n)$$

$$A \mapsto (A.x \mapsto A.x).$$

Here $\text{Aut}(\mathbb{R}^n)$ denotes the group of automorphisms of $\mathbb{R}^n$. Define the affine orthogonal group $AO(n)$ as the semi direct product of $\mathbb{R}^n$ and $O(n)$ with respect to $\varphi$, which is given by Equation (1.20). In symbols: $AO(n) := \mathbb{R}^n \rtimes_{\varphi} O(n)$. Every element contained in $AO(n)$ is of the form $(b,A)$. We interpret $b$ in $\mathbb{R}^n$ as a translation and $A$ in $O(n)$ as a rotation. Note that the group $AO(n)$ acts linearly on $\mathbb{R}^n$ by the law

$$(b,A).x = b + Ax \quad (x \in \mathbb{R}^n)$$

(1.21)

We can use the action $AO(n) \curvearrowright \mathbb{R}^n$, given by (1.21), to define the general notion of a cube.

**Definition 1.16 (n-dimensional cube)** Let $(b,A)$ be contained in $AO(n) = \mathbb{R}^n \rtimes_{\varphi} O(n)$, and let $\tau > 0$ be a positive number. The set $(b,A).(\tau \cdot I^n)$ is called n-dimensional cube with edge length $\tau$, center $b$, and orientation $A$.

The next definition formalizes the idea of a triangulation from the beginning of this chapter. Definition 1.17 is merely a reformulation of Definition 2.2.1 in [4, p. 54].
1.2. Standard triangulation of the n-dimensional unit cube

Definition 1.17 (Triangulation) A triangulation of a finite set \( A \subset \mathbb{R}^n \) is a set \( S \) consisting of \( n \)-dimensional simplices \( S \) with vertices in \( A \) such that:

- the union of these simplices is equal to \( \text{conv}(A) \)
- the intersection of each distinct pair of these simplices is either empty or a common l-face, where \( l \in \{0, \ldots, n-1\} \)

Since \( I^n = \text{conv}(\{0, 1\}^n) \) we misuse Definition 1.17 and call a triangulation of \( \{0, 1\}^n \) a triangulation of the \( n \)-dimensional unit cube \( I^n \). In the following Lemma 1.19 we want to prove that our sets \( S_\sigma \) are in fact a triangulation of \( I^n \). In order to prove this we need the notion of an extremal point, see [9, p. 162].

Definition 1.18 (Extremal point) Suppose \( C \) is a convex set in \( \mathbb{R}^n \). A point \( x \in C \) is called extremal if there exist no distinct points \( y, z \in C \) and \( 0 < t < 1 \) such that

\[
x = ty + (1-t)z.
\]

The fact that the set \( \delta \) consisting of all simplices \( S_\sigma \), see Corollary 1.12 for the definition, is a triangulation of \( I^n \) is the matter of the following Lemma 1.19.

Lemma 1.19 For every \( \tau, \sigma \in S_n \) the intersection \( S_\tau \cap S_\sigma \) is either empty or equal to a common l-face, where \( l \in \{0, \ldots, n\} \). Furthermore the union of all \( S_\sigma \), where \( \sigma \in S_n \), is equal to \( I^n \). In other words, the set \( \delta \) of simplices given by \( \{S_\sigma : \sigma \in S_n\} \) is a triangulation of the \( n \)-dimensional unit cube \( I^n \).

Proof At first, we will show that every point \( x = (x_1, \ldots, x_n) \in I^n \) is contained in some simplex \( S_\tau \). Let \( \sigma \in S_n \) be the permutation that arranges \( x_1, \ldots, x_n \) in an increasing order, i.e

\[
x_{\sigma^{-1}(1)} \leq \cdots \leq x_{\sigma^{-1}(n)}.
\]

So, \( x \) is contained in \( S_{\sigma^{-1}} \). Let \( \tau, \sigma \in S_n \) be arbitrary. The set \( S_\tau \cap S_\sigma \) is compact and convex. It is immediate that if \( x \in S_\tau \cap S_\sigma \) is extremal then \( \sigma^{-1}x \) is also an extremal point of \( S_{\sigma^{-1}} \). Since no other points of \( S_{\sigma^{-1}} \) than \( v_0, \ldots, v_n \) are extremal, it follows that the only extremal points of \( x \in S_\tau \cap S_\sigma \) lie in the set \( \{\sigma.v_i, \tau.v_i\}_{i=0}^n \). Therefore, using Equation (1.19), we conclude that for every extremal point \( p \) of \( S_\tau \cap S_\sigma \) there exists an \( i \) contained in \( \{0, \ldots, n\} \) such that \( p = \sigma.v_i = \tau.v_i \). Define

\[
I := \{i \in \{0, \ldots, n\} : \sigma.v_i \text{ is an extremal point of } S_\sigma \cap S_\tau\}
\]

Since \( \sigma.v_i = \tau.v_i \) for every \( i \in I \), we deduce that \( \text{conv}(\{\sigma.v_i\}_{i=1}^I) \) is equal to \( \text{conv}(\{\tau.v_i\}_{i=1}^I) \). Thus, the set \( \text{conv}(\{\sigma.v_i\}_{i=1}^I) \) is a common \( \{1\} \)-face of the simplices \( S_\tau \) and \( S_\sigma \). Given that every convex compact set is equal to the convex hull of its extremal points, see Theorem (A.2), we obtain that \( S_\tau \cap S_\sigma = \text{conv}(\{v_i\}_{i=1}^I) \). We can conclude the result. 

1.2. Standard triangulation of the n-dimensional unit cube

We call this triangulation $S$ of $I^n$, where we use Convention 1.14 for the vertex orders, the standard triangulation with grid length one.

For every $m \geq 1$ we can write $I^n$ as the union of $m^n$ smaller squares that do only intersect, if at all, on their boundaries. Namely,

$$I^n = \bigcup_{j \in \{0, \ldots, m-1\}^n} \left[ \frac{j_1}{m}, \frac{j_1+1}{m} \right] \times \cdots \times \left[ \frac{j_n}{m}, \frac{j_n+1}{m} \right].$$

Define $I^n_j := \left[ \frac{j_1}{m}, \frac{j_1+1}{m} \right] \times \cdots \times \left[ \frac{j_n}{m}, \frac{j_n+1}{m} \right]$. The image of every simplex $S_\sigma$ of the standard triangulation under the injective affine linear map

$$\psi_j : I^n \rightarrow I^n_j \quad x \mapsto \frac{1}{m} x + j,$$

is again a simplex with vertices $\psi_j(\sigma.v_0), \ldots, \psi_j(\sigma.v_n)$, see Lemma 1.6. Therefore the simplices $\{\psi_j(\sigma) : \sigma \in S_n\}$ triangulate the cube $I^n_j$. It is natural to use $\{\psi_j(\sigma.v_0), \ldots, \psi(\sigma.v_n)\}$ as the canonical vertex order of $S$, if $S$ is the image of the simplex $S_\sigma$. Thus, we have a canonical way to triangulate $I^n_j$. The set of simplices

$$\bigcup_{j \in \{0, \ldots, m-1\}^n} \{\psi_j(\sigma) : \sigma \in S_n\}$$

is a triangulation of the unit cube $I^n$. We denote the set (1.22) with $\xi(m; I^n)$ in order to make its dependence on $m$ visible.

**Definition 1.20** For all $m \geq 1$ the set (1.22) denoted by $\xi(m; I^n)$ is called standard triangulation with grid length $m$. For every $j$ in $\{0, \ldots, m-1\}^n$ we can canonically use the vertex order $\psi_j(\sigma.v_0), \ldots, \psi(\sigma.v_n)$ for $\psi_j(\sigma)$ contained in $\xi(m; I^n)$.

The $(n-1)$-faces of the simplices of every standard triangulation $\xi(m; I^n)$ are called the main faces of $\xi(m; I^n)$. If a main face happens to lie entirely on the topological boundary of $I^n$, we call it boundary face of $\xi(m; I^n)$. For each $0 \leq l < n-1$ every $l$-face of every simplex of the standard triangulation is called main edge of $\xi(m; I^n)$. We denote the set of main edges of $\xi(m; I^n)$ by $F^e$. For the sake of clarity we omit to indicate the underlying standard triangulation. The set of zero dimensional faces is called the set of vertices of $\xi(m; I^n)$.

We denote the complement in $I^n$ of the union of the main edges of the standard triangulation $\xi(m; I^n)$ by $\Theta^n_m$. Let $m = 1$. Even if we remove the main faces from $\Theta^n_1$ and denote this set by $\widetilde{\Theta}^n_1$, we see that

$$\widetilde{\Theta}^n_1 = \bigcup_{\sigma \in S_n} \sigma.S^n_0,$$  

(1.23)
1.2. Standard triangulation of the n-dimensional unit cube

since $S = \{S_\sigma : \sigma \in S_n\}$ is a triangulation of $I^n$. Hence $S_{id}$ is a fundamental domain of the action $S_n \curvearrowright \tilde{\Theta}^n_1$. The identity $\sigma.S_{id} = S_\sigma$ even implies that each $S_\sigma$ is a fundamental domain.
Chapter 2

Invariance of dimension

In the year 1911 L. E. J. Brouwer published in his paper “Beweis der Invarianz der Dimensionenzahl” [2, p. 161-165] a strongly geometrically orientated proof of the fact that for every point in the image of an injective continuous map, going from an non-empty open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$; and for every open neighborhood in $\mathbb{R}^n$ of this point, there exists a non-empty open set that is contained completely in the intersection of the image and the neighborhood. See Theorem 2.18 for the exact statement. Brouwer uses this statement to provide an elementary proof of the sophisticated Theorem of Invariance of Dimension, which says that for distinct positive integers $n$ and $m$ no non-empty open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$, see Theorem 2.19. Brouwer’s paper [2, p. 161-165] contains essentially the techniques of the mapping degree and simplicial approximation.

At the time the paper was considered as clever but inaccessible; cf. [10, p. 265]. Consult [6, p. 954 ff.] for a detailed historical overview.

The goal of this chapter is to work out carefully all steps of Brouwer’s proof in [2, p.161-165]. Basically, Brouwer approximates the injective continuous map in question with maps that are defined with respect to the standard triangulations we have constructed in the first chapter. Moreover, loosely speaking Brouwer establishes a Lemma, which we call Fundamental Lemma 2.1, which guarantees that a map $\varphi: I^n \to I^n$ having only small displacements cannot have an image that does not contain a $n$-dimensional cube, see Definition 1.16. In fact, setting up the prerequisite material we need in order to prove the Fundamental Lemma 2.1 turns out to be the major part of this chapter. Having the Fundamental Lemma 2.1 on hand, the proof of the Invariance of Dimension Theorem is found rather fast.
2.1 Fundamental Lemma

The goal of this section is to develop an adequate language which enables us to give a proof from first principles of the subsequent Fundamental Lemma 2.1.

**Lemma 2.1 (Fundamental Lemma)** Suppose $\phi: I^n \to I^n$ is continuous and the number

$$d := \max_{x \in I^n} \|\phi(x) - x\|_{\mathbb{R}^n}$$

is strictly smaller than $\frac{1}{2}$. Then every cube $K_e$ concentric to $I^n$ with edge length $2e < 2\left(\frac{1}{2} - d\right)$ is entirely contained in $\phi(I^n)$.

By construction, all cubes are homeomorphic to each other. Therefore, if we substitute the unit cube $I^n$ in the Fundamental Lemma 2.1 with an arbitrary cube with edge length $2s$, and the number $\frac{1}{2}$ in the Fundamental Lemma 2.1 with the number $s$, we can deduce that this adjusted Fundamental Lemma 2.1 remains valid. Loosely speaking, the Fundamental Lemma is valid for an arbitrary cube.

Before diving into the proof of the Fundamental Lemma 2.1 we need a bunch of new concepts and notations. We develop them in the following subsections and eventually come back to the Fundamental Lemma 2.1 and provide a proof in the subsection 2.1.4. As everything we develop serves the ultimate goal to allow a proof of the Fundamental Lemma 2.1, it seems appropriate to hold some notions fixed. For the whole section we fix a cube $K_e$ concentric to $I^n$ with edge length $2e$ and a map $\phi: I^n \to I^n$; both with properties as stated in the Fundamental Lemma 2.1. This means in particular that $\phi$ suffices Equation (2.1) and $2e < 2\left(\frac{1}{2} - d\right)$. Furthermore, we fix a standard triangulation $\xi(m; I^n)$ and an enumeration $x_1, \ldots, x_N$ of its vertices. Note $N = (m + 1)^n$. Recall that for every simplex $S$ in the standard triangulation $\xi(m; I^n)$ we canonically have a $(n + 1)$-tupel $k = (k_0, \ldots, k_n)$ contained in the set {$1, \ldots, N$}$^{n+1}$ such that the simplex $S$ has vertex order $(x_{k_0}, \ldots, x_{k_n})$.

### 2.1.1 Linear regularization

In this subsection we define the notion of a linear regularization. Roughly speaking, a linear regularization is a map from $I^n$ to $\mathbb{R}^n$ that maps any simplex of $\xi(m; I^n)$ into the convex hull of points in $\mathbb{R}^n$ that have been associated to the vertices of the simplex.

Recall that we already fixed a standard triangulation $\xi(m; I^n)$ and an enumeration $x_1, \ldots, x_N$ of its vertices. For each point $y = (y_1, \ldots, y_N)$ in $\mathbb{R}^{n\cdot N}$ we want to define a map $\sigma_y: I^n \to \mathbb{R}^n$. We define the map $\sigma_y$ on each simplex of the fixed standard triangulation $\xi(m; I^n)$ separately. The assignment
of the map \( \sigma_y \) restricted to the simplex \( S \) in \( \xi(m; I^n) \) is given by

\[
x = \sum_{i=0}^{n} \lambda_i \cdot x_{k_i} \mapsto \sum_{i=0}^{n} \lambda_i \cdot y_{k_i}, \tag{2.2}
\]

where \( x \) is a point of \( S \) and \( (k_0, \ldots, k_n) \in \{1, \ldots, N\}^{n+1} \) is the \((n+1)\)-tupel such that the simplex \( S \) has vertex order \((x_{k_0}, \ldots, x_{k_n})\). Since \( \xi(m; I^n) \) is a triangulation of the unit cube \( I^n \) the map \( \sigma_y \) is well defined.

**Definition 2.2** We call the map \( \sigma_y : I^n \to \mathbb{R}^n \), given by Equation (2.2) on each simplex \( S \), the linear regularization associated to the point \( y \) in \( \mathbb{R}^{n \cdot N} \).

Note that the map

\[
\| \cdot \|_* : \mathbb{R}^{n \cdot N} \to \mathbb{R},
\]

\[
(y_1, \ldots, y_N) \mapsto \max_{i \in \{1, \ldots, N\}} \|y_i\|_{\mathbb{R}^n} \tag{2.3}
\]

is a norm on \( \mathbb{R}^{n \cdot N} \). We use this norm to make the notation less unwieldy.

**Lemma 2.3** The map \( \sigma_y \) is continuous. Furthermore, there exists for every simplex \( S \) contained in \( \xi(m; I^n) \) a constant \( C_S > 0 \), such that for all \( u, v \in S \):

\[
\| \sigma_y(u) - \sigma_y(v) \|_{\mathbb{R}^n} \leq C_S \| y \|_* \| u - v \|_{\mathbb{R}^n}. \tag{2.4}
\]

**Proof** Let the simplex \( S \) in \( \xi(m; I^n) \) be arbitrary. Let \((x_{k_0}, \ldots, x_{k_n})\) be the vertex order of the simplex \( S \). Denote with \( \beta \) the barycentric coordinates of \( S \). The map

\[
p : \Delta^n \to \mathbb{R}^n,
\]

\[
(\lambda_0, \ldots, \lambda_n) \mapsto \sum_{j=0}^{n} \lambda_j y_{k_j}
\]

is a polynomial in each component; thus, the map \( p \) is in particular continuous. Hence on the simplex \( S \) the map \( \sigma_y = p \circ \beta^{-1} \) is continuous, as a composition of continuous maps. Since all simplices \( S \) contained in the standard triangulation \( \xi(m; I^n) \) are closed and their union is equal to \( I^n \), we can apply the Pasting Lemma, see A.11, and deduce that \( \sigma_y \) is continuous.

Now we prove the estimate (2.4). As \( S \) is a simplex, we know that the points \( x_{k_1} - x_{k_0}, \ldots, x_{k_n} - x_{k_0} \) are linearly independent and form a basis \( B \) of \( \mathbb{R}^n \). Let \( \| \cdot \|_1 \) be the 1-norm with respect to the basis \( B \). Recall that on the space \( \mathbb{R}^n \) all norms are equivalent, for this reason there exists a constant \( C_2 > 0 \) such that \( \| \cdot \|_1 \leq C_2 \| \cdot \|_{\mathbb{R}^n} \). Now let the points \( u, v \) in the simplex \( S \) be
2.1. Fundamental Lemma

arbitrary. Denote with \( \lambda_0, \ldots, \lambda_n \) the barycentric coordinates of \( u \) and with \( \lambda'_0, \ldots, \lambda'_n \) the barycentric coordinates of \( v \). We can conclude

\[
\| \sigma_y(u) - \sigma_y(v) \|_{\mathbb{R}^n} = \left\| \sum_{j=1}^{n} (\lambda_j - \lambda'_j)(y_{k_j} - y_{k'_j}) \right\|_{\mathbb{R}^n} \leq 2 \cdot \|v\| \sum_{j=1}^{n} |\lambda_j - \lambda'_j| \leq \|v\| C_S \|u - v\|_{\mathbb{R}^n},
\]

(2.5)

since the coordinates of \( u - v \) with respect to \( B \) are \( (\lambda_1 - \lambda'_1, \ldots, \lambda_n - \lambda'_n) \). \( \blacksquare \)

2.1.2 Non collapsed sets and points

From now on we fix an enumeration \( S_1, \ldots, S_{n! \cdot m^n} \) of the simplices of the standard triangulation \( \xi(m; I^n) \). Recall that we have already fixed an enumeration \( x_1, \ldots, x_N \) of the set of vertices of \( \xi(m; I^n) \). Let \( k^i = (k^i_0, \ldots, k^i_n) \) in \( (1, \ldots, N)^{n+1} \) be the \( (n+1) \)-tupel such that the \( i \)th simplex of the enumeration has the vertex order \( (x_{k^i_0}, \ldots, x_{k^i_n}) \).

For each \( i \) contained in the set \( \{1, \ldots, n! \cdot m^n\} \) let \( d_{k^i} : \mathbb{R}^{n \cdot N} \rightarrow \mathbb{R} \) be the map that assigns to each point \( y = (y_1, \ldots, y_N) \) in the set \( \mathbb{R}^{n \cdot N} \) the determinant of the matrix that has the points \( y_{k^i_0} - y_{k^i_1}, \ldots, y_{k^i_n} - y_{k^i_0} \) (in this order) as columns. Note that the real number \( d_{k^i} (y_1, \ldots, y_N) \) is exactly equal to zero if the points \( y_{k^i_1} - y_{k^i_0}, \ldots, y_{k^i_n} - y_{k^i_0} \) are linearly dependent. So, if \( d_{k^i} (y_1, \ldots, y_N) \) is equal to zero, then the convex hull of the points \( y_{k^i_0}, \ldots, y_{k^i_n} \) is not an \( n \)-dimensional simplex.

**Definition 2.4** We call

\[
M_{\xi} = \mathbb{R}^{n \cdot N} \setminus \bigcup_{i=1}^{n! \cdot m^n} d_{k^i}^{-1}(0)
\]

the set of non collapsed maps associated to \( \xi(m; I^n) \).

**Lemma 2.5** The set \( M_{\xi} \) of non collapsed maps associated to \( \xi(m; I^n) \) is open and dense in \( \mathbb{R}^{n \cdot N} \).

**Proof** Let the simplex \( S \in \xi(m; I^n) \) and the point \( y \) contained in \( \mathbb{R}^{n \cdot N} \) be arbitrary. Furthermore, let \( k = (k_0, \ldots, k_n) \) contained in the set \( \{1, \ldots, N\}^{n+1} \) be the \( (n+1) \)-tupel such that the simplex \( S \) has the vertex order \( (x_{k_0}, \ldots, x_{k_n}) \).

Clearly, the set \( d_{k}^{-1}((0, \infty)) \) is non-empty. We take a point \( z \) in the non-empty set \( d_{k}^{-1}((0, \infty)) \). Consider the map

\[
\gamma : [0, 1] \rightarrow \mathbb{R}
\]

\[
t \mapsto d_k(ty + (1-t)z).
\]
The map \( \gamma \) is a polynomial and on account of \( \gamma(0) \neq 0 \) we may deduce that \( \gamma \) is not the zero polynomial. But, a non zero polynomial has only finitely many zeros, hence for each open neighborhood \( U_y \) of \( y \) there must exist points that are contained in the intersection of the set \( \mathbb{R}^{n-N} \setminus d_k^{-1}(0) \) and the set \( U_y \). This is immediate, since the intersection \( W \) of the line segment \([z,y]\) and the open neighborhood \( U_y \) of the point \( y \) is an uncountable set; therefore, not all the images of the points contained in the intersection \( W \) under the map \( d_k \) can be equal to zero, for the map \( \gamma \) has only finitely many zeros. We have shown that the open set \( \mathbb{R}^{n-N} \setminus d_k^{-1}(0) \) is dense in \( \mathbb{R}^{n-N} \). Consequently, using Lemma A.3, we can conclude the proof by noticing that the set \( M_{\xi} \) must be dense and open, as it is the finite intersection of dense and open sets.

**Definition 2.6** Let \( S \) be a simplex in \( \xi(m;I^n) \) and \( y \) be a point in \( \mathbb{R}^{n-N} \), where \( N \) is as usual the number of vertices of the standard triangulation \( \xi(m;I^n) \). We use the notation \( S^y \) for the set \( \sigma_y(S) \). We call \( S^y \) the image of \( S \) under the linear regularization \( \sigma_y \).

By construction, the image of \( S \) under the linear regularization \( \sigma_y \), i.e. the set \( S^y \), is equal to the convex hull of the points \( y_{k_0}, \ldots, y_{k_\alpha} \). Recall that \((x_{k_0}, \ldots, x_{k_\alpha})\) is the vertex order of the simplex \( S \in \xi(m;I^n) \). If the point \( y \) happens to lie in the set \( M_{\xi} \) of non collapsed maps associated to \( \xi(m;I^n) \), the sets \( S^y \), where the simplex \( S \) is contained in standard triangulation \( \xi(m;I^n) \), are \( n \)-dimensional simplices with vertex sets \( V_{S^y} = \{y_{k_0}, \ldots, y_{k_\alpha}\} \). The fact that for every \( S \) in \( \xi(m;I^n) \) the image \( S^y \) is an \( n \)-dimensional simplex if \( y \) is contained in \( M_{\xi} \) is why we call \( M_{\xi} \) the set of non collapsed maps associated to \( \xi(m;I^n) \). Consequently, if the set \( S^y \) is not an \( n \)-dimensional simplex, we call it collapsed. Note that this can only happen if \( y \notin M_{\xi} \).

**Definition 2.7 (Non collapsed set)** Let \( y \) be a point in \( \mathbb{R}^{n-N} \) and \( S \) be a simplex of the standard triangulation \( \xi(m;I^n) \). The set \( S^y \) is called non collapsed if it is an \( n \)-dimensional simplex.

It is natural that if the simplex \( S \) in \( \xi(m;I^n) \) has vertex order \((x_{k_0}, \ldots, x_{k_\alpha})\), to calculate the sign of \( S^y \) with respect to the vertex order \((y_{k_0}, \ldots, y_{k_\alpha})\). Recall that the sign of \( S^y \) is equal to the signum of the determinant of the matrix that has the points \( y_{k_0} - y_{k_0}, \ldots, y_{k_\alpha} - y_{k_\alpha} \) (in this order) as columns, see Definition 1.10. In the proof of Lemma 2.14 we need the notion of associated sets.

**Definition 2.8 (Associated sets)** Let \( y, z \) be points in \( \mathbb{R}^{n-N} \) and \( S_i, S_j \) be simplices in \( \xi(m;I^n) \). We call the sets \( S_i^y \) and \( S_j^y \) associated if \( i \) is equal to \( j \).

As usual let \( \mathcal{F}_e \) denote the set of main edges of the standard triangulation...
ξ(m; I^n). Let y be a point in \( \mathbb{R}^{n-N} \). We define the set

\[
\mu_y = \left( \mathbb{R}^n \setminus \bigcup_{F \in \mathcal{F}^c} \sigma_y(F) \right) \cap \mathcal{K}_e^c.
\] (2.6)

The set \( \mu_y \) consists of those points in \( \mathcal{K}_e^c \) that are not contained in the image of a main edge.

**Lemma 2.9** The set \( \mu_y \) is open and connected in the subspace topology of \( \mathcal{K}_e^c \).

**Proof** Every main edge is a closed subset of a plane of codimension strictly greater than one. The notion of a plane is defined in Definition A.4. The image of this plane under the map \( \sigma_y \) is again a plane of codimension strictly greater than one. Let \( F \) denote the complement of the union of the images of theses planes under the map \( \sigma_y \). The set \( \mathcal{K}_e^c \) is convex and not a measure zero set. By considering the first part of the proof of Lemma A.6 closely we see that all we need to make the proof work is that \( \mathbb{R}^n \) is convex and not a measure zero set; as a result, Lemma A.6 can be adapted to conclude that \( \mathcal{K}_e^c \) intersected with \( F \) is path connected. Furthermore, we see that the set \( \mathcal{K}_e^c \cap F \) is open and dense in the subspace topology of \( \mathcal{K}_e^c \). Recall that the set \( \mu_y \) is the complement in \( \mathcal{K}_e^c \) of the image of the union of the main edges under the map \( \sigma_y \). We obtain the set \( \mu_y \) by adding open sets to the set \( F \cap \mathcal{K}_e^c \). Hence \( \mu_y \) is open and since \( F \cap \mathcal{K}_e^c \subset \mu_y \subset \mathcal{K}_e^c \cap \mathcal{K}_e^c \), we deduce that \( \mu_y \) is connected, as it is contained in closure of a connected set. \( \blacksquare \)

**Definition 2.10 (Non collapsed point)** Let \( y \) be a point in \( \mathbb{R}^{n-N} \) and the point \( z \) be contained in \( \mu_y \). The point \( z \) is called non collapsed if the set \( \sigma_y^{-1}(z) \) contains no points that lie on a main face of the standard triangulation \( \xi(m; I^n) \).

Conversely, if a point \( z \) in \( \mu_y \) happens to lie in the image of a main face under the map \( \sigma_y \), the point \( z \) is said to be collapsed. Observe that every main face of \( \xi(m; I^n) \) is a set of measure zero in \( I^n \), as it is contained in an \((n-1)\)-dimensional hyperplane and all planes of codimension greater than or equal to one have measure zero, see Lemma A.5. The map \( \sigma_y \) is Lipschitz continuous on each simplex of \( \xi(m; I^n) \), see Lemma 2.3, therefore the image of every main face is also a set of measure zero. Hence the union of these images is also of measure zero. Due to the fact that \( \mathcal{K}_e^c \) has a non zero measure, there must exist points in \( \mathcal{K}_e^c \) that are not contained in the union of the images of the main faces. To put this differently, the subset of \( \mu_y \) that consists of the non collapsed points is non-empty. For later references we state this as a lemma.

**Lemma 2.11** For all \( y \in \mathbb{R}^{n-N} \) the subset of non collapsed points in \( \mu_y \) is non-empty.
2.1. Fundamental Lemma

In the next lemma we state the crucial observation that every non collapsed point in \( \mu_y \) cannot be contained in a collapsed image set of a simplex under a linear regularization.

**Lemma 2.12** Suppose \( y \in \mathbb{R}^{n-N} \) and let \( z \in \mu_y \) be a non collapsed point. Furthermore, let \( S \in \xi(m, I^n) \) be arbitrary. If \( z \in S^y \), then the set \( S^y \) is non collapsed.

**Proof** Assume, for the sake of a contradiction, that the point \( z \) in \( \mu_y \) is contained in a collapsed image simplex \( S^y \) with vertex order \( (y_{k_0}, \ldots, y_{k_n}) \). On one hand, since \( z \) is non collapsed, we know that in every expression

\[
z = \sum_{i=0}^{n} \lambda_i y_{k_i}, \quad \text{where} \quad \sum_{i=0}^{n} \lambda_i = 1,
\]  

(2.7)

all the \( \lambda_i \)'s must be positive. Otherwise \( z \) would be a collapsed point. On the other hand, on account of Theorem 1.8, we can express \( z \) as

\[
z = \sum_{i=0}^{n} \lambda'_i y_{k_i},
\]  

(2.8)

where the \( \lambda'_i \)'s sum up to one, as desired, but at least one of them is equal to zero. This is possible due to the fact that \( S^y \) is collapsed and therefore not an \( n \)-dimensional simplex. This contradiction concludes the proof. \( \blacksquare \)

2.1.3 Degree modulo 2

We have seen in Lemma 2.11 that for each \( y \) in \( \mathbb{R}^{n-N} \) the set \( \mu_y \) contains non collapsed points. Furthermore, every non collapsed point in \( \mu_y \) is only contained, if at all, in those images \( S^y \) that happen to be \( n \)-dimensional simplices. Recall that we canonically assign to each \( S^y \) a sign, induced by the sign of the simplex \( S \). Non collapsed points are therefore somewhat special for they always exist and are only covered, if at all, by those sets of the \( S^y \)'s that are simplices. These observations serve as motivation for the next definition.

**Definition 2.13** We define for every point \( y \) in \( \mathbb{R}^{n-N} \) and every non collapsed point \( z \) contained in \( \mu_y \) the non negative integers \( p^+(z), p^-(z) \). Denote by \( p^+(z) \geq 0 \) the number of those sets of \( S^y_1, \ldots, S^y_n \) that are positively signed simplices and contain \( z \). Analogously, let \( p^-(z) \geq 0 \) denote the number of these sets that contain \( z \) and are negatively signed.

Note that the non negative integers \( p^+(z), p^-(z) \) depend a priori on the non collapsed \( z \) in \( \mu_y \) and on \( y \) in \( \mathbb{R}^{n-N} \) as well. In the progress of this subsection we see that if \( y \in \mathbb{R}^{n-N} \) lies in a certain subset of \( \mathbb{R}^{n-N} \) and \( z \) is a non collapsed point in \( \mu_y \), the number

\[
p^+(z) + p^-(z) \mod(2)
\]  

(2.9)
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does not depend on the choice of \( z \). Hence for every \( y \) contained in a certain subset of \( \mathbb{R}^{n \cdot N} \) Equation (2.9) produces a unique number, which we call \( n(y) \). Taking this for granted, observe that we constructed a map

\[
n: \text{dom}(n) \to \{0, 1\}
\]

\[
y \mapsto n(y).
\]

(2.10)

We call the number \( n(y) \) degree modulo 2 of the point \( y \) in \( \text{dom}(n) \subset \mathbb{R}^{n \cdot N} \).

In the remaining part of this subsection we deal with the question whether the degree modulo 2 is well defined and define the set \( \text{dom}(n) \) properly. First, we start with the construction of \( \text{dom}(n) \). Recall that the cube \( K_e \) has edge length \( 2e \) strictly smaller than 1. Moreover, we have fixed a triangulation \( \xi(m; I^n) \) and an enumeration \( x_1, \ldots, x_N \) of the elements of the set of vertices. We need that the number \( n(y) \) is independent of the non collapsed point in \( \mu_y \), we use for its computation. It turns out to be useful to restrict ourselves to points \( y \) that are near to the vertices of \( \xi(m; I^n) \). Let \( y \) be a point in \( \mathbb{R}^{n \cdot N} \) such that each component of \( y \) has distance smaller than \( \frac{1}{2} - e \) from the corresponding component of \( x = (x_1, \ldots, x_N) \). Such a point \( y \) is near to the point \( x \) with respect to the norm \( \| \cdot \|_* \). Furthermore, note that for such a point \( y \) the images of the boundary faces, i.e. main faces lying on the topological boundary of \( I^n \), under the map \( \sigma_y \) do not intersect with the cube \( K_e \). Indeed this property suffices to characterize \( \text{dom}(n) \). We define

\[
\text{dom}(n) := \left\{ y = (y_1, \ldots, y_N) \in \mathbb{R}^{n \cdot N} : \forall i \in \{1, \ldots, N\} : \| y_i - x_i \| < \frac{1}{2} - e \right\}.
\]

The set \( \text{dom}(n) \) is equal to the \( \| \cdot \|_* \)-ball around \( x \) with radius \( \frac{1}{2} - e \). Note that \( \text{dom}(n) \) depends on the set of vertices (and its enumeration) of \( \xi(m; I^n) \). Nevertheless, we omit this dependence in the notation for the sake of clarity.

Next, we deal with the well-definedness of the degree modulo 2. We do this in two independent steps. At first, we fix a non collapsed point \( z \) in \( \mu_y \) and observe that for arbitrary close neighborhoods of \( y \) we find a point \( y' \) contained in it such that the point \( z \) remains non collapsed with respect to \( y' \) and even the numbers (2.11) of \( y \) and \( y' \), both calculated with respect to \( z \), coincide.

**Lemma 2.14** Suppose the point \( y \) is contained in \( \text{dom}(n) \) and let \( z \) in \( \mu_y \) be non collapsed. Then there exists an \( \varepsilon > 0 \) such that the set \( B^{\| \cdot \|_*}_{\varepsilon} (y) \cap \text{dom}(n) \cap M_\xi \) is non-empty and if we calculate the number

\[
p^+(z) + p^-(z) \mod(2),
\]

(2.11)

once with respect to \( y \) and then with respect to some point \( y' \) contained in the set \( B^{\| \cdot \|_*}_{\varepsilon} (y) \cap \text{dom}(n) \cap M_\xi \), then these two numbers are well defined and coincide.
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Proof This proof is merely an application of the Technical Lemma, see Lemma A.16. Let \( y \) in \( \text{dom}(n) \) and the non collapsed \( z \) in \( \mu_y \) be arbitrary. Let \( F^y \) denote the image of an arbitrary main face of \( \xi(m; I^n) \) under the linear regularization \( \sigma_y \). Let \( \mathcal{F} \) be the set consisting of all \( F^y \). Due to the fact that all elements of \( \mathcal{F} \) are compact and the fact that the point \( z \) is non collapsed we may deduce that the number

\[
\varepsilon := \min_{F^y \in \mathcal{F}} \text{dist}(z, F^y) \quad (2.12)
\]

is strictly greater than zero, since the non collapsed \( z \) cannot lie in a collapsed set \( S^y \), see Lemma 2.12, and by the compactness of all the elements of the set \( \mathcal{F} \) this is the only possibility for the distances in Equation (2.12) to get zero. Here we have used Theorem 1.8, which implies that a collapsed simplex is equal to one of its main faces. Using Lemma A.10 we can deduce that the set \( M_\xi \cap \text{dom}(n) \) is dense in \( \text{dom}(n) \) with respect to the subspace topology induced by the standard topology. Since on \( \mathbb{R}^{n \cdot N} \) all norms are equivalent, there exists at least one \( y' \) contained in \( M_\xi \cap \text{dom}(n) \cap \text{M}_\xi \) arbitrarily.

Now we show that \( z \) is contained in \( \mu_{y'} \) and furthermore that \( z \) is non collapsed. Thus, the number (2.11) calculated with respect to \( y' \) is well defined. Choose \( S \) in \( \xi(m; I^n) \) arbitrarily. Let \( S^{y'} \) be the associated set to \( S^y \), see Definition 2.8. If we can show that \( z \) cannot be contained on the boundary of \( S^{y'} \), we have shown that \( z \in \mu_{y'} \) and that it is non collapsed. This is the case since if \( z \) is collapsed, it is contained in the image of a main face of \( \xi(m; I^n) \) under \( \sigma_{y'} \), and therefore also on the boundary of some \( S^{y'} \). We make a distinction of cases on the sign of the set \( S^y \), which can either be zero or non zero.

At first, we assume that the sign of \( S^y \) is equal to zero. Note that the sets \( \Lambda := V_{S^y} \) and \( \Lambda' := V_{S^{y'}} \) satisfy the hypotheses of the Lemma A.17 for \( \varepsilon \). Additionally, we know by the construction of \( \varepsilon \) that the point \( z \) has a distance greater than or equal to \( \varepsilon \) from every main face of \( S^y \). Hence we can apply Lemma A.17 and deduce that \( z \) is neither contained in \( S^y \), which was evident before, and more importantly that the point \( z \) is not contained in \( S^{y'} \). Hence in particular the point \( z \) is not contained in the image of a main face of \( S \) under \( \sigma_{y'} \).

Next, let the sign of \( S^y \) be non zero. Note that the sets \( S^y \) and \( S^{y'} \) satisfy the hypotheses of the Technical Lemma A.16 for \( \varepsilon \). As before, we know from the construction of \( \varepsilon \) that the point \( z \) has a distance greater than or equal to \( \varepsilon \) from every main face of \( S^y \). Using both conclusions of the Technical Lemma we see that the point \( z \) is, if at all, only contained in the interior of the simplex \( S^{y'} \) and not on its boundary. Hence \( z \) is non collapsed with respect to \( \sigma_{y'} \).
Now we are left to show that if we calculate the number (2.11) with regard to \( y \) and \( y' \) respectively the result remains the same. If \( S^y \) is collapsed, we know already that \( z \) can neither be contained in \( S^y \) nor in its associated set \( S^{y'} \). Thus, we only have to consider the pairs \( (S^y, S^{y'}) \), where \( S^y \) is non collapsed. Let \( v \) be an arbitrary vertex of a non collapsed set \( S^y \), so there exists exactly one main face \( F_v \) of \( S^y \) that does not contain \( v \). Note that \( F_v \) is contained in a \((n-1)\)-dimensional plane of \( \mathbb{R}^n \). See Definition A.4 for the notion of a plane. Let the point \( v' \) contained in the vertex set of \( S^{y'} \) denote the associated vertex to \( v \). By construction of \( \varepsilon \) the points \( v \) and \( v' \) lie on the same side of \( F_v \). So since \( v \) was arbitrary, we deduce that if \( S^y \) is non collapsed, then \( S^y \) and its associated set \( S^{y'} \) have the same sign. Now we may conclude the result, for the Technical Lemma guarantees us that the point \( z \) is either contained in both of the sets \( S^y, S^{y'} \) or in none of them. ■

**Lemma 2.15** Let \( y \) be a point in \( \text{dom}(n) \cap M_\xi \). Then for all \( z \in \mu_y \), which are non collapsed, the number given by

\[
p^+(z) + p^-(z) \mod(2)
\]

(2.13)

is constant.

**Proof** Let \( z_1, z_2 \) in \( \mu_y \) be two non collapsed points. Lemma A.8 and Lemma 2.9 imply that \( z_1, z_2 \) can be connected by a piecewise linear continuous map \( \gamma : [0,1] \rightarrow \mu_y \), having only finitely many break points. By the use of mollifiers we can “smooth out” the break points of \( \gamma \); therefore, we may assume that \( \gamma \) is smooth. Furthermore, we are able to redefine \( \gamma : [0,1] \rightarrow \mu_y \), if necessary, in a way such that the endpoints stay unchanged and the trace of \( \gamma : [0,1] \rightarrow \mu_y \) does only contain finitely many points that lie on images of main faces. The existence of a redefined map \( \gamma' : [0,1] \rightarrow \mu_y \) with these properties is an immediate consequence of the Transversality Theorem, see Theorem A.15. Choose \( 3\varepsilon > 0 \) to be strictly smaller than the distance between the trace of \( \gamma \) and the union of the images of the main edges under \( \sigma_y \); and the distance of the trace of \( \gamma \) and the boundary of the cube \( K_\varepsilon \). We set \( (t,s) \in [0,1] \times B_\varepsilon(0) \rightarrow F(t,s) = \gamma(t) + s \). Clearly: \( F(t,s) \in \mu_y \) for all \( (t,s) \). Let \( Z' \) denote the image of an arbitrary main face \( F \) under the map \( \sigma_y \). The set \( Z' \) minus the closure of the \( \varepsilon \) neighborhood of the union of the images of the main edges that lie on the main face \( F \) is a boundaryless submanifold of \( \mathbb{R}^n \) of dimension \((n-1)\). Let us denote this new set by \( Z \). Now, \( F \) and the restriction of \( F \) to the boundary of \([0,1] \times B_\varepsilon(0) \) are smooth and transversal to \( Z \). See Definition A.14 for the notion of transversality. Hence we can apply the Transversality Theorem A.15 to deduce that for almost every \( s \) in \( B_\varepsilon(0) \) the map \( t \mapsto \gamma_s(t) = \gamma(t) + s \) is transversal to \( Z \). We see by construction of \( \gamma \) and as a result of the Transversality Theorem that the trace of the map \( \gamma_s \) does only contain finitely many images of the main face \( F \) under the map \( \sigma_y \). Since this holds for almost every \( s \) in \( B_\varepsilon(0) \) and on account of the fact
that we have chosen the main face $F$ arbitrarily, we may choose an $s$ in $B_ε(0)$ such that the trace of map $t \mapsto γ_ε(t) = γ(t) + s$ does only contain finitely many points that lie on images of main faces. Let $γ' : [0, 1] → µ_y$ be the continuous map whose trace is equal to the union of the trace of $γ_ε$ and the line segment $[z_1, γ_ε(0)]$ and the line segment $[γ_ε(1), z_2]$. We see that the map $γ'$ is the desired redefinition of the map $γ$. Hence, we may assume from now on that the trace of the map $γ$ does only contain finitely many points that lie on images of main faces. The linear regularization $σ_y$ does not map the boundary faces of the standard triangulation $ξ(ι_m; 1^n)$ into the cube $K_ε$. Hence, the trace of $γ$ does not contain points that lie on images of boundary faces and by definition of the set $µ_y$ neither points that lie on images of main edges.

Let $t_1 < \cdots < t_m$ in $[0, 1]$ be the preimages of those points in the trace of $γ$ that lie on images of main faces. Note that the number $(2.13)$ is not defined for the points $γ(t_1), \ldots, γ(t_m)$. Define $t_0 := 0$ and $t_{m+1} := 1$. Clearly $t_1 \neq t_0, t_m \neq t_{m+1}$ and the number $(2.13)$ is constant on the sets $γ([t_0, t_1]), γ([t_1, t_2]), \ldots, γ([t_{m-1}, t_m]), γ([t_m, t_{m+1}])$, for $(2.13)$ can only change along $γ$ if the image of a main face is passed.

Since the point $y$ is contained in the set $M_ξ$ of non collapsed maps associated to the standard triangulation $ξ(ι_m; 1^n)$, the image of every main face, that is not a boundary face, is the common face of two distinct simplices $S^y_1, S^y_2$ and lies in a unique $(n−1)$-dimensional hyperplane of $R^n$. The notion of a $(n−1)$-dimensional hyperplane is defined in Definition A.4. Let $l$ be an arbitrary element of the set $\{1, \ldots, m\}$. The point $γ(t_l)$ is contained in $k \geq 1$ many images $F_1, \ldots, F_k$ of main faces. Denote with $H_l$ the unique $(n−1)$-dimensional hyperplane that contains $F_l$. For each $l$ in the set $\{1, \ldots, k\}$ we have that the set $R^n \setminus H_l$ has two open connected components $H^+_l, H^-_l$. Thus, the trace $γ([t_{l−1}, t_l])$ is either completely contained in $H^+_l$ or $H^-_l$, for every $l ∈ \{1, \ldots, k\}$. The same applies for the trace of $γ((t_l, t_{l+1}))$. Let $S^l_1, S^l_2 \in \{S^y_1, \ldots, S^y_m\}$ be the unique simplices that have the face $F_l$ in common. Note that the elements of $C := \{S^l_1, S^l_2\}_{l∈\{1,\ldots,k\}}$ are pairwise distinct.

Let $l ∈ \{1, \ldots, k\}$ be arbitrary. We have the following two possibilities:

1. the interior of the simplices $S^l_1, S^l_2$ lie on the same side of $H_l$ or
2. the interior of the simplices $S^l_1, S^l_2$ lie on different sides of $H_l$.

To conclude the result we are obligated to conduct an utterly boring distinction of cases. Before we begin with this distinction of cases, note that for each set $S^y$ not contained in $C := \{S^l_1, S^l_2\}_{l∈\{1,\ldots,k\}}$ we have

$$γ((t_{l−1}, t_l)) ∈ S^y \iff γ((t_l, t_{l+1})) ∈ S^y,$$  

(2.14)

since by definition of $C$ no face of $S^y$ can be touched or crossed on the line segment $[t_{l−1}, t_{l+1}]$. 

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2.1. Fundamental Lemma

Assume now that the first case is valid and that the traces $\gamma((t_{i-1}, t_i))$ and $\gamma((t_i, t_{i+1}))$ are both contained in the same connected component of $\mathbb{R}^n \setminus H_1$. The fact that $H_1$ is a $(n - 1)$-dimensional hyperplane implies

$$\gamma((t_{i-1}, t_i)) \in (S^1_1 \cap S^1_2) \iff \gamma((t_i, t_{i+1})) \in (S^1_1 \cap S^1_2). \quad (2.15)$$

If the traces $\gamma((t_{i-1}, t_i))$ and $\gamma((t_i, t_{i+1}))$ are contained in different connected components of $\mathbb{R}^n \setminus H_1$, we have the equivalence

$$\gamma((t_{i-1}, t_i)) \notin (S^1_1 \cap S^1_2) \iff \gamma((t_i, t_{i+1})) \in (S^1_1 \cap S^1_2). \quad (2.16)$$

Next, suppose we are in the second case and assume that the traces $\gamma((t_{i-1}, t_i))$ and $\gamma((t_i, t_{i+1}))$ are both contained in the same connected component of $\mathbb{R}^n \setminus H_1$. We have for every $m \in \{1, 2\}$

$$\gamma((t_{i-1}, t_i)) \in S^1_m \iff \gamma((t_i, t_{i+1})) \in S^1_m. \quad (2.17)$$

And finally, if $\gamma((t_{i-1}, t_i))$ and $\gamma((t_i, t_{i+1}))$ are contained in different connected components of $\mathbb{R}^n \setminus H_1$, we have for every $m \in \{1, 2\}$ the equivalence

$$\gamma((t_{i-1}, t_i)) \in S^1_m \iff \gamma((t_i, t_{i+1})) \notin S^1_m. \quad (2.18)$$

Since $l$ contained in $\{1, \ldots, k\}$ was arbitrary, the Equations (2.14), (2.15), (2.16), (2.17), (2.18) imply that the numbers $p^+(u) + p^-(u)$ and $p^+(u) + p^-(u)$ for the points $u$ in $\gamma((t_{i-1}, t_i))$ and $v$ in $\gamma((t_1, t_{i+1}))$ can only differ by a multiple of 2. Hence we obtain that (2.13) is constant on $\gamma((t_{i-1}, t_{i+1})) \setminus H_1$.

We have chosen $i$ contained in $\{0, \ldots, m\}$ arbitrarily; thus, we can conclude that (2.13) is constant on $\gamma((t_0, t_{m+1}))$. Since the number (2.13) is constant on $\gamma((t_0, t_1))$ and $\gamma((t_m, t_{m+1}))$, we may obtain that the numbers (2.13) of the two points $z_1, z_2$ are equal, as desired. ■

As a result of Lemma 2.15 the degree modulo 2 of all points contained in the intersection of $M_\xi$ and $\text{dom}(n)$ is well defined. By consulting Lemma 2.14 we know that every point $y$ of $\text{dom}(n)$ has a $\| \cdot \|_\gamma$-neighborhood on which the numbers (2.11) of all points that are contained additionally in the set $M_\xi$ coincide with the number (2.11) of the point $y$, if they are calculated both with respect to the same non collapsed point. To put this differently, for each non collapsed $z$ in $\mu_y$ the number (2.11) of $y$ is equal to the degree mod 2 of the points contained in the intersection of the $\| \cdot \|_\gamma$-neighborhood of $y$ and the set $M_\xi$. Hence two non collapsed points $z_1$ and $z_2$ in $\mu_y$ that lead to different numbers (2.11) cannot exist, for the density of $M_\xi$ in $\text{dom}(n)$, consider Lemma 2.5 and Lemma A.10; the well definedness of the degree modulo 2 of all points contained in the intersection of $M_\xi$ and $\text{dom}(n)$, see Lemma 2.15; and the consequences of Lemma 2.14, as mentioned in the previous two sentences; imply that there exists a point in $M_\xi \cap \text{dom}(n)$ having degree modulo two equal to both the numbers (2.11) of $z_1$ and $z_2$. Hence we have proven the subsequent Lemma.
Lemma 2.16 The degree modulo 2 given by Equation (2.10) is a well defined map.

Now we are able to show that the map $n$ is indeed constant.

Lemma 2.17 The map $n : \text{dom}(n) \to \{0, 1\}$ is a constant function with value 1.

Proof The point $x = (x_1, \ldots, x_n)$ is trivially contained in $\text{dom}(n)$. To begin, we show that $n(x)$ is equal to 1. Due to the fact that $\xi(m; I^n)$ is a triangulation of the unit cube and since the linear regularization $\sigma_x$ is the identity map from $I^n$ to $I^n$ we can deduce that a non collapsed point of $\mu_x$ is contained in exactly one of the simplices $S_1 \ldots, S_m$. Thus, $n(x) = 1$.

Let $x'$ be an arbitrary point in $\text{dom}(n)$. The set $\text{dom}(n)$, being a $\| \cdot \|_*$-ball with a certain radius centered at the point $x$, is convex. So the line segment $[x, x']$ is entirely contained in the set $\text{dom}(n)$. For every point $v$ on the line segment $[x, x']$ there exists an $\varepsilon_v > 0$, see Lemma 2.14, such that for all $w$ in $B_{\varepsilon} \| \cdot \|_*(v) \cap \text{dom}(n) \cap M_{\xi}$ the well defined numbers, see Lemma 2.16, $n(v)$ and $n(w)$ coincide. By shrinking $\varepsilon_v$, if necessary, we may assume that the $\| \cdot \|_*$-ball centered at $v$ with radius $\varepsilon_v$ is entirely contained in $\text{dom}(n)$. Now we cover the line segment $[x, x']$ with these $\| \cdot \|_*$-balls. The Lebesgue Number Lemma guarantees (cf. Lemma 27.5 in [8, p. 175-176]) the existence of a $\delta > 0$ such that every subset of the compact normed space $[x, x']$ (endowed with the norm $\| \cdot \|_*$) with diameter less than $\delta$ is contained in some member of the cover. Hence, using the fact that on finite dimensional vectorspaces norms are equivalent, there exists a $k \geq 1$ such that for every $j$ in $\{1, \ldots, k\}$ the function $n$ is constant on the set $[x + (x - x') \frac{j - 1}{k}, x' \frac{j}{k}]$. Since for every $j$ contained in $\{1, \ldots, k - 1\}$ the sets $[x + (x - x') \frac{j - 1}{k}, x' \frac{j}{k}]$ and $[x + (x - x') \frac{j}{k}, x' \frac{j + 1}{k}]$ intersect non trivially, we may conclude $n(x) = n(x')$. Hence $n$ is a constant function with value 1, as desired. \[\square\]

Lemma 2.17 is a really strong result, as it implies that the set $K_\varepsilon$ is contained in every image $\sigma_y(I^n)$, where $y$ in $\text{dom}(n)$. This can be seen as follows. There are two types of points in the set $\mu_y$, namely: collapsed points and non collapsed points. By definition, collapsed points are contained in the image of a main face under the map $\sigma_y$, therefore all collapsed points are contained in the set $\sigma_y(I^n)$. On the other hand, since for all non collapsed points $z$ in $\mu_y$ the number $p^z(z) + p^z(z)$ is not equal to zero, the point $z$ is contained in a simplex $S^*$ and therefore in particular in the image set $\sigma_y(I^n)$. We obtain that $\mu_y \subset \sigma_y(I^n)$. Since $\mu_y = K_\varepsilon$, this implies immediately $K_e \subset \sigma_y(I^n)$, for the set $\sigma_y(I^n)$ is closed.

2.1.4 Proof of the Fundamental Lemma

In this subsection we provide a proof of the Fundamental Lemma, stated in Lemma 2.1.
Proof Let $N_m = (m + 1)^n$ denote the number of vertices of the standard triangulation $\xi(m; I^n)$. For each $m \geq 1$ fix an enumeration $x_1, \ldots, x_{N_m}$ of the set of vertices of the standard triangulation $\xi(m; I^n)$. For every $m \geq 1$ define the point $y_m$ contained in the set $\mathbb{R}^n \cdot N_m$ as $y_m = (\varphi(x_1), \ldots, \varphi(x_{N_m}))$. It follows from the definition of $\varphi$, see Equation (2.1), that $y_m \in \text{dom}(n)$ for all $m \geq 1$. Note that $\varphi$ is uniformly continuous, for the space $I^n$ is compact. Hence we see that

$$\sigma y_m \Rightarrow \varphi. \tag{2.19}$$

The double arrow indicates that the convergence (2.19) is uniform. We know that the cube $K_e$ is contained in every $\sigma y_m(I^n)$, since $y_m \in \text{dom}(n)$ for every $m \geq 1$ and the map $n: \text{dom}(n) \to \{0, 1\}$ is constantly equal to one. Assume, for the sake of a contradiction, that there exists a point $v \in K_e$ that is not contained in $\varphi(I^n)$. Since $\varphi(I^n)$ is compact and $v \notin \varphi(I^n)$, we deduce that

$$\inf_{u \in \varphi(I^n)} \|u - v\|_{\mathbb{R}^n} > 0. \tag{2.20}$$

On account of the fact that $K_e$ is contained in every set $\sigma y_m(I^n)$ and because of the uniform convergence (2.19), we conclude that for $m$ large enough, there must exist a point $w \in I^n$, such that the distance between the points $v$ and $\varphi(w)$ is strictly smaller than (2.20). This is a contradiction. Thus, all points in $K_e$ are contained in $\varphi(I^n)$, as desired. $\blacksquare$

2.2 Proof of the Invariance of Dimension Theorem

Now we have everything on hand to prove Theorem 2.18 that allows us to deduce the Invariance of Dimension Theorem, see Theorem 2.19, as an immediate consequence. Let $B$ denote a subset of $\mathbb{R}^n$. Recall that a set $A \subset B$ is said to be nowhere dense in $B$, if for all non-empty open sets $U$ contained in the subspace topology of $B$ the set $U \setminus (\overline{A} \cap B)$ is non-empty. Equivalently, if we endow $B$ with the subspace topology the set $A \subset B$ is said to be nowhere dense in $B$ if the interior of the closure of the set $A$ is empty.

**Theorem 2.18** [2, p. 165] Let $D \subset \mathbb{R}^n$ be a non-empty open subset and let the map $\varphi: D \to \mathbb{R}^n$ be injective and continuous. Then for every point in the set $\varphi(D) \subset \mathbb{R}^n$ and every open set $V$ that contains this point there exists a non-empty open set that is entirely contained in the set $\varphi(D) \cap V$.

Note that Theorem 2.18 implies in particular that the image of an injective continuous map $\varphi$ which goes from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ is not nowhere dense in $\mathbb{R}^n$. Moreover, note that Theorem 2.18 does not tell us whether the set $\varphi(D)$ is open or not. We only have for every point $p$ in the set $\varphi(D)$ and every open neighborhood $V$ of this point the weaker statement that the intersection of $V$ and $\varphi(D)$ does contain a non-empty open set $B$. But, there is no need for the set $B$ to contain the point $p$. Nevertheless, the
stronger statement which claims that the set \( \varphi(D) \) is indeed open turns out to be true. This classical theorem, known in the literature as the *Invariance of Domain Theorem*, is due to L. E. J. Brouwer; cf. [3, p. 305-313]. As we dont use the full power of the Invariance of Domain Theorem we leave it with this remark and omit to provide a proof here. Now we establish a proof of the weaker Theorem 2.18.

**Proof** This proof closely follows Brouwer’s proof of Theorem 1 in his paper [2, p. 165].

Let \( v_0 \) be a point in the set \( \varphi(D) \) and let \( V \) be any open neighborhood of the point \( v_0 \). Let \( u_0 \) be the unique point in the set \( D \) such that \( \varphi(u_0) = v_0 \). Assume, for the sake of a contradiction, that for all non-empty open sets \( B \) in \( \mathbb{R}^n \) the set \( B \setminus V \cap \varphi(D) \) is non-empty. Denote with \( k \subset V \) a cube that contains \( v_0 \). Let \( 2s \) denote the edge length of the cube \( k \). Due to the fact that the map \( \varphi \) is continuous, we are able to find a cube \( K \subset D \) such that the point \( u_0 \) is in \( K \) and the image of \( K \) under \( \varphi \) is entirely contained in the interior of the cube \( k \). Let \( 2\varepsilon \) denote the edge length of the cube \( K \). By construction, the set \( \varphi(K) \) implies that for all \( \delta > 0 \) there exists an \( \varepsilon > 0 \) such that for all \( \varphi(u), \varphi(u') \in \varphi(K) \)

\[
\| \varphi(u) - \varphi(u') \|_{\mathbb{R}^n} < \varepsilon \Rightarrow \| u - u' \|_{\mathbb{R}^n} < \delta.
\]

(2.21)

Recall that \( e \) is half the edge length of the cube \( K \). Choose \( 0 < \varepsilon < e \) and \( \varepsilon < \varepsilon_\delta < \varepsilon \) accordingly such that Equation (2.21) holds. Fix \( m \geq 1 \) such that \( 2 \cdot \frac{2s \sqrt{m}}{\varepsilon} < \varepsilon_\delta \). We triangulate the cube \( k \) with the standard triangulation \( \xi(m; k) \). Let \( \mathcal{C} \) be the set of simplices of \( \xi(m; k) \) that contain at least one element of \( \varphi(K) \). Clearly, \( \mathcal{C} \) is non-empty. Let \( \{x_1, \ldots, x_M\} \) be the set that contains all vertices of the simplices in \( \mathcal{C} \). Denote with \( \mathcal{C}_{x_i} \) for all \( i \in \{1, \ldots, M\} \), the set that consists of all simplices in \( \mathcal{C} \) that have \( x_i \) as a vertex. Associate to \( x = (x_1, \ldots, x_M) \) the point \( y = (y_1, \ldots, y_M) \in K^M \), where for each \( i \) contained in the set \( \{1, \ldots, M\} \) we have

\[
\varphi(y_i) \in \bigcup_{S \in \mathcal{C}_{x_i}} S.
\]

(2.22)

As usual, let \( \sigma_y : \bigcup_{C \in \mathcal{C}} C \rightarrow \mathbb{R}^n \) be the linear regularization associated to the point \( y \). Note that Lemma 2.3 also holds for \( \sigma_y \). Now we define the map

\[
\psi : K \rightarrow K
\]

\[
x \mapsto \sigma_y(\varphi(x)).
\]

(2.23)

Note that since for each \( i \) in \( \{1, \ldots, M\} \) the point \( y_i \) is contained in \( K \), the map \( \psi \) is well defined. As a composition of continuous maps the map \( \psi \) is
2.2. Proof of the Invariance of Dimension Theorem

continuous. We have chosen \( m \) large enough such that:

\[
\forall v, v' \in \bigcup_{s \in e_s} S : \|v - v'\| < \varepsilon_e.
\]  (2.24)

Therefore, using Equation (2.21) - (2.24), we obtain

\[
\|x - \psi(x)\|_{\mathbb{R}^n} < \varepsilon \quad (\forall x \in K).
\]  (2.25)

Since the Fundamental Lemma 2.1 holds for arbitrary \( n \)-dimensional cubes, there exists a cube \( K_e \) concentric to \( K \) that is contained in the image of \( \psi(K) \). But \( \varphi(K) \) is compact and nowhere dense in \( k \) and therefore the image of \( \varphi(K) \) under the continuous map \( \sigma_y \) is nowhere dense in \( K \), see Lemma A.13. Hence we have obtained a contradiction. ■

Finally, we state and prove the Invariance of Dimension Theorem.

**Theorem 2.19 (Invariance of Dimension)** Let \( n \) and \( m \) be two distinct positive integers and let \( U \) be a non-empty open subset of \( \mathbb{R}^n \) and \( V \) be an open subset of \( \mathbb{R}^m \). Then there cannot exist a homeomorphism between \( U \) and \( V \).

**Proof** This proof is basically a reformulation of the proof of Theorem 2 in Brouwer’s paper [2, p. 165]. We may assume \( n > m \). Let \( U \) and \( V \) be as stated in the Theorem. Since \( U \) is open, we may choose a set \( C \) contained in \( U \) homeomorphic to a \( m \)-dimensional cube \( K \). Let \( \psi: K \to C \) denote the homeomorphism between \( K \) and \( C \). By construction, the set \( C \) is contained in a plane of codimension greater than or equal to one. Corollary A.7 implies that the set \( C \) is nowhere dense in \( U \), as it is contained in a \( m \)-dimensional plane. Assume that we are given a homeomorphism \( \varphi: U \to V \). The image under the map \( \varphi \) of the nowhere dense subset \( \psi(K^o) \) of \( U \) is nowhere dense in \( V \), this follows from the fact that the map \( \varphi \) is a homeomorphism, see Lemma A.13 for a complete proof. This is a contradiction to Theorem 2.18. Thus, as desired, there cannot exist such a homeomorphism \( \varphi \). ■

A few final remarks are in order. The proof of Theorem 2.19 is actually a proof of the fact that for any two distinct positive integers \( n > m \), any non-empty open subset \( U \) of \( \mathbb{R}^n \) and any open subset \( V \) of \( \mathbb{R}^m \) there cannot exist a injective continuous map between \( U \) and \( V \). The converse, i.e. the impossibility of the existence of an injective continuous map between \( U \) and \( V \) for \( m > n \), is not true. An easy counterexample is the inclusion map \( x \in \mathbb{R} \to (x,0) \in \mathbb{R}^2 \). But it can be shown that the image of an injective continuous map going from an open subset \( U \) of \( \mathbb{R}^n \) to an open subset \( V \) of \( \mathbb{R}^m \) is nowhere dense in \( V \), if \( n \) is strictly smaller than \( m \); cf. Theorem 3 in [2, p. 165].
Appendix A

Auxiliary results

A.1 Various facts from topology and analysis

Lemma A.1 Suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) is affine linear, i.e. \( f \) is of the form \( f(x) = Ax + b \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Let \( C \) be a convex set. Then the image set \( f(C) \) is convex as well.

Proof Let \( x' = Ax + b \) and \( y' = Ay + b \) be two points in \( f(C) \). Since \( C \) is convex, the line segment \([x, y]\) is entirely contained in \( C \). Hence we can conclude the proof by noticing that for all \( t \) in \([0, 1]\) the points \((1-t)x' + ty' = A((1-t)x + ty) + b\) are contained in \( f(C) \). ■

Theorem A.2 (Krein-Milman, easy version) Suppose \( K \subset \mathbb{R}^n \) is convex and compact. Let \( E \subset K \) be the set of its extremal points, see Definition 1.18. Assume \( E \) is finite. Then, \( \text{conv}(E) = K \).

Proof We use a more general version of the Krein-Milman Theorem to conclude our easy version. Let \( S \) denote the set of extremal points of a convex compact set \( T \), which is contained in a vector space over the reals. S. Lang shows in Theorem 1.5 in his book \([7, \text{p. 88}]\) that \( T \) is the smallest closed convex set containing all the elements of \( S \).

Note that we have seen in chapter one the equivalence of the Definitions 1.1 and 1.6 of the convex hull of a finite set. Since the set of extremal points \( E \) is assumed to be finite, this implies that the Definitions 1.1 and 1.6 of the convex hull of \( E \) yield the same set. Therefore, putting things together, \( K = \text{conv}(E) \), as desired. ■

Lemma A.3 The finite intersection of open and dense subsets of \( \mathbb{R}^n \) is again open and dense.

Proof Since openness is preserved under finite intersections, we are left to show that the finite intersection of open dense subsets of \( \mathbb{R}^n \) is dense. We
do this by induction. Assume \( \{U_i\}_{i=1}^N \) is a finite set consisting of open and dense subsets of \( \mathbb{R}^n \). Suppose that the intersection \( U \) of those sets is open and dense. Let \( U_{N+1} \) be an arbitrary open and dense subset of \( \mathbb{R}^n \). We have to show that \( U \cap U_{N+1} \) is dense. Let \( x \) be a point in \( \mathbb{R}^n \) and \( B \) an open neighborhood of \( x \). Due to the fact that \( U \) is dense, there exists a point \( x' \) contained in the intersection of \( U \) and \( B \). Since \( U \) and \( B \) are open, their intersection is open as well and we may use the density of the set \( U \cap U_{N+1} \) to obtain a point \( x'' \) in \( U_{N+1} \) contained in the intersection of \( U \) and \( B \). By construction, \( x'' \) is contained both in \( U \) and \( U_{N+1} \) and additionally in the open neighborhood \( B \) of \( x \). Hence \( U \cap U_{N+1} \) is dense, as desired. \( \blacksquare \)

**Definition A.4** The preimage of zero of an affine linear map \( f: \mathbb{R}^n \to \mathbb{R}^m \) with assignment \( f(x) = Ax + b \), where \( A \) has rank \( m \), is called plane of codimension \( m \). Besides, a plane of codimension of 1 is called \((n-1)\)-dimensional hyperplane.

Note that the rank assumption on \( A \) implies that \( m \leq n \).

**Lemma A.5** Every plane of codimension \( m \geq 1 \) is a set of measure zero.

**Proof** This is an immediate consequence of Sard’s Theorem; nonetheless, we give a proof from first principles. We may assume \( m < n \). Otherwise the claim follows immediately. Let \( H \) be a plane of codimension greater than or equal to one. We may write \( H = p + W \), where \( p \) is a point in \( \mathbb{R}^n \) and \( W \) is a \((n - m)\)-dimensional subspace of \( \mathbb{R}^n \). This is well-known from linear algebra. Fix a basis \( w_1, \ldots, w_{n-m} \) of \( W \). Let \( \Phi: \mathbb{R}^{n-m} \to W \) be the unique linear map such that for all \( i \) in \( \{1, \ldots, n-m\} \): \( \Phi(e_i) = w_i \). Here \( \{e_i\}_{i=1}^{n-m} \) is the standard basis of \( \mathbb{R}^{n-m} \). Let \( i: \mathbb{R}^{n-m} \to \mathbb{R}^n \) denote the inclusion map and \( T_p: \mathbb{R}^n \to \mathbb{R}^n \) the translation by the vector \( p \). By construction, we have \( H = T_p \circ i \circ \Phi(\mathbb{R}^{n-m}) \). Let \( x, y \) be points in \( \mathbb{R}^{n-m} \). It follows that

\[
\|T_p \circ i \circ \Phi(x) - T_p \circ i \circ \Phi(y)\|_{\mathbb{R}^n} = \|\Phi(x) - \Phi(y)\|_W \leq \|\Phi\| \cdot \|x - y\|_{\mathbb{R}^{n-m}}. \tag{A.1}
\]

Here \( \| \cdot \| \) is the induced operator norm and \( \| \cdot \|_W \) denotes the restriction of the standard norm of \( \mathbb{R}^n \) to \( W \). From now on we denote the map \( T_p \circ i \circ \Phi \) by \( F \). Let \( C_R \) denote the closed cube centered at the origin with edge length \( 2R \), i.e. \( C_R = [-R, R]^{n-m} \). Fix \( k \geq 1 \). For every \( x \) in \( \mathbb{R} \cdot 2^{-k} \cdot (-2^k, \ldots, 0, \ldots, 2^k - 1)^{n-m} =: J_k \) let \( c_x \) denote the cube \( [x, x + R \cdot 2^{-k}]^{n-m} \). Clearly

\[
C_R = \bigcup_{x \in J_k} c_x \tag{A.2}
\]
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by construction. Let \( L^n \) denote the \( n \)-dimensional Lebesgue measure. Now

\[
L^n(F([cR])) \leq \sum_{x \in J_k} L^n(F(c_x)) \leq \sum_{x \in J_k} \sqrt{n-m} \cdot L^n(B_1(0)) \cdot \|\Phi\|^n \cdot \mathbb{R}^n \cdot (2^{-k})^n = (A.3)
\]

The second inequality in (A.3) follows, since the estimate (A.1) implies that \( F(c_x) \) is contained in a ball of radius \( \sqrt{n-m} \cdot \|\Phi\| \cdot \mathbb{R} \cdot 2^{-k} \) centered at \( F(x) \).

The constant \( C \) in the estimate (A.3) is independent of \( k \); therefore, by letting \( k \) tend to infinity we can conclude that \( F(C_R) \) is a set of measure zero. Since \( F(C_1) \subset F(C_2) \subset \cdots \) and

\[
H = F(\mathbb{R}^m) = \bigcup_{l=1}^{\infty} F(C_l),
\]

we can use that \( L^n \) is continuous from below and obtain

\[
L^n(H) = \lim_{l \to \infty} L^n(F(C_l)) = 0,
\]

as desired. \[\square\]

Lemma A.6 Let \( \{H_k\}_{k=1}^{N} \) be a finite set of planes in \( \mathbb{R}^n \) of codimension strictly greater than one. Then the set

\[
\mathbb{R}^n \setminus \bigcup_{k=1}^{N} H_k
\]

is dense, open and path connected.

Proof By definition, every plane \( H \) of codimension \( m > 1 \) is equal to the preimage of zero of an affine linear map

\[
f: \mathbb{R}^n \to \mathbb{R}^m \quad x \mapsto Ax + b,
\]

where the \( m \times n \) matrix \( A \) has rank \( m > 1 \). Let the point \( x \) be contained in the set \( \mathbb{R}^n \) minus \( H \). Let \( x' \) in \( H \) be the unique orthogonal projection of \( x \) onto \( H \). Let \( w_1, \ldots, w_{n-m} \) be independent vectors such that the set \( x' + \text{span}(w_1, \ldots, w_{n-m}) \) is equal to the preimage of zero of the map \( f \). This is well-known from linear algebra. Denote the set \( \text{span}(w_1, \ldots, w_{n-m}) \) by \( U \). The set

\[
A_x(H) := \bigcup_{t \in [-1,-\infty)} x + t(x-x') + U
\]

(A.5)
is contained in the plane $x' \oplus \text{span}(U \cup (x' - x))$. Since the plane $x' \oplus \text{span}(U \cup (x' - x))$ has a codimension greater than or equal to 1, this implies that the set $A_x(H)$ is a set of measure zero, see Lemma A.5.

An easy calculation shows that for all points $y$ in the set $A_x(H)$ the line segment $[x, y]$ and the plane $H$ have a point in common. On the other hand, assume that the line segment $[x, y]$ contains a point $z$ of $H$. We can write $z = x' + \tilde{x}$, where the point $\tilde{x}$ is contained in $U$. There exists a $t_0 \in [-1, -\infty)$ such that $y = x + t_0(x - x' - \tilde{x}) = x + t_0(x - x') + t_0\tilde{x}$. It follows that $y$ is contained in $A_x(H)$; therefore, we have shown that the line segment $[x, y]$ is entirely contained in $\mathbb{R}^n$ minus $H$ if an only if $y$ is not contained in $A_x(H)$. For each $x$ contained in $(A.4)$ let

$$A_x := \bigcup_{l=1}^{N} A_x(H_l).$$

Since a finite union of measure zero sets has measure zero, it follows that $A_x$ is a measure zero set. Let $x$ and $y$ be two points contained in the set $(A.4)$. The set $A_x \cup A_y$ has measure zero in $\mathbb{R}^n$; as a result, there exists a point $z$ not contained in $A_x \cup A_y$. We deduce that the line segments $[x, z]$ and $[z, y]$ are contained entirely in $(A.4)$. In fact, we have shown that $(A.4)$ is path connected.

Next, we show that $\mathbb{R}^n$ without a plane $H$ of codimension $m > 1$ is open and dense. Note that each component function of $f$ is a non zero polynomial. This implies that for every $x$ contained in $H$ there cannot exist an open neighborhood $B$ of $x$ that is entirely contained in $H$. Hence the plane $H$ is nowhere dense. Due to the fact that $H$ is also closed, we may use the equivalence (A.7) in Corollary A.7 and deduce that $\mathbb{R}^n$ without $H$ is open and dense. Lemma A.3 implies that the set $(A.4)$ is open and dense.

**Corollary A.7** Every plane $H$ in $\mathbb{R}^n$ of codimension greater than or equal to one is closed and nowhere dense.

**Proof** Let $U$ be a subset of $\mathbb{R}^n$. By elementary set theory we know

$$U \text{ open, dense } \iff U^c \text{ closed, nowhere dense.}$$

Let $H$ be a plane in $\mathbb{R}^n$ of codimension greater than or equal to one. In the second part of the proof of Lemma A.6 we have shown that $H$ is closed and nowhere dense. In the reasoning we have not used that the codimension of this plane needs to be strictly greater than one, hence the same arguments also work for a plane of codimension one. Using (A.7) we may conclude the result.

**Lemma A.8** If $U$ is an open and connected subset of $\mathbb{R}^n$, then every two points $x, y \in U$ can be connected by a continuous piecewise linear map $\gamma: [0, L] \to U$ with only finitely many breakpoints.
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**Proof** If \( U \) is empty, there is nothing to prove. Let \( x \) be a point in \( U \). Denote by \( U_1 \) the set consisting of those points in \( U \) that can be connected with \( x \) by a continuous piecewise linear map \( \gamma: [0, L] \to U \) with only finitely many breakpoints. Let \( U_2 \) denote the sets consisting of those points in \( U \) that cannot be connected with \( x \) by a continuous piecewise linear map \( \gamma: [0, L] \to U \) with only finitely many breakpoints. The fact that \( U \) is open and the convexity of \( \| \cdot \| \)-balls suffice to show immediately that both \( U_1 \) and \( U_2 \) are open. Since \( U \) is connected and \( x \) is contained in \( U_1 \) we can conclude that \( U_2 \) is empty. Thus, the statement follows. ■

**Lemma A.9** The union of two connected sets \( A, B \) that have a point in common is connected.

**Proof** Let \( C, D \) be two disjoint non-empty open sets such that \( A \cup B = C \cup D \). To put it differently, assume that \( A \cup B \) is not connected. Note that neither the set \( A \) nor \( B \) can have points both in \( C \) or \( D \), for \( A \) and \( B \) are connected. But \( A \) and \( B \) have a point in common and therefore \( A \) and \( B \) are both contained exclusively either in \( C \) or \( D \). This is a contradiction to the fact that both \( C \) and \( D \) are non empty. ■

**Lemma A.10** [8, p. 95] Let \( Y \) be a subspace of \( X \); let \( A \) be a subset of \( Y \); let \( \overline{A} \) denote the closure of \( A \) in \( X \). Then the closure of \( A \) in \( Y \) is equal to \( \overline{A} \cap Y \).

**Proof** See Theorem 17.4 in [8, p. 95 ] for a complete proof. ■

**Lemma A.11 (Pasting Lemma)** Let \( \{ X_i \}_{i=1}^N \) be a finite set of closed subsets of a topological space \( X \), whose union is equal to \( X \). Let \( Y \) be a topological space and suppose that for every \( i \) in \( \{ 1, \ldots, N \} \) the restriction onto \( X_i \) of the map \( f: X \to Y \) is continuous. Then, \( f \) is continuous.

**Proof** This proof closely follows Munkres proof of Theorem 18.3 in his wonderful book [8, p. 108]. Let \( C \) be a closed subset of \( Y \). Now

\[
f^{-1}(C) = \bigcup_{i=1}^N f^{-1}\big|_{X_i}(C)
\]

by elementary set theory. Since for every \( i \) in \( \{ 1, \ldots, N \} \) the map \( f \) is continuous when restricted to \( X_i \) we may deduce that every set \( f^{-1}\big|_{X_i}(C) \) is closed. Hence \( f^{-1}(C) \) as a finite union of closed sets is closed, as desired. ■

**Definition A.12 (Nowhere dense)** A subset \( A \) of a topological space \( B \) is said to be nowhere dense in \( B \), if the closure \( \overline{A} \) of \( A \) has an empty interior.

In the special case where \( B \) is a subset of \( \mathbb{R}^n \) endowed with the subspace topology, we see that \( A \subset B \) is nowhere dense in \( B \), if for all non-empty open sets \( U \) contained in the subspace topology of \( B \) the set \( U \setminus (\overline{A} \cap B) \) is non-empty.
Lemma A.13 Let $f: X \to Y$ be a continuous map between Hausdorff spaces, and let $K$ be a nowhere dense subset of $X$.

1. If the set $K$ is compact, then the set $f(K)$ is nowhere dense in $Y$.
2. If $f$ is a homeomorphism, then the set $f(K)$ is nowhere dense in $Y$.

Proof Throughout the proof we may assume that $X$ and $Y$ are non-empty.

1. We prove the contrapositive of the given statement. So, we assume that $f(K)$ is not nowhere dense in $Y$. Since $K$ is compact, the map $f$ is continuous, and $Y$ is a Hausdorff space we obtain that $f(K)$ is closed in $Y$. Since $f(K)$ is not nowhere dense in $Y$ there exists a non empty open subset $U$ of $Y$ such that $U$ is completely contained in $f(K) = f(K)$. As a result, $K$ is not nowhere dense in $X$, since the non empty open set $f^{-1}(U)$ is completely contained in $K = K$.

2. If we are able to prove $f(K) = f(K)$, we may use similar methods as in the proof of the first part and deduce the desired result immediately. Hence we are left to show $f(K) = f(K)$. Let $g$ be a continuous map between topological spaces. It is well-known from general topology, cf. Theorem 18.1 (2) in [8, p. 104], that for every subset $A$ of the domain of $g$, one has $g(A) \subset g(A)$. Since the homeomorphism $f$ is in particular continuous, we obtain $f(K) \subset f(K)$. On the other hand, the map $f^{-1}: Y \to X$ is continuous as well. Therefore, we have $f^{-1}(f(K)) \subset f^{-1}(f(K)) = K$. By elementary set theory, we see that $f(K) \subset f(K)$ and conclude the proof.

Definition A.14 [5, p. 28] Let $f: X \to Y$ be a smooth map between smooth manifolds, and let $Z$ be a submanifold of $Y$. We say that the map $f$ is transversal to the submanifold $Z$, we write $f \independent Z$, if for all points $x$ in $X$ the tangent space of $Y$ at $f(x)$ is equal to the direct sum of the image of the tangent space of $X$ at $x$ under the tangent map of $f$ and the tangent space of $Z$ at $f(x)$.

Theorem A.15 (Transversality Theorem) [5, p. 68] Suppose $F: X \times S \to Y$ is a smooth map of manifolds, where only $X$ has boundary, and let $Z$ be any boundaryless submanifold of $Y$. If both $F$ and the restriction of $F$ to the boundary of $X \times S$ are transversal to $Z$, then for almost every $s$ in $S$, both $x \in X \mapsto F(x, s)$ and $x \in \partial X \mapsto F(x, s)$ are transversal to $Z$.

A.2 Technical Lemma

In the proofs of Lemma 2.14 and 2.15 we need to ensure that points contained sufficiently in the interior of a simplex also stay contained in simplices with vertices that have positions slightly altered from the positions of the original simplex. This rather technical requirement is embodied in the following Technical Lemma A.16.

**Lemma A.16 (Technical Lemma)** Suppose that $S$ is an $n$-dimensional simplex with vertex set $\{x_0, \ldots, x_n\} \subset \mathbb{R}^n$, and assume that the set $S' = \text{conv}\{x_0', \ldots, x_n'\} \subset \mathbb{R}^n$. Fix $\epsilon > 0$. We assume $\|x_i - x'_i\|_{\mathbb{R}^n} < \epsilon$ for each $i$ in the set $\{0, \ldots, n\}$. Let $x$ be contained in $\mathbb{R}^n$. Assume that the distance between the point $x$ and every set $E_i := \text{conv}(x_0, \ldots, \hat{x}_i, \ldots, x_n)$ is at least $\epsilon$, then we have the following equivalence:

$$x \in S' \iff x \in S.$$  \hfill (A.8)

If in addition the set $S'$ happens to be a simplex and if $x$ is contained in $S$, $x$ must lie in the interior of $S'$.

**Proof** Unfortunately, the statement is not symmetric in $S$ and $S'$, since the point $x$ only suffices conditions with respect to the boundary of $S$. This lack of symmetry enforces us to deal in the proof of the first statement with both directions separately. Observe that for the first statement the set $\text{conv}(S')$ is not required to be a simplex. We begin with the proof of the equivalence (A.8).

\[ \Longleftarrow: \] Suppose we are given a point $x' \in S'$ such that the distance between the point $x'$ and the set $E_i$ is greater than or equal to $\epsilon$ for all $i \in \{0, \ldots, n\}$. We need to show that $x' \in S$; accordingly, assume for the sake of a contradiction that $x' \not\in S$. Consider the map

$$F: \mathbb{R}^n \to \mathbb{R}^n$$

$$x_0 + \sum_{i=1}^{n} \lambda_i(x_i - x_0) \mapsto x'_0 + \sum_{i=1}^{n} \lambda_i(x'_i - x'_0).$$  \hfill (A.9)

On account of the linear independence of the vectors $x_1 - x_0, \ldots, x_n - x_0$, there exist unique numbers $\lambda_1, \ldots, \lambda_n$ for every point $x$ in $\mathbb{R}^n$ such that $x = x_0 + \lambda_1(x_1 - x_0) + \cdots + \lambda_n(x_n - x_0)$. Thus, the map $F$ is well defined. Observe

$$\|x - F(x)\| < \epsilon \quad \forall x \in S:$$  \hfill (A.10)

and that $F$ is affine linear. There exists at least one $x$ in $S$ such that $F(x) = x'$ and furthermore $\|x - x'\| < \epsilon$. Since $x \in S$ and $x' \not\in S$, we see that there exists a point $z \in [x, x']$ such that $z$ lies on the boundary of $S$. This is immediate since the interior of $S$ and the set $S^c$ are non empty open sets.
A.2. Technical Lemma

We deduce that $d(x', z) < \varepsilon$, since $z$ is contained on the line segment $[x, x']$ and $\|x - x'\| < \varepsilon$. Note that with a little effort one can show that $\partial S = \bigcup_{i=0}^{n} E_i$. This leads to a contradiction, since it implies that $z \in \partial S$ must be contained in a set $E_i$ and therefore the distance from $x'$ to $E_i$ turns out to be strictly smaller than $\varepsilon$, in contradiction to our hypotheses on $x'$.

"$\Rightarrow$": Let $x \in S$ be such that the distance from $x$ to $E_i$ is greater than or equal to $\varepsilon$ for all $i \in \{0, \ldots, n\}$. Unfortunately, a reasoning as in the preceding direction does no longer work, since our point $z$ would lie on the boundary of $S'$ and to obtain a contradiction in the same spirit as before, we would need a point on $\partial S$.

Nevertheless, there is a nice geometrical idea that does work. To begin, we show with similar methods as before that $\delta S = \bigcup_{i=0}^{n} E_i$. Observe that if a point $y$ in $B_{\varepsilon}(x)$ is not contained in $S$, there exists a point $z$ on the line segment $[x, y]$ such that $z$ is contained on the boundary of $S$. This leads to a contradiction, since $\partial S = \bigcup_{i=0}^{n} E_i$ implies that $z$ must be contained in a set $E_i$ and therefore, with the same reasoning as in the preceding direction, it follows that the distance between the point $x$ and the set $E_i$ must be strictly smaller than $\varepsilon$. Hence we have shown that $B_{\varepsilon}(x) \subset S$.

Now we show that $x$ is contained in $S'$. Assume for the sake of a contradiction that $x \notin S'$. Using Lemma A.1 we see that $F(B_{\varepsilon}(x)) \subset S'$ is convex. Note that we can apply Lemma A.1 for $F$ is affine linear and $B_{\varepsilon}(x)$ is convex. The convexity and closedness of $F(B_{\varepsilon}(x))$ imply that there exists a unique point $x' \in F(B_{\varepsilon}(x))$ such that $\text{dist}(x, F(B_{\varepsilon}(x))) = d(x, x')$. Note that $x \neq x'$. We parametrise the ray, which goes through the point $x$ and starts at the point $x'$, as follows:

$$
\gamma: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n
$$

$$
t \mapsto x' + t(x - x').
$$

An easy calculation shows that for all $t \in \mathbb{R}_{\geq 0}$ the point $x'$ minimizes the distance from $\gamma(t)$ to $F(B_{\varepsilon}(x))$. In other words,

$$
\text{dist}(\gamma(t), F(B_{\varepsilon}(x))) = d(\gamma(t), x') \quad \forall t \in \mathbb{R}_{\geq 0}:
$$

$$
(\text{A.11})
$$

Since the trace of $\gamma$ has points in $B_{\varepsilon}(x)$ and $B_{\varepsilon}(x)^c$, elementary geometry shows us that there exists a unique $t_0 > 1$ such that $\gamma(t_0) \in \partial B_{\varepsilon}(x)$. Equation (A.11) implies that $\text{dist}(\gamma(t_0), F(B_{\varepsilon}(x))) = d(\gamma(t_0), x') = \varepsilon + d(x, x') > \varepsilon$. This contradicts Equation (A.10), since $F(\gamma(t_0)) \in F(B_{\varepsilon}(x))$ and therefore $\|\gamma(t_0) - F(\gamma(t_0))\| \geq \text{dist}(\gamma(t_0), F(B_{\varepsilon}(x))) > \varepsilon$. Hence $x \in S'$, as desired. The first part follows.

Now we assume that $S'$ is a simplex and that $x$ satisfies the usual hypotheses and is contained in $S$. We know from the first part that $x$ must be contained
A.2. Technical Lemma

in $S'$. Since $S$ and $S'$ are simplices, the map $F$ has an inverse, which we call $F^{-1}$. It is immediate that

$$
\|x' - F^{-1}(x')\| < \varepsilon \quad \forall x' \in S':
$$

(A.12)

A priori it is possible that $x$ is contained on the boundary of $S'$. If $x$ lies on the boundary of $S'$, the point $F^{-1}(x)$ is contained on the boundary of $S$ and therefore in particular contained in some of the sets $E_i$. Equation (A.12) implies $\|x - F^{-1}(x)\| < \varepsilon$. This leads to a contradiction to the hypotheses on $x$ since we have $\text{dist}(x', E_i) < \varepsilon$, where $E_i$ is the set that contains the point $F^{-1}(x')$. This contradiction concludes the proof. ■

Lemma A.16 depends heavily on the assumption that requires $\text{conv}(S)$ to be an $n$-dimensional simplex. If we drop this assumption about $\text{conv}(S)$ we obtain that our point, which suffices the hypotheses of the lemma, cannot be contained in neither of the sets $\text{conv}(S), \text{conv}(S')$. The exact statement is embodied in the next lemma.

Lemma A.17 Suppose that $A = \{x_0, \ldots, x_n\}$ and $A' = \{x'_0, \ldots, x'_n\}$ are finite subsets of $\mathbb{R}^n$ respectively, and assume that the convex hull of $A$ is not an $n$-dimensional simplex. Let $\varepsilon > 0$, assume that for each $i$ in the set $\{0, \ldots, n\}$ we have $\|x_i - x'_i\|_{\mathbb{R}^n} < \varepsilon$. Let $x$ be contained in $\mathbb{R}^n$. Assume the the distance between the point $x$ and every set $E_i := \text{conv}(x_0, \ldots, \hat{x}_i, \ldots, x_n)$ is at least $\varepsilon$, then $x$ is neither contained in $\text{conv}(A)$ nor contained in $\text{conv}(A')$.

Proof The proof of this statement is much easier than the one of the previous Lemma A.16. Basically we only need to apply Lemma 1.8.

Let $x$ in $\mathbb{R}^n$ be such that the distance from $x$ to $E_i$ is greater than or equal to $\varepsilon$ for each $i \in \{0, \ldots, n\}$. In particular, $x$ cannot be contained in one of the sets $E_i$. We assume that $x$ is an element of $\text{conv}(A)$. Since the set $\text{conv}(A)$ is not an $n$-dimensional simplex, we can apply Lemma 1.8 and conclude that $x$ must be contained in at least one of the sets $E_i$. This leads to a contradiction. Therefore $x$ cannot be contained in $\text{conv}(A)$.

Now assume for the sake of a contradiction that $x'$ is in $\text{conv}(A')$ and $x'$ suffices the hypotheses of the lemma. We define the map $F$ as in the previous proof, see Equation (A.9). Recall that there exists at least one point $x$ in $\text{conv}(A)$ such that $F(x) = x'$. Note that since $x$ is contained in $\text{conv}(A)$, there exists a set $E_i$ such that $x$ is contained in it. Furthermore, we know $\|x - x'|_{\mathbb{R}^n} < \varepsilon$. Hence we can deduce that in contradiction to our assumptions on $x'$ the distance from $x'$ to $E_i$ must be strictly smaller than $\varepsilon$. Thus, we may conclude the proof. ■


