151-0590-00 **Regelungstechnik II** (FS 2008) **Musterlösung**

Übung 8 **Linear Quadratic Regulators**

C.B, 15. Mai 2008

Aufgabe 1 (Sufficient Conditions)

For the problem to be well-posed the expressions x^TQx and u^TRu within the cost function must never be negative, for any x and u respectively. This condition is satisfied only if $Q \geq 0$ and $R \geq 0$. If these conditions are not satisfied a solution may exist, but will not make sense because there is the possibility for the integration of negative values in the cost function . Further conditions are that $R > 0$ and that $Q^T = Q$ and $R^T = R$ which are related to solving the Riccati equation.
For a solution to the problem to be guaranteed some other conditions must be satisfied. These are that the pair $\{A, B\}$ is completely controllable and the pair $\{A, \bar{C}\}$ completely observable, where $\bar{C} = \sqrt{Q}$. This second set of conditions are related to the cost function containing sufficient information about every state variable for the formulation of a meaningful trade-off problem.

a) $Q = Q^T \longrightarrow$ YES
$Q \geq 0 \longrightarrow$ YES
$R = R^T \longrightarrow$ YES
$R > 0 \longrightarrow$ YES
$\{A, B\}$ completely controllable $\longrightarrow$ NO
$\{A, \bar{C}\}$ completely observable where $\bar{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ YES
$\longrightarrow$ The problem is well posed but does not guarantee a solution.

b) $Q = Q^T \longrightarrow$ YES
$Q \geq 0 \longrightarrow$ YES
$R = R^T \longrightarrow$ YES
$R > 0 \longrightarrow$ YES
$\{A, B\}$ completely controllable $\longrightarrow$ YES
$\{A, \bar{C}\}$ completely observable where $\bar{C} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ YES
$\longrightarrow$ The problem is well posed and guaranties a solution.

c) $Q = Q^T \longrightarrow$ NO
$Q \geq 0 \longrightarrow$ NO
$R = R^T \longrightarrow$ YES
$R > 0 \longrightarrow$ YES
$\{A, B\}$ completely controllable $\longrightarrow$ YES
$\{A, \bar{C}\}$ completely observable $\longrightarrow$ N/A
$\longrightarrow$ The problem is not well posed and does not guarantee a solution.

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Aufgabe 2 (LQR-Problem)

a) Given the system:

$$A = \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Stability depends on the real part of the eigenvalues of A being positive.

$$\det[sI - A] = \det \begin{bmatrix} -2-s & 0 \\ -2 & -s \end{bmatrix} = -s(-2-s) = s^2 + 2s \overset{!}{=} 0$$

$s_1 = 0$ and $s_2 = -2 < 0$.

$\rightarrow$ The system is marginally stable.

b) Matrices Q and R of the quadratic cost function are $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 1$

The symmetric Matrix $\Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{bmatrix}$ is found by solving the associated Riccati equation:

$$\begin{aligned}
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \Phi B R^{-1} B^T \Phi - \Phi A - A^T \Phi - Q \\
&= \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{bmatrix} - \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \phi_1^2 & \phi_1\phi_2 \\ \phi_1\phi_2 & \phi_2^2 \end{bmatrix} + \begin{bmatrix} 4\phi_1 + 4\phi_2 & 2\phi_2 + 2\phi_3 \\ 2\phi_2 + 2\phi_3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Solving (Φ is positive definite):

i) $\phi_2^2 - 1 = 0 \quad \longrightarrow \quad \phi_2 = +1 \text{ or } -1 \longrightarrow -1$

ii) $\phi_1^2 + 4\phi_1 - 5 = 0 \quad \longrightarrow \quad \phi_1 = -5 \text{ or } 1 \longrightarrow 1$

iii) $\phi_1\phi_2 + 2\phi_2 + 2\phi_3 - 1 = 0 \quad \longrightarrow \quad \phi_3 = 1.5$

The unique solution is: $\Phi = \begin{bmatrix} 1 & -1 \\ -1 & 1.5 \end{bmatrix}$

c) $K = R^{-1} B^T \Phi = 1 \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1.5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$

d) The poles of the closed-loop system are the poles of $[A - BK]$.

$$\det[sI - (A - BK)] = \det \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix} = s(s+3) + 2 = s^2 + 3s + 2 \overset{!}{=} 0$$

$s_1 = -2 < 0$ and $s_2 = -1 < 0$.

$\rightarrow$ The system is asymptotically stable.

Aufgabe 3 (Cart and Inverted Pendulum, Matlab/Simulink)

a) Matlab Code:

```
x0=[0.1 0 0 0]; % Initial Conditions

Q=[100000000 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1]; R=0.1; % Weighting matrices

[K]=lqr(A,B,Q,R); % Controller
```

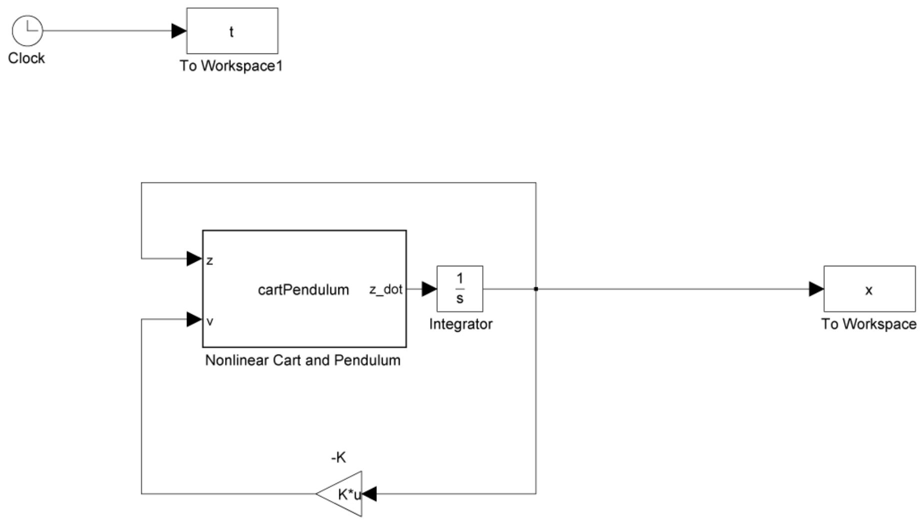

Abbildung 1: Simulink Model

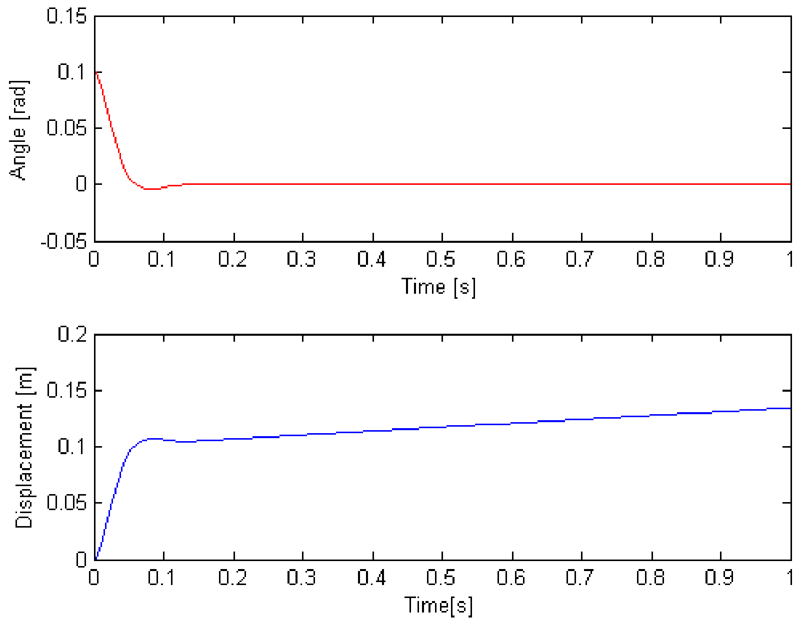

Abbildung 2: Response

b) Matlab Code:

```
x0=[0 0 0 0]; % Initial Conditions
Q=[1 0 0 0; 0 1 0 0; 0 0 1000 0; 0 0 0 1]; R=0.1; % Weighting matrices

gamma=1000000; % Define integration tuning parameter

A_tilde=[A zeros(4,1); -C 0];
B_tilde=[B; 0];
C_tilde=[sqrt(Q) zeros(4,1); zeros(1,4) gamma];
K_tilde=lqr(A_tilde,B_tilde,C_tilde'*C_tilde,R);

K=K_tilde(1:4);
KI=-K_tilde(5);
```

A physical limitation to the tracking performance of the cart is that the controller has to reverse the cart at the start of the ramp in order to create the correct angle in the pendulum to accelerate in the forward direction. This is an example of the controller “seeing into the future”.

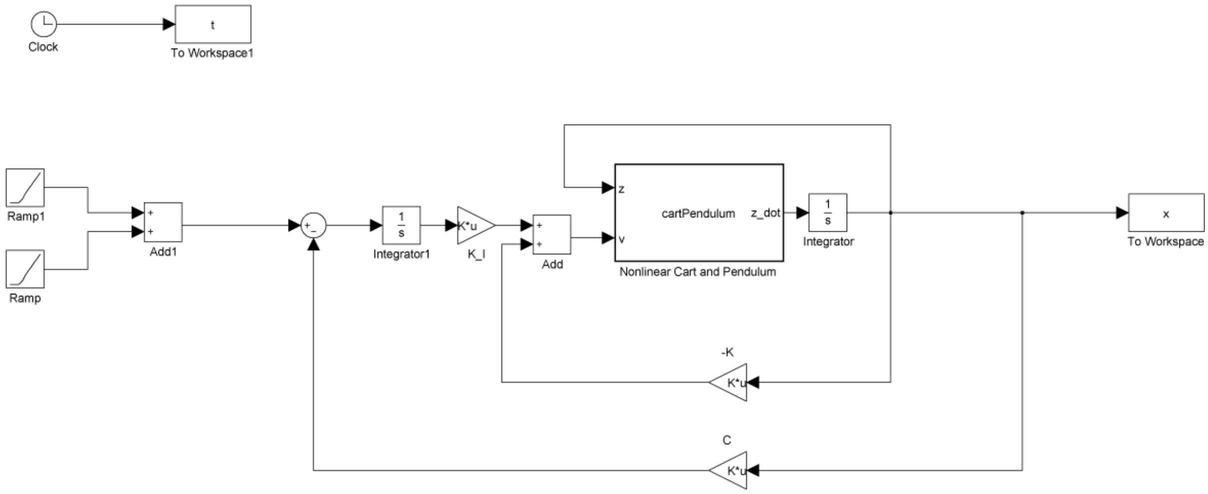

Abbildung 3: Simulink Model

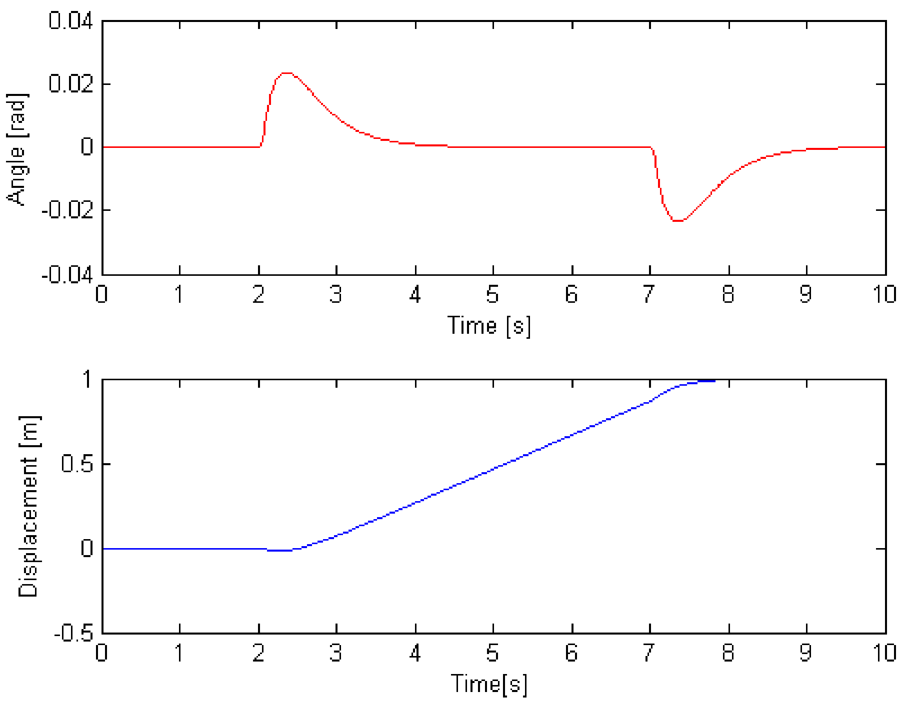

Abbildung 4: Response