HFDT Summary

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2 General

HF: $300Mhz - 3GHz$ $\lambda : 1m - 1mm \rightarrow$ distributed circuits
$\lambda = \frac{c}{f} = \frac{c_0}{\sqrt{\epsilon_r} f}$

2.1 Skin effect

$\delta_s = \frac{1}{\sqrt{\pi f \mu_0 \mu_r \sigma}}$, $\sigma = conductivity = \frac{1}{\rho}$
Cylindrical conductor: $R_{DC} = \frac{l}{\pi r^2 \sigma}$ $R_{AC} = \frac{l}{2\pi r \delta_s \sigma}$
$\frac{R_{AC}}{R_{DC}} = \frac{r}{2\delta_s}$

2.2 Ohm's Law

What	Symbol
Resistance	R
Reactance	X
Impedance	$Z = R + j \cdot X$
Conductance	$G = \frac{Re(Z)}{Re(Z)^2 + Im(Z)^2}$
Susceptance	$B = \frac{-Im(Z)}{Re(Z)^2 + Im(Z)^2}$
Admittance	$Y = \frac{1}{Z} = G + j \cdot B$

	Imp.[Z]	Adm.[Y]	Differential	Energy
Res	R	$\frac{1}{R}$	$U = R \cdot I$	-
Cap	$\frac{1}{j \cdot \omega \cdot C}$	$j \cdot \omega \cdot C$	$i(t) = C \cdot \frac{du(t)}{dt}$	$\frac{1}{2} CU^2$
In	$j \cdot \omega \cdot L$	$\frac{1}{j \cdot \omega \cdot L}$	$u(t) = L \cdot \frac{di(t)}{dt}$	$\frac{1}{2} LI^2$

2.3 Decibels & Neper

$\frac{P}{P_0}(dB) = 10 \cdot log(\frac{P}{P_0}) = 20 \cdot log(\frac{V}{V_0})$
$dB \rightarrow P_0 = 1W, dBm \rightarrow P_0 = 1mW$ Neper: $\frac{P}{P_0}(Np) = \frac{1}{2} \cdot ln(\frac{P}{P_0})$
$1Np = 10loge^2 = 8.686dB$

2.4 Power

Only real power is dissipated $p(t) = u(t) \cdot i(t)$
$P_{peak} = Re(VI^*) = VI^* cos(\theta)$
$P_{avg} = \frac{1}{2} Re(VI^*) = \frac{1}{2} VI^* cos(\theta)$
Maximum Power Transfer: $Z_{Load} = Z^*_{Generator}$

2.5 Loss in AC circuits

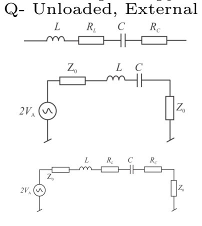
Insertion Loss: IL $= 10\log(\frac{P_{L1}}{P_{L2}})$ dB
$= \frac{Power-without-2-port}{Power-with-2-port}$ dB
Transducer Loss TL $= 10\log(\frac{P_A}{P_{L2}})$ dB
$= \frac{Max.-Power-available}{Power-with-2-port}$ dB
Series: TL $= |1 + \frac{Z}{2Z_0}|^2$ Parallel: TL $= |1 + \frac{Y}{2Y_0}|^2$

2.6 Resonators

Resonance Frequency: Admittance equal zero.
A circuit with several L and C has several resonance freq.
normal: $\omega_0 = \frac{1}{\sqrt{LC}}$

2.7 Quality Factor

$Q = \frac{avergaeenergystored}{averageenergydissipated} = \frac{X}{R}$
Q- Unloaded, External and Loaded



$Q_U = \frac{\omega_o L}{R_L + R_C} = \frac{1}{\omega_o C(R_L + R_C)}$

$Q_E = \frac{\omega_o L}{2Z_0} = \frac{1}{\omega_o C(2Z_0)}$

$Q_L = \frac{\omega_o L}{R_L + R_C + 2Z_0}$

$\frac{1}{Q_L} = \frac{1}{Q_U} + \frac{1}{Q_E}$

2.7.1 Parallel Resonator

$Q = \frac{1}{\omega_0 LG} = \frac{\omega_0 C}{G}$ same as serial, replace $R = \frac{1}{G}, B = \frac{1}{X}$

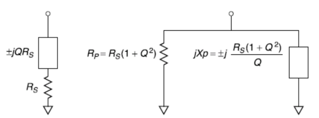
2.7.2 Bandwidth

$Q = \frac{f_0}{\Delta f} = \frac{\omega_0}{\Delta \omega}$; $B = \frac{1}{Q}$ for high Q only $(B \cdot Q = 1)$!
higher Q means narrower Bandwidth!

2.7.3 Fano's Limit

Matching narrowband, only works for the designed frequency. Fano's limit calculates the minimum obtainable Γ_{MIN} over the selected bandwith:
$\omega_0 = \sqrt{\omega_1 \omega_2}$, $f_0 = \frac{\omega_0}{2\pi}$, $\Delta f = \frac{\omega_2 - \omega_1}{2\pi}$
$\Gamma_{MIN} \geq e^{-(\frac{\pi}{Q})(\frac{f_0}{\Delta f})}$

2.8 Series to parallel equivalent circuit



$R_P = R_S(1 + Q^2)$; $X_P = \frac{R_S(1+Q^2)}{Q} = \frac{R_P}{Q}$

$Z = R_S + jX = R_S(1 + jQ)$
$Y = \frac{1 - jQ}{R_s(1+Q^2) = G - jB}$

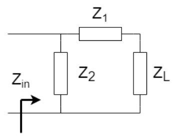
3 Q-Matching

To increase bandwith use multiple stages with lower Q use n sections: $1 + Q^2 = \sqrt[n]{R} \rightarrow$ do the transformation n times

3.1 Low to high Resistance

R_P = target Resistance; $R_S = R_L$

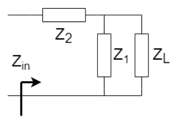
1. Find target Q ($R_P = R_S(1 + Q^2)$); $Q = \sqrt{\frac{R_P}{R_S} - 1}$
2. Add L or C (X_{LC}) in series (X_S is the target reactance and we want to archieve it with the help of the series reactance)
   $\rightarrow X_{LC} + X_L \stackrel{!}{=} \pm X_S = \pm Q \cdot R_S$
   If $X_S = \pm X_L$ no L or C element has to be added.
3. Convert to parallel($X_P = \frac{R_s(1+Q^2)}{Q}$, keep sign)
4. Add C or L in parallel to cancel out (resonate) the reactive part (X_P)
5. Check input impedance!



3.2 High to low Resistance

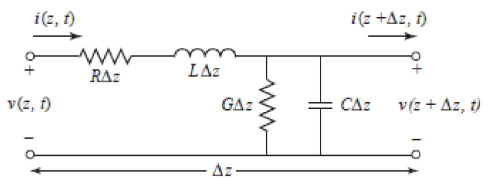
$Q_L = \frac{X_L}{R_L}$; R_S = target Resistance

1. Convert to parallel representation
   $R_{PL} = R_L(1 + Q_L^2); X_{PL} = \frac{R_L(1+Q_L^2)}{Q_L}$ Sign of imag. part stays the same!
2. Find target Q: $Q = \sqrt{\frac{R_{PL}}{R_S} - 1}$
3. Add L or C (X_{LC}) in parallel, X_P has sign of X_{PL}
   $X_P = \frac{R_{PL}}{Q} \rightarrow X_P \stackrel{!}{=} \frac{X_{PL} X_{LC}}{X_{PL} + X_{LC}}$
   $\rightarrow X_{LC} = \frac{X_P X_{PL}}{X_{PL} - X_P}$
4. Convert to series representation $X_S = Q \cdot R_S$ (keep the sign)
5. Add L or C in series to cancel out (resonate) the reactive part (X_S)
6. Check input impedance!



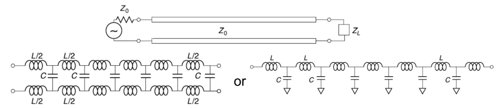
4 Transmission Line

- Device/circuit dimensions are in the same order as λ ($l > \frac{\lambda}{10}$)
- Lumped Elements are not used at HF because of parasitic behaviour
- One piece of transmission line with length Δz can be modeled as a lumped element circuit $\rightarrow$ TL = periodic circuit of infinitely many of those elements



Characteristics:

1. Uniform Crosssection
2. Separation between the two conductors $<< \lambda$
3. Characteristic Impedance Z_0 (= voltage to current ratio traveling in a direction)
4. distributed behaviour



4.1 Telegraphers equations & some formulas

Voltage and Current are not constant, it is a superposition of a forward $\rightarrow$ and reflected $\leftarrow$ wave. We get a standing wave pattern. For $\Delta z \rightarrow 0$: $\frac{\delta V}{\delta z} = -(R + j\omega L) \cdot I$; $\frac{\delta I}{\delta z} = -(G + j\omega C) \cdot V$
Current: $I = I_I e^{-\gamma z} - I_R e^{\gamma z} \Longrightarrow + \leftarrow$
Voltage: $V = V_I e^{-\gamma z} + V_R e^{\gamma z} \Longrightarrow + \leftarrow$
propagation const: $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$
Phase constant: $\beta = \frac{rad}{m}$; $\lambda = \frac{2*\pi}{\beta}$; $\beta z = \omega t$
attenuation constant: $\alpha = \frac{Np}{m}$ low loss $\rightarrow \alpha \approx \frac{R}{2Z_0}$
The attenuation const. α defines the loss of the TL often given in dB.
$Z_0 = \frac{V_I}{I_I} = \frac{V_R}{I_R} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$ complex!
Wavelength: $\lambda = \frac{2\pi}{\beta}$

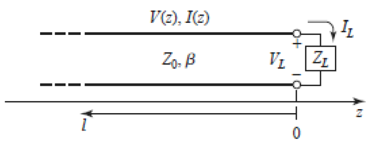
4.2 Lossless Line

$\beta = \omega\sqrt{LC}$, $\alpha = 0$, $Z_0 = \sqrt{\frac{L}{C}}$
Incident + Reflected wave:
Current: $I = I_I e^{-j\beta z} - I_R e^{j\beta z} = I_I e^{-j\beta z}[1 - \Gamma_L e^{2j\beta z}]$
Voltage: $V = V_I e^{-j\beta z} + V_R e^{j\beta z} = V_I e^{-j\beta z}[1 + \Gamma_L e^{2j\beta z}]$
Phase caused by a TL: $\theta = \beta l = \frac{2\pi}{\lambda} l = \frac{2\pi}{\nu_p} l = \frac{\omega l}{\nu_p}$
if lossy, replace β with γ

4.3 Phase & Group Velocities

phase velocity: $\nu_p = \frac{\omega}{\beta} = \lambda f = \frac{c_0}{\sqrt{\epsilon_r}} = \frac{1}{\sqrt{LC}}$
$e^{j(\omega t - \beta z)} = e^{j\omega(t - z/\nu_p)}$
group velocity: $\nu_G = \frac{d\omega}{d\beta} = \frac{1}{1 - (\frac{\omega}{\nu_p})(\frac{d\nu_p}{d\omega})}$

4.4 Reflection Coefficient



$\Gamma = |\Gamma|\angle\phi = |\Gamma|e^{j\phi}$
Derivation at z=0 (load):
$V(0) = V_I(0) + V_R(0) = I(0) \cdot Z_L$
$I(0) = I_I(0) - I_R(0) = \frac{1}{Z_0}(V_I(0) - V_R(0))$
$V_I(0) + V_R(0) = \frac{1}{Z_0}(V_I(0) - V_R(0)) \cdot Z_L$
$\Rightarrow V_R(0) = \frac{Z_L - Z_0}{Z_L + Z0} V_I(0)$
$\Gamma(z) = \frac{V_R(z)}{V_I(z)} = \frac{I_R(z)}{I_I(z)} = \Gamma(0)e^{2j\beta z}$ (lossless)
$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z0}$
$\Gamma_{in} = \frac{Z_{in} - Z_G}{Z_{in} + Z_G} = \Gamma_L e^{-2\gamma l} == \Gamma_L e^{-2l(\alpha + j\beta)}$
No reflection $\rightarrow \Gamma = 0$

$V(z) = V_I(z)(1 + \Gamma(z))$
$I(z) = I_I(z)(1 - \Gamma(z))$

$P_{avg} = \frac{1}{2}\frac{|V_{I0}|^2}{Z_0}(1 - |\Gamma(0)|^2)$

Return Loss $RL = -10\log(|\Gamma|^2) \quad dB$, $|\Gamma| = 10^{-\frac{RL}{20}}$
(the fraction of power reflected from the load)
Mismatch loss $= -10\log(1 - |\Gamma|^2) \quad dB$
(the fraction of power absorbed at the load)

4.5 Voltage Standing Wave Ratio (VSWR)

Max. Voltage of standing wave: $V_{max} = |V_I|(1 + |\Gamma|)$
Min. Voltage of standing wave: $V_{min} = |V_I|(1 - |\Gamma|)$
$VSWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$
$\Gamma = \frac{VSWR-1}{VSWR+1} = 10^{\frac{-RL(dB)}{20}}$

- Distance between maxima (minima) is $\frac{\lambda}{2}$
- Distance between maxima and minima is $\frac{\lambda}{4}$

Just for lossless:

- Because voltage and current on the line aren't constant, impedance looking into the transmission line will vary with position x: $Z(x) = \frac{V(x)}{I(x)} = Z_0\frac{1+\Gamma(x)}{1-\Gamma(x)}$
- Since $\Gamma(x) = \frac{Z(x)-Z_0}{Z(x)+Z0}$ and $\Gamma_L = \frac{Z_L-Z_0}{Z_L+Z0}$ we get $Z(x) = Z_0\frac{Z_L+jZ_0 tan\beta x}{Z_0+jZ_L tan\beta x}$

4.6 Input Impedance

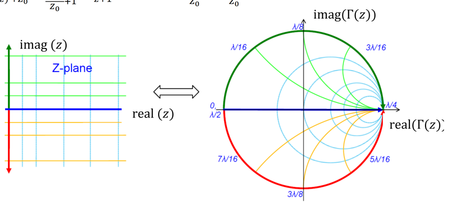
- At input z = -l we get: $Z_{in} = Z_0\frac{Z_L+jZ_0 tan\theta}{Z_0+jZ_L tan\theta}$
- with loss: $Z_{in} = Z_0\frac{Z_L+Z_0 tanh\gamma l}{Z_0+Z_L tanh\gamma l}$

Special Cases (lossless):

- impedance matched: $Z_L = Z_0 \rightarrow Z_{in} = Z_0$
- lambda half: $l = n\frac{\lambda}{2} \rightarrow Z_{in} = Z_L$
- lambda quarter: $l = \frac{\lambda}{4} \rightarrow Z_{in} = \frac{Z_0^2}{Z_L}$
- shorted TL: $Z_{in} = jZ_0 tan\theta \rightarrow$ inductive for $\theta < 90$, capacitive for $90 < \theta < 180$
- open TL: $Z_{in} = -jZ_0 cot\theta \rightarrow$ capacitive for $\theta < 90$, inductive for $90 < \theta < 180$

5 Smith Chart

$\Gamma(z) = \frac{Z(z)-Z_0}{Z(z)+Z_0} = \frac{\frac{Z(z)}{Z_0}-1}{\frac{Z(z)}{Z_0}+1} = \frac{z-1}{z+1}$ where $z = \frac{Z(z)}{Z_0} = \frac{R+jX}{Z_0} = r + x$



Addmitance Smith chart: $\Gamma = \frac{1-y}{1+y}$

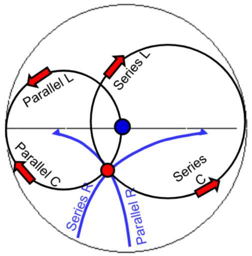
- At a fixed frequency, movement from the load toward the generator results in a clockwise rotation on the Smith Chart.
- As frequency increases, reflection coefficients always rotate clockwise

5.1 Matching

- There are two types:
  1. impedance matching (no reflection)
  2. complex conjugate matching (max power is delivered to the load)
- Matching can be done with:
  - Lumped LC elements
  - Sections of Transmission line (stub with open or short circuit, series TL)
  - Mix of both
- Matching narrowband, only works for the designed frequency. Fano's limit calculates the minimum obtainable Γ_{MIN} over the selected bandwidth:

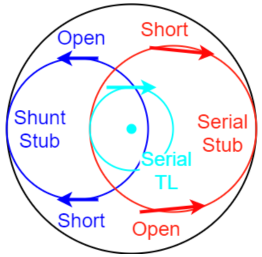
$\Gamma_{MIN} \geq e^{-(\frac{\pi}{Q})(\frac{f_0}{\Delta f})}$

5.2 LC Matching



High to Low: first Parallel (stub) then Serial
Low to High: first Serial then Parallel (stub)

5.3 Transmission Line (TL) Matching



5.3.1 Serial TL

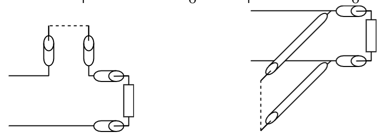
- Making TL to the load longer $\rightarrow$ rotate around center point in clockwise direction on const. Γ (radius) $\rightarrow$ lossless.
- If Loss is respected the line spirals towards center (matched point) of the smith chart.
- $Z_0 = \sqrt{Z_{max}Z_{min}}$

Match any complex load to any complex generator:
We want: $Z_{in} = R_G - jX_G \stackrel{!}{=} Z_0\frac{Z_L+jZ_0 tan\theta}{Z_0+jZ_L tan\theta} \rightarrow$ solve for complex Z_0; often $\theta = \beta l$ (lossless)

5.3.2 Stub TL

theoratically both (open or short) can be used as cap/induc. but the TL would have to be $> \pi/2$ and therefor it takes more area
$\beta = \frac{2\pi}{\lambda}$

	open circuit	short ciruit
Impedance	$Z = -jZ_0 cot\beta l$	$Z = jZ_0 tan\beta l$
Admittance	$Y = \frac{jtan\beta l}{Z_0}$	$Y = \frac{-jcot(\beta l)}{Z_0}$

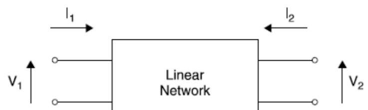


Serial Stub (impedance) Shunt Stub (admittance)

5.4 Quarter wave transformer

TL with a length of $\frac{\lambda}{4}$ and a characteristic impedance which has to be defined.
deriviation:
$Z_{in} = Z_0\frac{Z_L+jZ_0 tan\beta l}{Z_0+jZ_L tan\beta l} = Z_0\frac{Z_0}{Z_L}$ for $l = \frac{\lambda}{4}$
$\rightarrow Z_{in} = \frac{Z_0^2}{Z_L} \Rightarrow Z_0 = \sqrt{Z_{in}Z_L}$
to make the transformation more wideband by doing the transformation in $n\frac{\lambda}{4}$ steps, each providing a $\sqrt[n]{\frac{Z_L}{Z_{in}}}$ transformation.

6 Matrix Network Analysis



$V_1 = Z_{11}I_1 + Z_{12}I_2$
$V_2 = Z_{21}I_1 + Z_{22}I_2$
Impedance: $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$

$Z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0}$ $Z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0}$

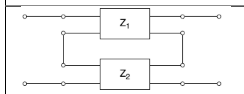
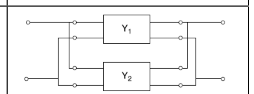
$Z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0}$ $Z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0}$

Admittance: $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

$Y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$ $Y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0}$

$Y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0}$ $Y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0}$

6.1 Serial and Parallel

Serial	Parallel
	
$Z_T = Z_1 + Z_2$	$Y_T = Y_1 + Y_2$

6.2 Special Networks

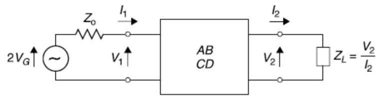
6.2.1 Reciprocal Network

Its elements are bilateral and linear (not containing any active devices, ferrites, plasma..) $Z_{12} = Z_{21}$; $Y_{12} = Y_{21}$

6.2.2 Symmetric Network

$Z_{11} = Z_{22}$; $Z_{12} = Z_{21}$

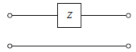
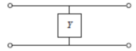
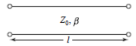
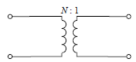
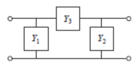
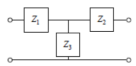
6.3 ABCD (transmission) Matrix



$A = \frac{V_1}{V_2}\Big|_{I_2=0}$ $B = \frac{V_1}{I_2}\Big|_{V_2=0}$

$C = \frac{I_1}{V_2}\Big|_{I_2=0}$ $D = \frac{I_1}{I_2}\Big|_{V_2=0}$

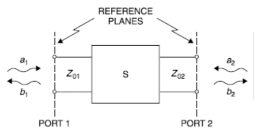
ABCD matric of the cascade connection of several two port network can be found just by multiplying the ABCD matrices of individual two ports:
$\begin{pmatrix} A_T & B_T \\ C_T & D_T \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} (A_1A_2 + B_1C_2) & (A_1B_2 + B_1D_2) \\ (C_1A_2 + D_1C_2) & (C_1B_2 + D_1D_2) \end{pmatrix}$
Reciprocal: $AD - BC = 1$
Symmetrical: $A = D$
Lossless: $|S_{11}|^2 + |S_{21}|^2 = 1$ and $|S_{21}|^2 + |S_{22}|^2 = 1$

Circuit	*ABCD* Parameters	
	$A = 1$ $C = 0$	$B = Z$ $D = 1$
	$A = 1$ $C = Y$	$B = 0$ $D = 1$
	$A = \cos\beta\ell$ $C = jY_0\sin\beta\ell$	$B = jZ_0\sin\beta\ell$ $D = \cos\beta\ell$
	$A = N$ $C = 0$	$B = 0$ $D = \frac{1}{N}$
	$A = 1 + \frac{Y_2}{Y_3}$ $C = Y_1 + Y_2 + \frac{Y_1Y_2}{Y_3}$	$B = \frac{1}{Y_3}$ $D = 1 + \frac{Y_1}{Y_3}$
	$A = 1 + \frac{Z_1}{Z_3}$ $C = \frac{1}{Z_3}$	$B = Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}$ $D = 1 + \frac{Z_2}{Z_3}$

Input Impedance:
$V_1 = AV_2 + BI_2$; $I_1 = CV_2 + DI_2$
$Z_{in} = \frac{AV_2+BI_2}{CV_2+DI_2} = \frac{A(V_2/I_2)+B}{C(V_2/I_2)+D} = \frac{AZ_L+B}{CZ_L+D}$
Insertion Loss: $IL = \frac{1}{4}|A + \frac{B}{Z_0} + CZ_0 + D|^2$ ($Z_L = Z_0 = real$)

6.4 Scattering Matrix (S-Parameters)

S_{ii} is the reflection coefficient seen looking into port i when all other ports are terminated in matched loads
S_{ij} is the transmission coefficient from port j to port i when all other ports are matched loads
Instead of measuring voltages and currents at the ports (for Z,Y, ABCD), S parameters are obtained by measuring intensities of the incident and reflected waves (at certain conditions: matching is achieved!)
Reciprocal: $[S] = [S]^T \to$ symmetric
Lossless: $|S_{11}|^2 + |S_{21}|^2 = 1$, **transmitted power**: $|S_{12}|^2 = |S_{21}|^2$
dissipated power = 1 - transmitted power



b_i: reflected wave, a_i: incident wave

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$a_1 = \frac{V_{I1}}{\sqrt{Z_{01}}}$ $b_1 = \frac{V_{R1}}{\sqrt{Z_{01}}}$

$a_2 = \frac{V_{I2}}{\sqrt{Z_{02}}}$ $b_2 = \frac{V_{R2}}{\sqrt{Z_{02}}}$

Reflection:

$$S_{11} = \frac{b_1}{a_1}\bigg|_{a_2=0} = \frac{V_{R1}}{V_{I1}}\bigg|_{V_{I2}=0} = \Gamma_1\bigg|_{V_{I2}=0}$$

$$S_{22} = \frac{b_2}{a_2}\bigg|_{a_1=0} = \frac{V_{R2}}{V_{I2}}\bigg|_{V_{I1}=0} = \Gamma_2\bigg|_{V_{I1}=0}$$

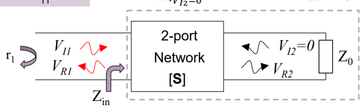
Transmission:

$$S_{21} = \frac{b_2}{a_1}\bigg|_{a_2=0} = \frac{V_{R2}}{V_{I1}}\bigg|_{V_{I2}=0}$$

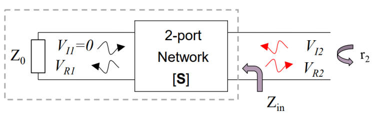
$$S_{12} = \frac{b_1}{a_2}\bigg|_{a_1=0} = \frac{V_{R1}}{V_{I2}}\bigg|_{V_{I1}=0}$$

Even though Z, Y, and ABCD parameters cannot be practically measured at RF frequencies they can be obtained by conversion from the measured S parameters

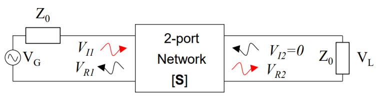
Determining S_{11} $\quad S_{11} = \frac{V_{R1}}{V_{I1}}\Big|_{V_{I2}=0} = \Gamma_1\Big|_{V_{I2}=0} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$



Determining S_{22} $\quad S_{22} = \frac{V_{R2}}{V_{I2}}\Big|_{V_{I1}=0} = \Gamma_2\Big|_{V_{I1}=0} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$



Determining S_{21} $\quad S_{21} = \frac{V_{R2}}{V_{I1}}\Big|_{V_{I2}=0} = \frac{2V_L}{V_G}$



S_{12} is similar to S_{21}

$$S_{21} = \frac{b_2}{a_1}\bigg|_{a_2=0} = \frac{V_2^2}{V_1^+}\bigg|_{Z_0 at port 2}$$

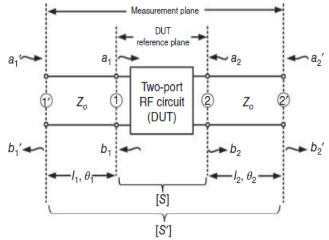
often: $V_2^- = V_2$ when the load is matched; $V_1^+ = V_1 - S_{11}V_1^+ \to$ solve for V_1^+

6.4.1 S parameters of a TL

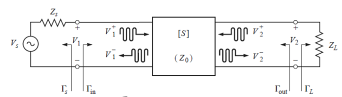
$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{pmatrix}$$

6.4.2 Problems when determining S parameters

- Measurements of individual circuits, components or devices often cannot be done directly at individual ports
- As a result, the actual measurement at reference planes are different from those of the interested RF device under test (DUT)
- However, S parameters of a DUT can be obtained from the measured S parameters with a simple calculation: shifting of the reference planes



$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} S'_{11}e^{j2\theta_1} & S'_{12}e^{j(\theta_1+\theta_2)} \\ S'_{21}e^{j(\theta_1+\theta_2)} & S'_{22}e^{j2\theta_2} \end{pmatrix}$$ When load is mismatched:



$$\begin{cases} \Gamma_L = \frac{a_2}{b_2} = \frac{V^+{}_2}{V^-{}_2} = \frac{Z_L - Z_0}{Z_L + Z_0} \\ \Gamma_S = \frac{a_1}{b_1} = \frac{V^+{}_1}{V^-{}_1} = \frac{Z_S - Z_0}{Z_S + Z_0} \end{cases} \qquad \begin{cases} \Gamma_{out} = \frac{b_2}{a_2} = \frac{V^-{}_2}{V^+{}_2} = \frac{1}{\Gamma_L} = \frac{Z_{out} - Z_0}{Z_{out} + Z_0} \\ \Gamma_{in} = \frac{b_1}{a_1} = \frac{V^-{}_1}{V^+{}_1} = \frac{1}{\Gamma_s} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \end{cases}$$

$$\begin{cases} V_1^- = S_{11}V_1^+ + S_{12}V_2^+ = S_{11}V_1^+ + S_{12}\Gamma_L V_2^-, \\ V_2^- = S_{21}V_1^+ + S_{22}V_2^+ = S_{21}V_1^+ + S_{22}\Gamma_L V_2^-. \end{cases} \longrightarrow \begin{cases} \Gamma_{in} = \frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} \\ \Gamma_{out} = \frac{V_2^-}{V_2^+} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S} \end{cases}$$

If $Z_L = Z_0$ and $Z_S = Z_0$ $\longrightarrow \begin{cases} \Gamma_L = \Gamma_S = 0 \\ \Gamma_{out} = S_{22} \text{ and } \Gamma_{in} = S_{11} \end{cases}$

6.4.3 S parameters

TL: $\begin{pmatrix} V_R^1 \\ V_R^2 \end{pmatrix} = \begin{pmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{pmatrix} \begin{pmatrix} V_I^1 \\ V_I^2 \end{pmatrix}$

7 Amplifier Design

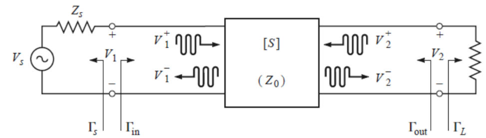
7.1 Stability

$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}$ $\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1-S_{11}\Gamma_S}$ Oscillation is possible if either the input or output port reflection coefficient has a magnitude larger than 1

7.1.1 Types of stability

unconditional stability:
The network is unconditionally stable if $|\Gamma_{in}| < 1$ and $|\Gamma_{out}| < 1$ for all passive source and load impedances ($|\Gamma_s < 1|$ and $|\Gamma_L| < 1$)
conditional stability:
The network is conditionally stable if $|\Gamma_{in}| < 1$ and $|\Gamma_{out}| < 1$ only for a certain range of passive source and load impedances. This case is also referred to as potentially unstable

7.1.2 Determining Stability



$|\Gamma_{in}| = |S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}| \overset{!}{<} 1$

$|\Gamma_{out}| = |S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1-S_{11}\Gamma_S}| \overset{!}{<} 1$

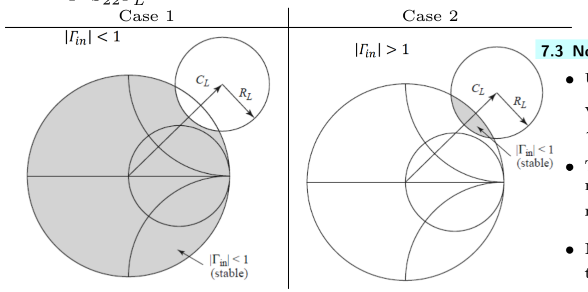
7.1.3 Stability Circles

Output Stability circle: $(\Gamma_{out} = 1)$
Center: $C_L = \frac{(S_{22} - \Delta S_{11}^*)^*}{|S_{22}|^2 - |\Delta|^2}$
Radius: $R_L = \left|\frac{S_{12}S_{21}}{|S_{22}|^2 - |\Delta|^2}\right|$
Input Stability circle: $(\Gamma_{in} = 1)$
Center: $C_S = \frac{(S_{11} - \Delta S_{22}^*)^*}{|S_{11}|^2 - |\Delta|^2}$
Radius: $R_S = \left|\frac{S_{12}S_{21}}{|S_{11}|^2 - |\Delta|^2}\right|$
Analyzing stability circles:

- We find the output stability circle
- The center of the Smith Chart is Z_0
- Consider the load to be $Z_L = Z_0 \to \Gamma_L = 0$

Case 1: if $|S_{11}| < 1$ then from $\Gamma_{in} = \frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}$; $|\Gamma_{in}| < 1$

Case 2: if $|S_{11}| > 1$ then from $\Gamma_{in} = \frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}$; $|\Gamma_{in}| > 1$



7.1.4 Test for unconditional Stability (K-Δ Test)

$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1$

$|\Delta| = |S_{11}S_{22} - S_{12}S_{21}| < 1$
both have to be satisfied simultaneously, if the condition is not satisfied, stability circles should be constructed for the designed frequency and input and output matching network should be designed away from the unstable regions

7.2 Gain

For a two-port network characterized by its S parameter matrix and source and load impedances Z_S and Z_L we can define three types of gain:

- **Voltage Gain**: $A_v = \frac{S_{21}\Gamma_L + S_{21}}{1 - S_{22}\Gamma_L + S_{11}(1 - S_{22}\Gamma_L) + S_{21}\Gamma_L S_{12}}$
- **Power gain** is the ratio of power dissipated in the load to the power delivered to the input of the two-port network $G_P = \frac{P_L}{P_{in}} = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{(1-|\Gamma_{in}|^2)|1-S_{22}\Gamma_L|^2}$
- **Available power gain** is the ratio of the power available from the two-port $G_A = \frac{P_{avn}}{P_{avs}} = \frac{|S_{21}|^2(1-|\Gamma_S|^2)}{(1-|\Gamma_{out}|^2)|1-S_{11}\Gamma_S|^2}$

7.2.1 Transducer Power Gain

G_T is the ratio of the power delivered to the load to the power available from the source

$G_T = \frac{P_L}{P_{avs}} = \frac{|S_{21}|^2(1-|\Gamma_S|^2)(1-|\Gamma_L|^2)}{|1-\Gamma_{in}\Gamma_S|^2|1-S_{22}\Gamma_L|^2}$

- **Input matching gain**: $G_S = \frac{(1-|\Gamma_S|^2)}{|1-\Gamma_{in}\Gamma_S|^2}$, if transistor is unilateral then $S_{12} = 0$ and $G_S = \frac{(1-|\Gamma_S|^2)}{|1-S_{11}\Gamma_S|^2}$
- **Transistor gain**: $G_0 = |S_{21}|^2$
- **Output matching gain**: $G_L = \frac{(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2}$

For unilateral transistor:
$G_{Smax} = \frac{1}{1-|S_{11}|^2}$ $(\Gamma_S = S_{11}^*)$, $g_S = \frac{G_S}{G_{Smax}}$
$G_{Lmax} = \frac{1}{1-|S_{22}|^2}$ $(\Gamma_L = S_{22}^*)$, $g_L = \frac{G_L}{G_{Lmax}}$
Constant input gain circle: $C_S = \frac{g_S S_{11}^*}{1-(1-g_S)|S_{11}|^2}$,
$R_S = \frac{\sqrt{1-g_S}(1-|S_{11}|^2)}{1-(1-g_S)|S_{11}|^2}$
Constant output gain circle: $C_L = \frac{g_L S_{22}^*}{1-(1-g_L)|S_{22}|^2}$,
$R_L = \frac{\sqrt{1-g_L}(1-|S_{22}|^2)}{1-(1-g_L)|S_{22}|^2}$

7.3 Noise

- Unbiased resistor generates a random thermal noise voltage $P_n = (\frac{V_n}{2R})^2; R = \frac{V_n^2}{4R} = kTB; V_n = \sqrt{4kTBR}$
- The measure of signal degradation in the signal-to-noise ration between input and output is called the noise figure: $F = \frac{S_I/N_I}{S_O/N_O}$
- Noise figure of the cascaded system is determined by the Friis formula: $F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} \cdots$

- The noise figure of a two port amplifier can be expressed as: $F = F_{min} + \frac{R_N}{G_S}|Y_S - Y_{S,opt}|$
  - source admittance: $Y_s = G_S + jB_S$
  - source admittance that results in the minimum noise figure$Y_{s,opt}$
  - minimum noise figure when admittance is $Y_{s,opt}$: F_{min}
  - equivalent noise resistance: R_N
- instead of $Y_s, Y_{s,opt}$ we can use $\Gamma_S, \Gamma_{S,opt}$: $F = F_{min} + \frac{4R_N}{Z_0}\frac{|\Gamma_S - \Gamma_{s,opt}|^2}{(1-|\Gamma_s|^2)|1+\Gamma_{s,opt}|^2}$
- We can define a noise figure parameter N: $N = \frac{|\Gamma_S - \Gamma_{s,opt}|^2}{1-|\Gamma_s|^2} = \frac{F-F_{min}}{4R_N/Z_0}|1+\Gamma_{s,opt}|^2$
- circle solution: $C_F = \frac{\Gamma_{opt}}{N+1}$, $R_F = \frac{\sqrt{N(N+1-|\Gamma_{opt}|^2)}}{N+1}$

7.4 Amplifier Design Techniques

7.4.1 Design for specific gain and best possible noise

1. Draw noise circles for several noise figures F close to F_{min}
2. Find the sum of the gains for the input and output matching networks
3. Take into account that maximum gain you can get from the output is $10log\frac{1}{1-|S_{22}|^2}$
4. Draw several G_S and choose which input matching intersects with the smallest noise circle
5. For a fixed G_S now you can calculate G_L and then draw G_L circle
6. Choose Γ_S and Γ_L where G_S and G_L circles intersect the least noise figure circle and are close as possible to center of the Smith chart

7.4.2 Design for specific noise and best possible gain

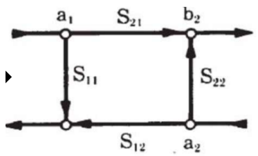
1. $\Gamma_L = \Gamma^*_{out}$ we can maximize it because noise figure does not depend on output. If unilateral : $\Gamma_L = S^*_{22}$
2. Draw the noise circle for desired F
3. Find maximum input matching circuit gain G_S for which the noise circle and input gain circle have 1 common point. That point will be the desired Γ_S
4. Amplifier gain is now:
   $G_T(dB) = 10log|S_{21}|^2 + 10log\frac{1-|\Gamma_S|^2}{|1-\Gamma_{in}\Gamma_S|^2} + 10log\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}$
   if unilateral:
   $G_{TU}(dB) = 10log|S_{21}|^2 + 10log\frac{1-|\Gamma_S|^2}{|1-S_{11}\Gamma_S|^2} + 10log\frac{1}{1-|S_{22}|^2}$

7.4.3 Design for minimum noise and best possible gain

1. $\Gamma_L = \Gamma^*_{out}$ if unilateral: $\Gamma_L = S^*_{22}$
2. Minimum noise occurs for $\Gamma_S = \Gamma_{s,opt}$
3. Amplifier gain is now:
   $G_T(dB) = 10log|S_{21}|^2 + 10log\frac{1-|\Gamma_S|^2}{|1-\Gamma_{in}\Gamma_S|^2} + 10log\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}$
   if unliteral:
   $G_{TU}(dB) = 10log|S_{21}|^2 + 10log\frac{1-|\Gamma_S|^2}{|1-S_{11}\Gamma_S|^2} + 10log\frac{1}{1-|S_{22}|^2}$

7.5 Signal Flow Graphs (SFG)

- A SFG is a directed graph in which nodes represent system variables, and branches (edges, arcs, or arrows) represent functional connections between pairs of nodes.
- SFGs are used to represent the signal flow in electronic networks.
- The SFG here is used in association with our S-parameters!
- each variable (a_1, a_2, b_1, b_2) is marked at a node
- S-Params are marked as branches



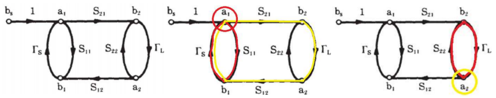
for ex: $b_2 = a_1S_{21} + a_2S_{22}$; $b_1 = a_1S_{11} + a_2S_{12}$

Mason's Rule states:

$$\frac{b_1}{b_s} = T = \frac{(P_1[1-\sum L(1)^{(1)}+\sum L(2)^{(1)}-\sum L(3)^{(1)}...]+P_2[1-\sum L(1)^{(2)}+\cdots]+\cdots)}{1-\sum L(1)+\sum L(2)-\sum L(3)+\cdots}$$

Where:

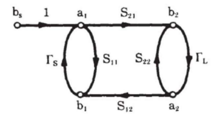
- $\sum L(1)^{(1)}$ is the sum of all **first order loops** that do not touch the first path between the variables
- $\sum L(2)^{(1)}$ is the sum of all **second order loops** that do not touch the first path between the variables
- $\sum L(1)^{(2)}$ is the sum of all **first order loops** that do not touch the second path between the variables
- The denominator is 1 – the sum of all first order loops, plus the sum of all second order loops, minus the third order loops, and so on…

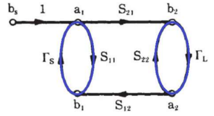


- First Order Loop: is defined as the product of the branches encountered in a journey **starting from a node** and moving in the direction of the arrows **back to that original node.**

  Example: Starting at node a_1: $S_{11}\Gamma_s$ $S_{21}\Gamma_L S_{12}\Gamma_s$
  Starting at node a_2: $S_{22}\Gamma_L$

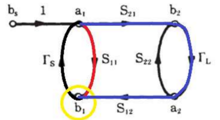
Any of the other loops we encounter includes one of these three first order loops!





Second Order Loop: is defined as the product of any **two non-touching first order loops.** Of the three first order loops just found only $S_{11}\Gamma_s$ and $S_{22}\Gamma_L$ do not touch.

Third Order Loop: is the product of any three non-touching first order loops. (no third order loop for this example)



Let's find the value of b_1. Since b_s is the only independent variable, its value will determine the value of all other variables in the network.

To determine b_1, we first have to identify the paths leading to b_1 from b_S:

- S_{11}
- $S_{21}\Gamma_L S_{12}$

Let's identify the non-touching loops with respect to the paths just found:

- $\boldsymbol{S_{11}}$ → First order loop $S_{22}\Gamma_L$ have no nodes or branches in common
- $\boldsymbol{S_{21}\Gamma_L S_{12}}$ touches all of the network's first order loop → no non-touching loops

It is now time to use **Mason's Rule** to determine the ratio of the variables b_1 and b_s.

Let's apply this rule to our case:

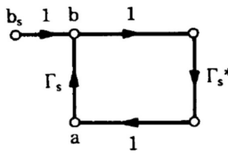
$$\frac{b_1}{b_s} = T = \frac{(S_{11}[1-S_{22}\Gamma_L]+S_{21}\Gamma_L S_{12}[1])}{1-(S_{11}\Gamma_s+S_{22}\Gamma_L+S_{21}\Gamma_L S_{12}\Gamma_s)+S_{11}\Gamma_S S_{22}\Gamma_L}$$

Although the example exploits a rather simple network, the expressions can become complicate.

Mason`s Rule provides a systematic approach to determine various transfer functions (usable for far more complex systems, as the error correction model for a VNA seen during the lecture).

Power Available from the Source

Power available from the source is the power delivered to a conjugate-matched load ($\Gamma_L = \Gamma_s^*$).



$P_{avs} = |b|^2 - |a|^2$

Applying **Mason's Rule**:

$b = \frac{b_s}{1-\Gamma_s\Gamma_s^*}$ $a = \frac{b_s\Gamma_s^*}{1-\Gamma_s\Gamma_s^*}$

So:

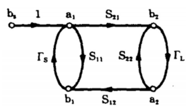
$$P_{avs} = \frac{|b_s|^2(1-|\Gamma_s|^2)}{(1-|\Gamma_s|^2)^2} = \frac{|b_s|^2}{1-|\Gamma_s|^2}$$

Voltage Gain

For a two-port network:

$$A_V = \frac{V_{out}}{V_{in}}$$

Recalling the total voltage on a Tline is $V = V^+ + V^-$:



$$\frac{a_1}{b_s} = \frac{1[1-S_{22}\Gamma_L]}{1-(S_{11}\Gamma_s+S_{22}\Gamma_L+S_{21}\Gamma_L S_{12}\Gamma_s)+S_{11}\Gamma_s S_{22}\Gamma_L}$$
$$\frac{b_1}{b_s} = \frac{S_{11}[1-S_{22}\Gamma_L]+S_{21}\Gamma_L S_{12}[1]}{1-(S_{11}\Gamma_s+S_{22}\Gamma_L+S_{21}\Gamma_L S_{12}\Gamma_s)+S_{11}\Gamma_s S_{22}\Gamma_L}$$
$$\frac{a_2}{b_s} = \frac{S_{21}\Gamma_L[1]}{1-(S_{11}\Gamma_s+S_{22}\Gamma_L+S_{21}\Gamma_L S_{12}\Gamma_s)+S_{11}\Gamma_s S_{22}\Gamma_L}$$
$$\frac{b_2}{b_s} = \frac{S_{21}[1]}{1-(S_{11}\Gamma_s+S_{22}\Gamma_L+S_{21}\Gamma_L S_{12}\Gamma_s)+S_{11}\Gamma_s S_{22}\Gamma_L}$$

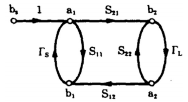
$$A_V = \frac{a_2 + b_2}{a_1 + b_1} \Longrightarrow A_V = \frac{\frac{a_2}{b_s}+\frac{b_2}{b_s}}{\frac{a_1}{b_s}+\frac{b_1}{b_s}} = \frac{S_{21}+S_{21}\Gamma_L}{1(1-S_{22}\Gamma_L)+S_{11}(1-S_{22}\Gamma_L)+S_{21}\Gamma_L S_{12}}$$

Transducer Power Gain

The **Transducer power gain** is defined as the power delivered to a load divided by the power available from a source.

$$G_T = \frac{P_{del}}{P_{avs}} = \frac{\boxed{|b_2|^2(1-|\Gamma_L|^2)}}{|b_s|^2/(1-|\Gamma_s|^2)}$$

$$\boxed{\begin{aligned} P_{del} &= P_{inc} - P_{refl} \\ &= |a|^2 - |b|^2 \end{aligned}}$$



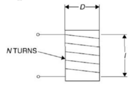
$$\frac{b_2}{b_s} = \frac{S_{21}[1]}{1-(S_{11}\Gamma_s+S_{22}\Gamma_L+S_{21}\Gamma_L S_{12}\Gamma_s)+S_{11}\Gamma_s S_{22}\Gamma_L}$$

$$G_T = \frac{|S_{21}|^2(1-|\Gamma_s|^2)(1-|\Gamma_L|^2)}{|(1-S_{11}\Gamma_s)(1-S_{22}\Gamma_L)-S_{21}\Gamma_L S_{12}\Gamma_s|^2} \xrightarrow{S_{12}\cong 0} G_{TU} = |S_{21}|^2\frac{(1-|\Gamma_s|^2)}{|(1-S_{11}\Gamma_s)|^2}\frac{(1-|\Gamma_L|^2)}{|(1-S_{22}\Gamma_L)|^2}$$

8 Apendix

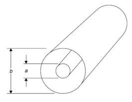
8.1 Designing Components

Spiral Coil:



$L = \frac{n^2r^2}{9r+10l}[\mu H] \to L = \frac{n^2r}{29}$, if $l = 2r$, for higher Q

Wire Inductance:



$L = \frac{\mu_0\mu_r}{2\pi}ln\frac{b}{a}[\frac{H}{m}] \to$ set $\frac{b}{a} = 50 =$ distant ground plane

Plate Capacitor:
$C = \frac{\epsilon_0\epsilon_r A}{d}[F]$

8.2 Slotted T-line

1. A known load e.g. Short is connected:
   a) $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$, here: $Z_L = 0$
   b) $SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \infty$
   c) Distance between 2 minima = $\frac{\lambda}{2}$
2. The unknown load is connected:
   a) $SWR = \frac{1+|\Gamma|}{1-|\Gamma|} \to |\Gamma|$
   b) $e^{j(\theta_L - 2\beta l)} = -1$; $l = l_{minshort} - l_{minload} > 0$
   c) $\Gamma_L = |\Gamma|e^{j\theta_L}$
   d) $Z_L = Z_0(\frac{1+\Gamma_L}{1-\Gamma_L})$

You can also solve from 2.a) with the help of the smith chart: just move "l" towards load.

8.3 Critical Length for Digital Interconnects

$l_c = \frac{t_r \cdot v}{2}$, t_r = rise time, v = wave speed
also used: $l_c = \frac{t_r \cdot v}{1.5}$

8.4 Transmission matrix

The T-matrices are multiplied in the same way as the ABCD parameters when networks are cascaded.

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \frac{S_{12}S_{21}-S_{11}S_{22}}{S_{21}} & \frac{S_{11}}{S_{21}} \\ \frac{-S_{22}}{S_{21}} & \frac{1}{S_{21}} \end{pmatrix}$$

reciprocal: $S_{12} = S_{21} \to T_{11}T_{22} - T_{12}T_{21} = 1$