Lösung E&M Frühling 2015

Andreas Biri (abiri@ethz.ch)

10. Juni 2015

1 Vektor- und Skalarpotentiale

1.

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q^t}{r^2}\log(q)\ \mathbf{e_r}$$
$$\mathbf{H}(\mathbf{r},t) = 0$$

2. -

2 Reflexion an einem dielektrischen Film

1.

$$\mathbf{E_{in}}(\mathbf{r},t) = Re\left\{E_0\ e^{-ik_0z}e^{-i\omega t}\right\}\mathbf{n_x}$$
$$\mathbf{E_{in}}(\mathbf{r}) = \mathbf{E_0}e^{-ik_0z}$$

2.

$$\mathbf{E_{diel}}(\mathbf{r}) = \mathbf{E_{diel,1}}e^{ik_dz} + \mathbf{E_{diel,2}}e^{-ik_dz}$$
$$\mathbf{E_{vac}}(\mathbf{r}) = \mathbf{E_{vac,1}}e^{ik_0z} + \mathbf{E_{vac,2}}e^{-ik_0z}$$

We can rewrite $k_d = \frac{\omega}{c}\sqrt{\varepsilon} = k_0\sqrt{\varepsilon}$.

3.

$$\mathbf{H_{diel}}(\mathbf{r}) = \frac{1}{\omega\mu_0}\left(\left[\begin{bmatrix}0\\0\\k_d\end{bmatrix}\times\mathbf{E_{diel,1}}\right]e^{ik_dz} + \left[\begin{bmatrix}0\\0\\-k_d\end{bmatrix}\times\mathbf{E_{diel,2}}\right]e^{-ik_dz}\right)$$
$$\mathbf{H_{vac}}(\mathbf{r}) = \frac{1}{\omega\mu_0}\left(\left[\begin{bmatrix}0\\0\\k_0\end{bmatrix}\times\mathbf{E_{vac,1}}\right]e^{ik_0z} + \left[\begin{bmatrix}0\\0\\-k_0\end{bmatrix}\times\mathbf{E_{vac,2}}\right]e^{-ik_0z}\right)$$

4.

$$
\begin{aligned}
E_1^{\|} = E_2^{\|}: \quad & \mathbf{E_{vac,1}} e^{ik_0L} + \mathbf{E_{vac,2}} e^{-ik_0L} && = \left(\mathbf{E_{diel,1}} e^{ik_dL} + \mathbf{E_{diel,2}} e^{-ik_dL}\right) && (1) \\
E_2^{\|} = E_3^{\|}: \quad & 0 && = \left(\mathbf{E_{diel,1}} e^{0} + \mathbf{E_{diel,2}} e^{0}\right) && (2) \\
H_1^{\|} = H_2^{\|}: \quad & \mathbf{H_{vac,1,y}} e^{ik_0L} + \mathbf{H_{vac,2,y}} e^{-ik_0L} && = \left(\mathbf{H_{diel,1,y}} e^{ik_dL} + \mathbf{H_{diel,2,y}} e^{-ik_dL}\right) && (3)
\end{aligned}
$$

These boundary conditions can be used to derive the following equation:

$$
\mathbf{E_{ref}}(\mathbf{r}, t) = \frac{\sqrt{\varepsilon}^{-1} \; i\tan(k_dL) + 1}{\sqrt{\varepsilon}^{-1} \; i\tan(k_dL) - 1} e^{-2ik_0L} \; \mathbf{E_0} \tag{4}
$$

5.

$$
I_{ref}(r) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} |\mathbf{E_0}|^2 = I_{in}
$$

The reflected intensity is independent of L and is equal to the incoming intensity. This follows as the material for $z < 0$ is perfectly reflecting, wherefore no energy can exit on this site, and as we only consider lossless materials.

6.

$$
\begin{aligned}
W_{electric} &= 2\varepsilon_0\varepsilon |\mathbf{E_{diel}}|^2 * \sin^2(k_dz) \\
W_{magnetic} &= 2\varepsilon_0\varepsilon |\mathbf{E_{diel}}|^2 * \cos^2(k_dz) \\
W_{tot} &= W_{electric} + W_{magnetic} = 4\varepsilon_0\varepsilon |\mathbf{E_{diel}}|^2
\end{aligned}
$$

By using the known formulae (3) and (4) we find:

$$
\mathbf{E_{diel}} = \frac{e^{-ik_0L}}{\cos(k_dL)(i\tan(k_dL) - \sqrt{\varepsilon})} \mathbf{E_0}
$$

7.

$$
L_n = \frac{\lambda_0}{2\sqrt{\varepsilon}}(n+1), \quad n \in \mathbb{N}
$$

3 Phase eines zeitharmonischen Feldes

Setting the amplitude into the Helmholtz equation and using its imaginary part yields:

$$
\Delta\phi(r) = \nabla^2\phi(r) = div \; (\; \nabla\phi(r) \;) = 0
$$

4 Strahlung einer Dipolschleife

This question is identical to Task 1 in the exercise sheet 8 from FS2015 EFUW.

1. $$\mathbf{E}^{(1)}(\mathbf{r}) = \omega^2 \mu_0 \mu \, p \frac{e^{ikr}}{4\pi r^3} \begin{bmatrix} r^2 - x^2 \\ -xy \\ -xz \end{bmatrix}$$

2. $$\mathbf{E}^{(1)}(\mathbf{r}) = \omega^2 \mu_0 \mu \, p \frac{e^{ik(r+\frac{dy}{r})}}{4\pi r^3} \begin{bmatrix} r^2 - x^2 \\ -xy \\ -xz \end{bmatrix}$$

3. $$\mathbf{E}^{(2)}(\mathbf{r}) = -\omega^2 \mu_0 \mu \, p \frac{e^{ik(r-\frac{dy}{r})}}{4\pi r^3} \begin{bmatrix} r^2 - x^2 \\ -xy \\ -xz \end{bmatrix}$$
$$\mathbf{E}^{(1,2)}(\mathbf{r}) = \omega^2 \mu_0 \mu \, p \frac{e^{ikr}}{4\pi r^3} \begin{bmatrix} r^2 - x^2 \\ -xy \\ -xz \end{bmatrix} 2i \sin(k\frac{dy}{r})$$

4. $$\mathbf{E}^{(3,4)}(\mathbf{r}) = -\omega^2 \mu_0 \mu \, p \frac{e^{ikr}}{4\pi r^3} \begin{bmatrix} -xy \\ r^2 - y^2 \\ -yz \end{bmatrix} 2i \sin(k\frac{dx}{r})$$

5. Use the small-angle approximation $\sin(k\frac{dy}{r}) = \sin(\frac{2\pi dy}{\lambda r}) \approx k\frac{dy}{r}$.

6. $$E_r = 0$$
$$E_\theta = 0$$
$$E_\varphi = -2i\omega^2 \mu_0 \, p \, \frac{e^{ikr}}{4\pi r} kd \sin\theta$$
As the field components are zero except for E_φ, the field is transversal.

7. The four electrical dipole form a current loop, which seems just like the source of a magnetic dipole in the far field. As the loop is situated in the xy-plane and flows counter-clockwise, the magnetic dipole points in the positive z-direction.

8. $$I = \frac{\omega^4 \mu_0 m^2}{32 \, \pi^2 c^3 r^2} \sin^2\theta$$
$$\bar{P} = \frac{\omega^4 \mu_0 m^2}{12 \, \pi c^3} = 4d^2\omega^2 \; Z^2 \bar{P}_{electric}$$
As the radiated power scales with d^2, it is proportional to the area of the current loop.