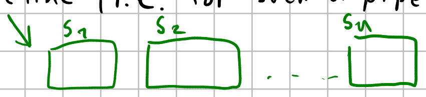
For a pipeline with exactly one execution unit per stage, prove that

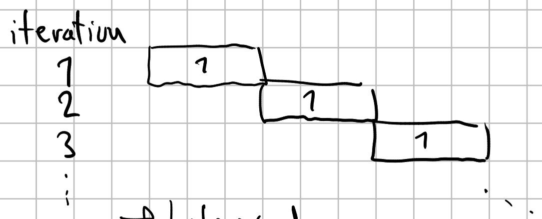
pipeline balanced <=> no stage longer than the first

$\Leftarrow$: Prove for any such pipeline, i.e. for such a pipeline of any length



induction over length

n=1:

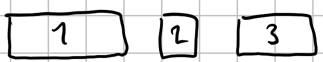


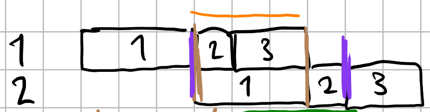
→ balanced

I.H.: Assume statement holds for some n, i.e. for any pipeline of length n where no stage is longer than stage 1, the pipeline is balanced.

n → n+1:

Proof idea:





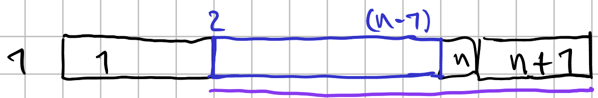
does stage 2 have to wait? $d_2 < d_1 \Rightarrow$ no

does stage 3 have to wait? $\cancel{d_2} + d_3 < d_1 + \cancel{d_2} \Rightarrow$ no

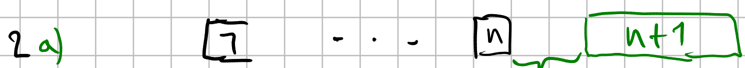
Consider a pipeline of length n+1:

To show: all iterations have the same latency.

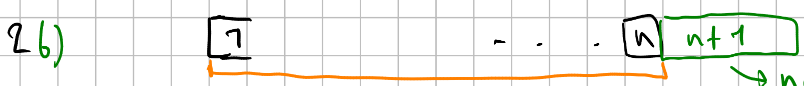
We look at iter. 1 and 2.



when does iteration 2 finish stage n?



bubble → unbalanced



→ no bubble

By I.H. we know that the pipeline $1, \dots, n$ is balanced $\overset{\nabla}{\cdot}$

$\Rightarrow$ duration of is $d_1 + d_2 + \dots + d_n$ $\quad$ ($d_i :=$ duration of stage i)

duration of is $d_2 + d_3 + \dots + d_{n+1}$

$\Rightarrow$ $-$ $= d_1 - d_n \geq 0$

$\Rightarrow$ stage n ends after $n+1$ in the prev. iteration

$\Rightarrow$ duration of iteration 2 is $d_1 + \dots + d_n + d_{n+1} =$ duration of iter. 1

Then the same argument can be applied for all other iterations

(could do this with induction to be more precise)

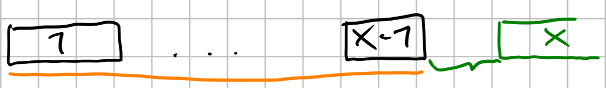
$\Rightarrow$: We prove it indirectly:

there exists a stage longer than the first $\Rightarrow$ pipeline unbalanced

Let x be the first stage longer than the first

$d_1 < d_x$,

1. 

2. a)  $\rightarrow$ unbalanced

2. b)  $\rightarrow$ ok

Since x is the first longer stage and $d_1 \geq d_i \ \forall i \in \{2, \dots, x\}$,

the pipeline $1, \dots, x-1$ is balanced $\overset{\nabla}{\cdot}$

$\Rightarrow$ $= d_1 + d_2 + \dots + d_{x-1}$

$= d_2 + d_3 + \dots + d_x$

$\Rightarrow$ $-$ $= d_x - d_1 > 0$

$\Rightarrow$ stage x will have to wait in iter. 2

$\Rightarrow$ iter 2. longer than iter. 1

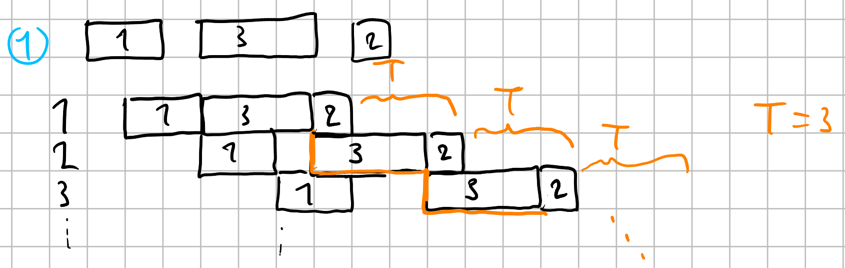
$\Rightarrow$ pipeline unbalanced

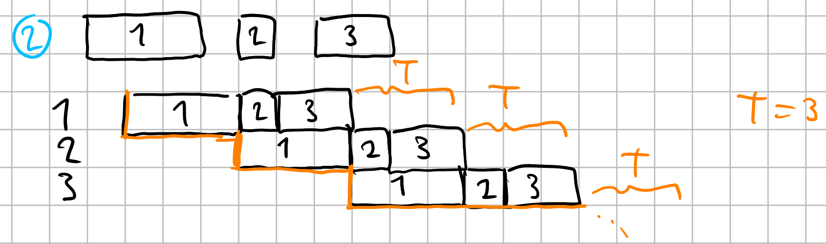
$\square$

Prove that (in a pipeline with 1 execution unit per stage)

$$\text{throughput} = \frac{1}{\text{length of longest stage}}$$

Examples:





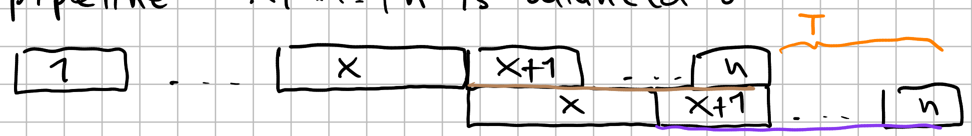
Let x the longest stage.

•: We see that x in iteration $k+1$ starts right when iter. k finishes with x (†)

Assuming (†) were always true:

If we look at the execution from stages x to n, then this is the execution of the pipeline with stages $x, \dots, n$. Since $d_x \geq d_i \ \forall i$, the pipeline $x, \dots, n$ is balanced $\overset{\nabla}{\circ}$

$\Rightarrow$



$\square = \square$ because balanced

$$\Rightarrow T = d_x + \underline{d_{x+1} + \dots + d_n} - (\underline{d_{x+1} + \dots + d_n})$$

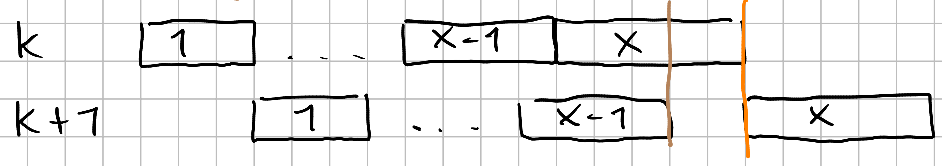
$$= \underline{d_x}$$

and T same for every iteration

Now we prove (†):

The earliest x can start in iter. $k+1$ is when iter. k finishes x

$\Rightarrow$ we need to show that iter. $k+1$ always finishes stages $1, \ldots, x-1$ before ($\le$) iter. k finishes x.



Trick: Consider the pipeline $0, 1, \ldots, x, \ldots, n$ with $d_0 = d_x$.

This pipeline is balanced, since $d_0 = d_x \ge d_i \; \forall i$!

pipeline balanced $\Rightarrow$ duration is sum of stages:

$$= d_0 + d_1 + \ldots + d_{x-1}$$
$$= d_x + d_1 + \ldots + d_{x-1}$$
$$=$$

$\Rightarrow$ We finish $x-1$ in iter. $k+1$ exactly when we finish x in iter. k

Since $d_0 \ge d_1$, adding stage 0 only delayed the execution of stages $1, \ldots, x-1$ by $d_0 - d_1$ in iter. $k+1$ compared to the original pipeline.

$\Rightarrow$ stages $1, \ldots, x-1$ in iter. $k+1$ can finish before iter. k finishes x.

$\Rightarrow$ (†) proven

$\square$