ETH zürich

Exercise Session 07 – Functors, Lambdas

Data Structures and Algorithms

These slides are based on those of the course, but were adapted and extended by the teaching assistant Adel Gavranović

Today's Schedule

- Intro
- Follow-up
- Learning Objectives
- Quadtrees
- Code-Example
- Higher Order Functions
  - Function Signature Notation
- Tree Recap
  - Recap 2-3 Trees
  - Recap Red-Black Trees
- Outro

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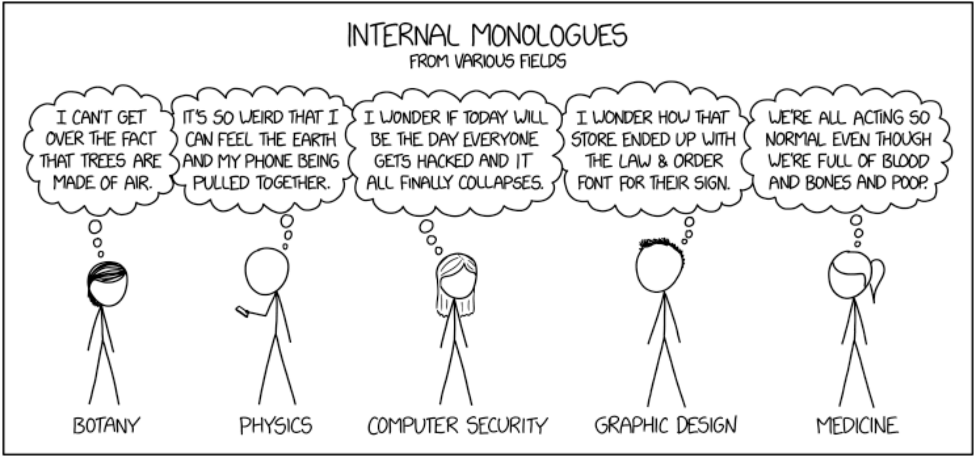
- Exercise Session Material
- Adel's Webpage
- Mail to Adel

Comic of the Week



▸ xkcd

1. Intro

Intro

- Welcome back!

2. Follow-up

Follow-up from last exercise session

- Red-Black tree from last session (S06)

Questions regarding **code** expert from your side?

3. Learning Objectives

Learning Objectives

Quadtrees

- ☐ Understand what *quadtrees* are where they are used
- ☐ Understand the minimization problem behind *quadtrees*

Red-Black Trees

- ☐ Be able to perform basic operations on the most common trees, in particular Red-Black trees

Functors

- ☐ Know what *Functors* are
- ☐ Understand how *Functors* work
- ☐ Know where and how to use *Functors*

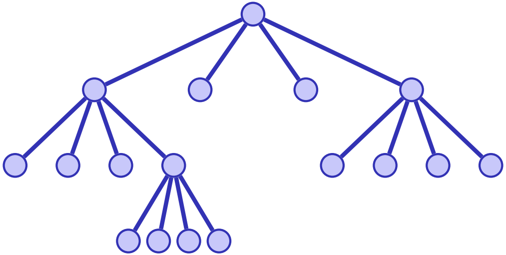
Lambdas

- ☐ Know what *Lambdas* are
- ☐ Understand how *Lambdas* work
- ☐ Know where and how to use *Lambdas*

4. Quadtrees

Quadtrees

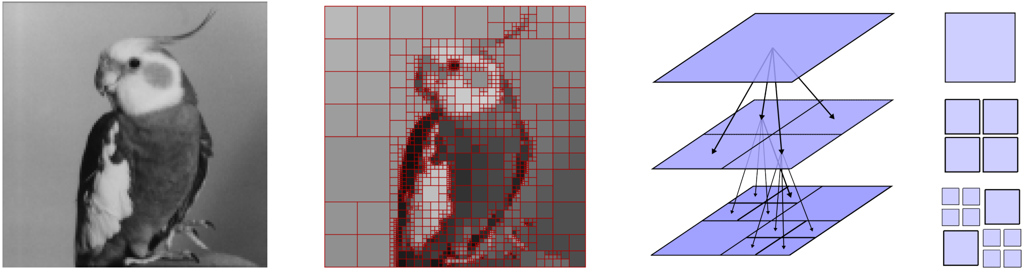
Quadtrees are trees where each node has at most **four** children.



Main application: Image processing.

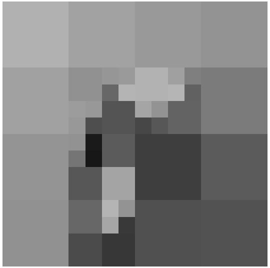
Quadtrees for Image Compression

Insight: (1) Divide image recursively into four regions, (2) map the regions to nodes in a quadtree und (3) assign each leaf the average color of its region.



Quadtrees for Image Compression

When and where to stop the recursion?

Too early? Is this better? **Question**: Should we stop when each node is mapped to a pixel? **Answer**: We would get the original image but gain no storage efficiency.

Quadtrees for Image Compression

We want

- as close approximation as possible, und
- as few nodes as possible.

This can be expressed as an optimization problem:

$$H_\gamma(T, \boldsymbol{y}) := \gamma \cdot \underbrace{|L(T)|}_{\text{Number of leaves}} + \underbrace{\sum_{r \in L(T)} \|\boldsymbol{y}_r - \boldsymbol{\mu}_r\|_2^2}_{\text{Cumulative approximation error of all leaves}}$$

where T is a quadtree, $\boldsymbol{y}$ is the image data, and $\gamma \geq 0$ is a regularization parameter. For a given γ we seek the optimal solution $\arg\min_T H_\gamma(T, \boldsymbol{y})$.

Quadtrees for Image Compression

$$H_\gamma(T, \boldsymbol{y}) := \gamma \cdot \underbrace{|L(T)|}_{\text{Number of leaves}} + \underbrace{\sum_{r \in L(T)} \|\boldsymbol{y}_r - \boldsymbol{\mu}_r\|_2^2}_{\text{Cumulative approximation error of all leaves}}$$

Question: What is the effect of a low value of γ?

Answer: Improves the approximation at the expense of increasing the size of the quadtree.

Algorithm: Minimize($\boldsymbol{y}$,r,γ)

Input: Image data $\boldsymbol{y} \in \mathbb{R}^S$, rectangle $r \subset S$, regularization $\gamma > 0$.
Output: $\min_T \gamma|L(T)| + \|\boldsymbol{y} - \boldsymbol{\mu}_{L(T)}\|_2^2$

if $|r| = 0$ **then return** 0
$m \leftarrow \gamma + \sum_{s \in r}(y_s - \mu_r)^2$
if $|r| > 1$ **then**
  Split r into r_{ll},r_{lr},r_{ul},r_{ur}
  $m_1 \leftarrow \mathrm{Minimize}(y, r_{ll}, \gamma)$; $m_2 \leftarrow \mathrm{Minimize}(y, r_{lr}, \gamma)$
  $m_3 \leftarrow \mathrm{Minimize}(y, r_{ul}, \gamma)$; $m_4 \leftarrow \mathrm{Minimize}(y, r_{ur}, \gamma)$
  $m' \leftarrow m_1 + m_2 + m_3 + m_4$
else
  $m' \leftarrow \infty$
if $m' < m$ **then** $m \leftarrow m'$
return m

Code-Example

Quadtrees on **code** expert

- Region-Point Quadtree

6. Higher Order Functions

Motivation

- Overarching goal: make code generic, thus reusable
- Templates so far: make code parametric in the data it operates on, e.g.
  - `Pair<T>` for all types `T`
  - `print<C>` for all iterable containers `C`
- Now: make code parametric in the algorithms it uses, e.g.
  - `filter(container, predicate)`
  - `apply(signal, transformation/filter)`
  - `leader_election(participants, protocol)`
  - `navigation_system(map, shortest_path_algorithm)`
  - `Button("Save").onClick(handle_click_event)`

Callables and Higher-Order Functions

```
// generic filter function
template <typename C, typename P>
C filter(const C& src_data, P pred) {
  C data;

  for (const auto& e : src_data)
    if (pred(e)) data.push_back(e);

  return data;
}
```

- `pred` must be callable (applicable, invocable), i.e., something function-like
- In C++:
  - free or member function
  - lambda function
  - functor (object with `operator()`)
  - `std::function` object
  - function pointers [not discussed]

Functions taking or returning functions are called **higher-order functions**.

C++Functors

```
// generic filter function
template <typename C, typename P>
C filter(const C& src_data, P pred) {
  C data;
  for (const auto& e : src_data)
    if (pred(e)) data.push_back(e);
  return data;
}
```

```
// stateful predicate as functor
template <typename T>
struct AtLeast {
  T min;

  AtLeast(T m): min(m) {};

  bool operator()(T i) const {
    return min <= i;
  }
};
```

- A functor
  - is it's an object that implements `operator()`
  - combines state (since an object) with callability (with the `operator()`)
- Objects of type `AtLeast<T>` are callable with one `T` argument.

```
std::vector<int> data = {-1,0,1,2,-2,4,5,-3};
selection1 = filter(data, AtLeast(-1));
// = {-1,0,1,2,4,5}
selection2 = filter(data, AtLeast(4));
// = {4,5}
```

Lambda Expressions Translate to Functors

```
std::vector<int> data = {-1,0,1,2,-2,4,5,-3};

auto selection1 = filter(data, [](int e) { return -2 <= e; });
auto selection2 = filter(data, [](int e) { return e != 0; });
```

```
struct lambda1 {
  bool operator()(int e) const {
    return -2 <= e;
  }
};

struct lambda2 {
  bool operator()(int e) const {
    return e != 0;
  }
};
```

- C++compiler generates functors from lambda expressions
- Lambdas are not essential, but “merely” convenient

Lambda Expression Syntax

Most general shape:

$$\underbrace{[e_1, \ldots, e_n]}_{\text{captures}}\ \underbrace{(T_1 x_1, \ldots, T_n x_n)}_{\text{parameters}}\ \underbrace{\texttt{->}\ \texttt{R}}_{\text{return type}}\ \underbrace{\texttt{\{ stmt \}}}_{\text{body}}$$

Captures declare context variables the lambda’s body can access. Syntax examples:

- `[]` no context access
- `[x]` `x` is copied (and `const`)
- `[&x]` `x` is accessible by reference
- `[x, &y]` `x` is copied, `y` is referenced
- `[&]` all necessary variables are automatically referenced
- `[=]` all necessary variables are automatically copied
- `[&, x]` all necessary variables are referenced, except `x`, which is copied
- `[=, &x]` all necessary variables are copied, except `x`, which is referenced

Functors

1. Write down the functor that corresponds to the lambda
2. Use the functor in the filter(...) expression

```cpp
unsigned count = 0;
int min = 3;
std::vector<int> data = {4,-2,0};
data = filter(data, [&, min](int e) {
  ++count; return min <= e;
});
```

Solution

```cpp
class lambda1 {
  unsigned& count;
  int min;
public:
  lambda1(unsigned& c, int m):
    count(c), min(m) {}
  bool operator()(int e) const {
    ++count;
    return min <= e;
  }
};
```

```cpp
unsigned count = 0;
int min = 3;
std::vector<int> data = {4,-2,0};

data = filter(data, lambda1(count, min));
```

Functors

- Observe that the lambda now uses the `auto` type placeholder for its argument

```
data = filter(data, [](auto e) { return 0 <= e; });
```

- Question: How is this reflected by the generated functor?
- Solution:

```
class lambda2 {
public:
  lambda2() {}

  template <typename T>
  bool operator()(T e) const {
    return 0 <= e;
  }
};
```

6.1 Function Signature Notation

not exam relevant

Function Signature Notation

- In the context of functional programming, function signatures are often expressed in a mathematics-inspired notation
- Convention today: upper-case letters denote type parameters, lower-case names denote concrete types
- Examples:
  - $f_1 : A \to \text{int}$ function from any type to integer
  - $f_2 : A \times A \times A \to \text{bool}$ function from three A's to boolean
  - $f_3 : A \times (A \to B) \to B$ "higher-order function" (with two arguments)
  - $f_4 : \texttt{vec<A>} \times (A \to B) \to \texttt{vec<B>}$ higher-order function involving vectors
  - $f_5 : (A \times A \to B) \times A \to ((A \to B) \to \text{bool})$ taking and returning a function

Function Signature Notation: Example 1

- Task: Write down a function with signature $f_2 : A \times A \to \text{bool}$
- Solution:

```
template <typename A>
bool eq(A a1, A a2) {
  return a1 == a2;
}
```

Function Signature Notation: Example 2

- Task: Write down a function with signature $f_2 : A \times (A \to B) \to B$
- Solution 1:

```
template <typename A, typename F>
auto apply1(A a, F a_to_b) {
  return a_to_b(a);
}

int i1 = apply1('a', [](char c) { return c - 65; });
```

- Observations
  - type parameter `B` is only implicitly given, as `F`'s return type
  - template type parameters inferred at call-site

Function Signature Notation: Example 2

- Task: Write down a function with signature $f_2 : A \times (A \to B) \to B$
- Solution 2:

```
template <typename A, typename B>
B apply2(A a, std::function<B(A)> a_to_b) {
  return a_to_b(a);
}

int i2 = apply2('a', std::function([](char c) { return c - 65; }));
```

- Observations
  - type parameter `B` is explicit
  - but we need to wrap the lambda in a `std::function`
  - template type parameters inferred at call-site

Function Signature Notation: Example 2

- Task: Write down a function with signature $f_2 : A \times (A \to B) \to B$
- ~~Solution~~ Attempt 3

```
template <typename A, typename F, typename B>
B apply3(A a, F a_to_b) {
  return a_to_b(a);
}

int i3 = apply3<char, ???, int>('a', [](char c) { return c - 65; });
```

- Observations
  - type parameter `B` is explicit
  - but not directly connected to return type of `F`
- Problem: At call-site, `B` can't be inferred. We can explicitly instantiate `B` - but now we'd have to do that for `F` as well, which we can't.

Function Signature Notation: Example 3

- Task: Write down a function with signature
  $f_2 : A \times (A \times A \to B) \to (A \to B)$
- Solution:

```
template <typename A, typename F>
auto bind(A a1, F aa_to_b) {
  return [=](A a2) { return aa_to_b(a1, a2); };
}

std::string planet = "Mars";
auto f = bind([](auto s1, auto s2) { return s1 + s2; }, planet);
```

- Question: how to use `f`?
- Answer:

```
std::cout << f(" is the fourth planet from the sun.");
```

Function Signature Notation: Example 3

- Task: Write down a function with signature
  $f_2 : A \times (A \times A \to B) \to (A \to B)$
- Solution:

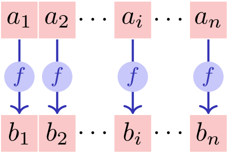
```
template <typename A, typename F>
auto bind(A a1, F aa_to_b) {
  return [=](A a2) { return aa_to_b(a1, a2); };
}

std::string planet = "Mars";
auto f = bind([](auto s1, auto s2) { return s1 + s2; }, planet);
```

- Question: What would happen if the capture were `[&]` instead of `[=]`?
- Answer: The returned lambda would capture argument `a1` by reference, but `a1` is removed from memory when the call to `bind()` terminates. Calling `f` would thus result in undefined behaviour.

A Prominent Higher Order Function

- Consider the function $m : \texttt{vec<A>} \times (A \to B) \to \texttt{vec<B>}$
- Given the signature above, what could function m do?
- Visual hint:



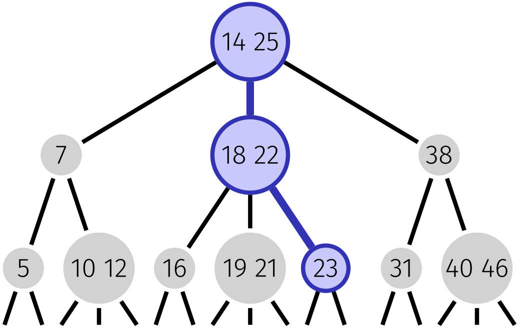
- Task: Implement the function in $\mathrm{C{+}{+}}$
- Solution:

```
template <typename A, typename B>
std::vector<B> map(std::vector<A> as, std::function<B(A)> f) {
  std::vector<B> result;
  for (const auto& a : as)
    result.push_back(f(a));
  return result;
}
```

7. Tree Recap

7.1 Recap 2-3 Trees

Searching

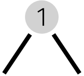


$\texttt{search}(23) \rightarrow \texttt{found}$

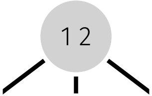
2-3 Tree: Insertion

Insert the keys $1, \ldots, 7$ into an (initially empty) 2-3-tree. Draw the tree after every step ($\texttt{split}/\texttt{propagate}$, $\texttt{join}$, $\ldots$).

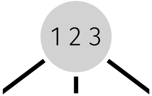
2-3 Tree: Insertion



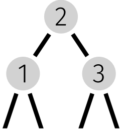
`insert(1)`: new node



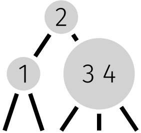
`insert(2)`: `join`



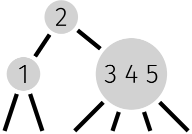
`insert(3)`: 4-node



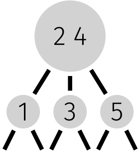
`insert(3)`: `split/propagate`



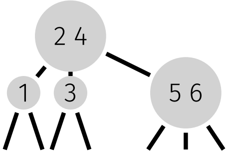
`insert(4)`: `join`



`insert(5)`: 4-node

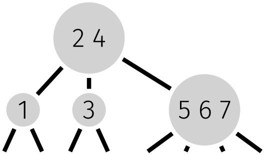


`insert(5)`: `split/propagate`

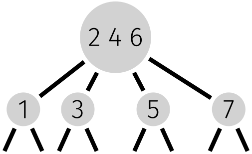


`insert(6)`: `join`

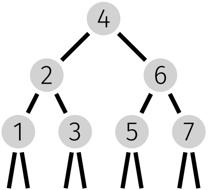
2-3 Tree: Insertion



`insert(7)`:
4-node

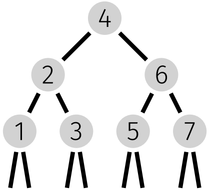


`insert(7)`:
`split/propagate`



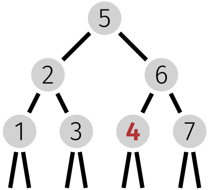
`insert(7)`:
`split/propagate`

2-4 Tree: Deletion

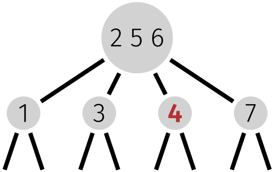


Delete key 4 from the resulting tree.

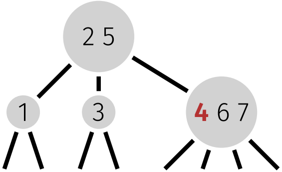
2-4 Tree: Deletion



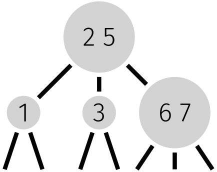
1. swap



2. create 4-node at root



3. combine with sibling

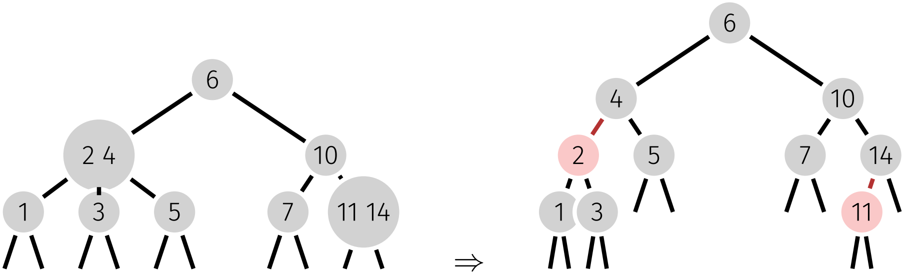


4. delete key

7.2 Recap Red-Black Trees

Red-Black Trees

Draw the following 2-3 tree as a red-black tree.



Red-Black Trees: True or False?

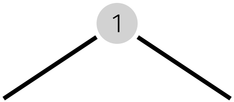
1. right spine (path going right from root) has length $\lceil \log_2(n+1) \rceil$.
   Correct, since there are no right-leaning red edges and we have perfect black balance.
2. the number of red edges is at most the number of black edges.
   Wrong, a tree with 2 nodes and one edge must have a red edge but not black edge.
3. All nodes in the left subtree of a node are smaller than the node.
   Correct, since a red-black tree is a search tree.

Red-Black Trees: Insertion

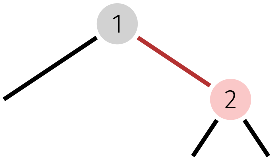
Insert the numbers $1, \ldots, 7$ into an (initially empty) red-black tree and draw the tree after every step.

Compare your steps with your result for the 2-3 tree before.

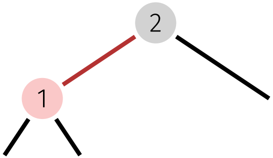
Red-Black Trees: Insertion



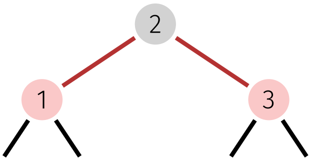
`insert(1)`



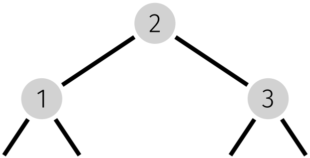
`insert(2): add`



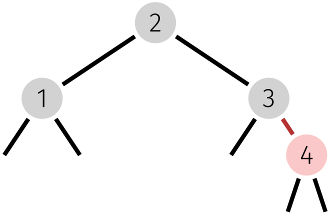
`insert(2): rotate_left`
(since right child red)



`insert(3): add`

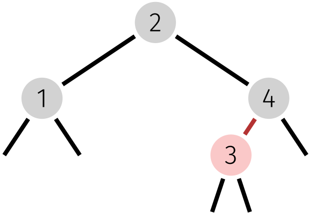


`insert(3): push_up`
(two red children)

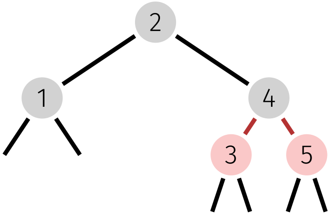


`insert(4): add`

Red-Black Trees: Insertion

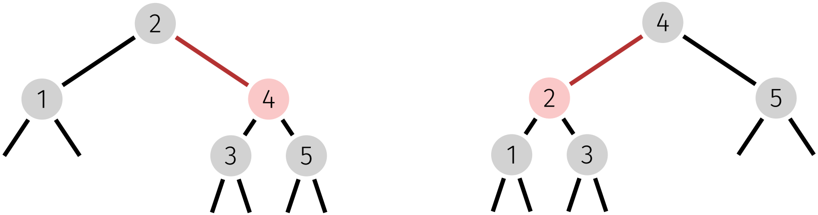


`insert(4): rotate_left`
(since right child red)



`insert(5): add`

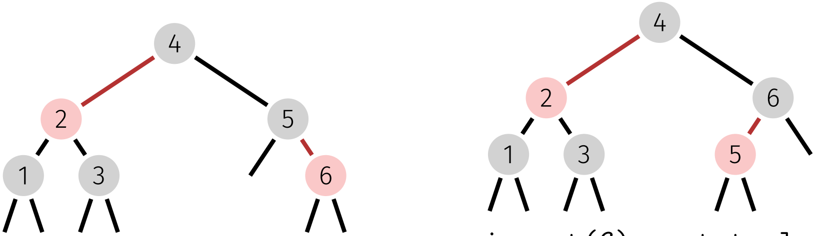
Red-Black Trees: Insertion



`insert(5): push_up`
(two children red)

`insert(5): rotate_left`
(since right child red)

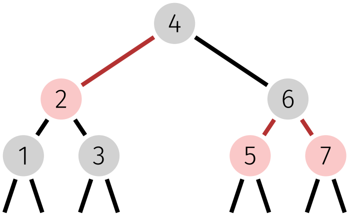
Red-Black Trees: Insertion



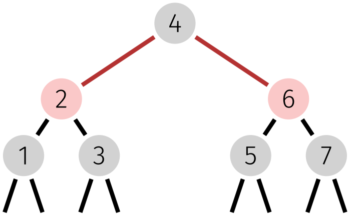
`insert(6): add`

`insert(6): rotate_left`
(since right child red)

Red-Black Trees: Insertion

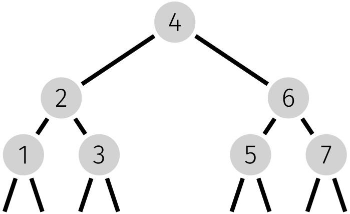


`insert(7): add`



`insert(7): push_up`
(since two children red)

Red-Black Trees: Insertion



`insert(7): push_up`

(since both children red)

8. Outro

General Questions?

See you next time

Have a nice week!

PVK Demand Analysis

- **Chemie**
- **D&A**