Datastructures and Algorithms

Single Source Shortest Paths Algorithms (A*, Bellman-Ford),
All Pairs Shortest Paths Algorithms (Floyd-Warshall)

Overview

- Learning Objectives
- Recap Theory
  - Shortest Paths
  - All Pairs Shortest Paths
- Code-Expert Example

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Material

Webpage

Mail

1. Follow-up

Follow-up from last session

- Welcome back!

2. Feedback regarding **code** expert

General things regarding **code** expert

- If anyone needs more XP for unlocking the Bonus Exercises, let me know via (your ETH) mail!

Any questions regarding **code** expert on your part?

3. Learning Objectives

Objectives

Understand how and why…

- ☐ …the A* algorithm,
- ☐ …the Bellman-Ford algorithm,
- ☐ …and the Floyd-Warshall algorithm

work and when to use which.

4. Summary

Getting on the same page

- What did you cover in the lectures?
- Did you cover Minimum Spanning Trees?
- Did you cover Directed Acylic Graphs?
- Did you cover Topological Sorting?

5. Recap Theory

5. Recap Theory

5.1. Shortest Paths

A*-Algorithm($G, s, t, \hat{h}$)

From the Lecture

Input: Positively weighted Graph $G = (V, E, c)$, starting point $s \in V$, end point $t \in V$, estimate $\widehat{h}(v) \leq \delta(v, t)$

Output: Existence and value of a shortest path from s to t

foreach $u \in V$ **do**
  $d[u] \leftarrow \infty$; $\widehat{f}[u] \leftarrow \infty$; $\pi[u] \leftarrow$ null

$d[s] \leftarrow 0$; $\widehat{f}[s] \leftarrow \widehat{h}(s)$; $U \leftarrow \{s\}$
while $U \neq \emptyset$ **do**
  $u \leftarrow$ ExtractMin$_{\widehat{f}}(U)$
  if $u = t$ **then return** success
  foreach $v \in N^+(u)$ with $d[v] > d[u] + c((u, v))$ **do**
    $d[v] \leftarrow d[u] + c((u, v))$; $\widehat{f}[v] \leftarrow d[v] + \widehat{h}(v)$; $\pi[v] \leftarrow u$
    $U \leftarrow U \cup \{v\}$

return failure

Properties

- The A*-Algorithm is an extension of the Dijkstra algortihm by a distance heuristic $\hat{h}$.
- A* is Dijkstra if $\hat{h} \equiv 0$
- Underestimation: $\forall v \in V: \hat{h}(v) \leq \delta(v, t)$
  If $\hat{h}$ underestimates the real distance, the algorithm works correctly.
- Monotonicity: $\forall (u, v) \in E: \widehat{h}(u) \leq c((u, v)) + \widehat{h}(v)$
  If $\hat{h}$ is monotone in addition, then the algorithm works efficiently.

General Weighted Graphs

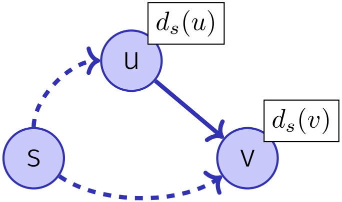
Relax(u, v) ($u, v \in V$, $(u, v) \in E$)

if $d_s(v) > d_s(u) + c(u, v)$ **then**

  $d_s(v) \leftarrow d_s(u) + c(u, v)$

  return true

return false



Problem: cycles with negative weights can shorten the path, a shortest path is not guaranteed to exist.

Dynamic Programming Approach (Bellman)

Induction over number of edges $d_s[i,v]$:
Shortest path from s to v via maximally i edges.

$$d_s[i,v] = \min\left\{ d_s[i-1,v], \min_{(u,v)\in E} \{d_s[i-1,u] + c(u,v)\} \right\}$$

$$d_s[0,s] = 0, d_s[0,v] = \infty \ \forall v \neq s.$$

Algorithm Bellman-Ford(G, s)

Input: Graph $G = (V, E, c)$, starting point $s \in V$
Output: If return value true, minimal weights d for all shortest paths from s, otherwise no shortest path.

foreach $u \in V$ **do**
    $d_s[u] \leftarrow \infty$; $\pi_s[u] \leftarrow$ null
$d_s[s] \leftarrow 0$;
for $i \leftarrow 1$ **to** $|V|$ **do**
    $f \leftarrow$ false
    foreach $(u, v) \in E$ **do**
        $f \leftarrow f \vee \mathrm{Relax}(u, v)$
    if $f =$ false **then return** true
return false;

Quiz: Single Source Shortest Paths

$n := |V|, m := |E|$

problem	method	runtime	dense $m \in \mathcal{O}(n^2)$	sparse $m \in \mathcal{O}(n)$
$c \equiv 1$	BFS	$\mathcal{O}(m+n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n)$
DAG[1] ($c \in \mathbb{R}$)	Top-Sort	$\mathcal{O}(m+n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n)$
$c \geq 0$	Dijkstra	$\mathcal{O}((m+n)\log n)$	$\mathcal{O}(n^2 \log n)$	$\mathcal{O}(n \log n)$
general ($c \in \mathbb{R}$)	Bellman-Ford	$\mathcal{O}(m \cdot n)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^2)$

[1] Directed Acyclic Graph

5. Recap Theory

5.2. All Pairs Shortest Paths

DP Algorithm Floyd-Warshall(G)

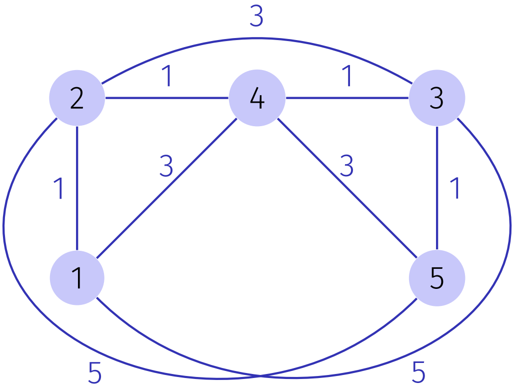
Input: Acyclic Graph $G = (V, E, c)$
Output: Minimal weights of all paths d

$d^0 \leftarrow c$
for $k \leftarrow 1$ **to** $|V|$ **do**
    for $i \leftarrow 1$ **to** $|V|$ **do**
        for $j \leftarrow 1$ **to** $|V|$ **do**
            $d^k(v_i, v_j) = \min\{d^{k-1}(v_i, v_j), d^{k-1}(v_i, v_k) + d^{k-1}(v_k, v_j)\}$

Runtime: $\Theta(|V|^3)$

Remark: Algorithm can be executed with a single matrix d (in place).

Example



adjacency matrix $M = c$

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 5 & 3 & \infty \\
2 & 1 & 0 & 3 & 1 & 5 \\
3 & 5 & 3 & 0 & 1 & 1 \\
4 & 3 & 1 & 1 & 0 & 3 \\
5 & \infty & 5 & 1 & 3 & 0
\end{array}
$$

Example

$$
\begin{array}{c}
k = 1 \\
\begin{array}{ccccc}
0 & 1 & 5 & 3 & \infty \\
1 & 0 & 3 & 1 & 5 \\
5 & 3 & 0 & 1 & 1 \\
3 & 1 & 1 & 0 & 3 \\
\infty & 5 & 1 & 3 & 0
\end{array} \\
d^0
\end{array}
\quad
\begin{array}{c}
k = 2 \\
\begin{array}{ccccc}
0 & 1 & 5 & 3 & \infty \\
1 & 0 & 3 & 1 & 5 \\
5 & 3 & 0 & 1 & 1 \\
3 & 1 & 1 & 0 & 3 \\
\infty & 5 & 1 & 3 & 0
\end{array} \\
d^1
\end{array}
\quad
\begin{array}{c}
k = 3 \\
\begin{array}{ccccc}
0 & 1 & 4 & 2 & 6 \\
1 & 0 & 3 & 1 & 5 \\
4 & 3 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 3 \\
6 & 5 & 1 & 3 & 0
\end{array} \\
d^2
\end{array}
\quad
\begin{array}{c}
k = 4 \\
\begin{array}{ccccc}
0 & 1 & 4 & 2 & 5 \\
1 & 0 & 3 & 1 & 4 \\
4 & 3 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 2 \\
5 & 4 & 1 & 2 & 0
\end{array} \\
d^3
\end{array}
\quad
\begin{array}{c}
k = 5 \\
\begin{array}{ccccc}
0 & 1 & 3 & 2 & 4 \\
1 & 0 & 2 & 1 & 3 \\
3 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 2 \\
4 & 3 & 1 & 2 & 0
\end{array} \\
d^4
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccc}
0 & 1 & 5 & 3 & \infty \\
1 & 0 & 3 & 1 & 5 \\
5 & 3 & 0 & 1 & 1 \\
3 & 1 & 1 & 0 & 3 \\
\infty & 5 & 1 & 3 & 0
\end{array} \\
d^1
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc}
0 & 1 & \mathbf{4} & \mathbf{2} & \mathbf{6} \\
1 & 0 & 3 & 1 & 5 \\
\mathbf{4} & 3 & 0 & 1 & 1 \\
\mathbf{2} & 1 & 1 & 0 & 3 \\
\mathbf{6} & 5 & 1 & 3 & 0
\end{array} \\
d^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc}
0 & 1 & 4 & 2 & \mathbf{5} \\
1 & 0 & 3 & 1 & \mathbf{4} \\
4 & 3 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & \mathbf{2} \\
\mathbf{5} & \mathbf{4} & 1 & \mathbf{2} & 0
\end{array} \\
d^3
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc}
0 & 1 & \mathbf{3} & 2 & \mathbf{4} \\
1 & 0 & \mathbf{2} & 1 & \mathbf{3} \\
\mathbf{3} & \mathbf{2} & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 2 \\
\mathbf{4} & \mathbf{3} & 1 & 2 & 0
\end{array} \\
d^4
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc}
0 & 1 & 3 & 2 & 4 \\
1 & 0 & 2 & 1 & 3 \\
3 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 2 \\
4 & 3 & 1 & 2 & 0
\end{array} \\
d^5
\end{array}
$$

Shortest Path for each Pair?

$$
\begin{array}{cccc}
M & D := d^5 & & D'' \\
\begin{matrix} 0 & 1 & 5 & 3 & \infty \\ 1 & 0 & 3 & 1 & 5 \\ 5 & 3 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 3 \\ \infty & 5 & 1 & 3 & 0 \end{matrix}
&
\begin{matrix} 0 & 1 & 3 & 2 & 4 \\ 1 & 0 & 2 & 1 & 3 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 3 & 1 & 2 & 0 \end{matrix}
&
\begin{matrix} 0 & 1 & 3 & 2 & 4 \\ 1 & 0 & 2 & 1 & 3 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 3 & 1 & 2 & 0 \end{matrix}
&
\begin{matrix} 0 & 1 & 3 & 2 & 4 \\ 1 & 0 & 2 & 1 & 3 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 3 & 1 & 2 & 0 \end{matrix}
\end{array}
$$

Question: Can we use the computed matrix D to determine the shortest path between each pair of nodes?

Direct connections $i \to j$ where $M[i,j] = D[i,j]$ (cf markings in D' above)

Could try to run the algorithm backwards. Example $1 \to 3$ above in D''. Find, with decreasing k, the first fitting candidate.

Complicated and inefficient.

Idea

Memorize the best k in the algorithm for each node pair (i, j) in a matrix M.
Start with matrix of existing direct connections (edges)

Example

$$k = 1$$

$$D = \begin{array}{ccccc} 0 & 1 & 5 & 3 & \infty \\ 1 & 0 & 3 & 1 & 5 \\ 5 & 3 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 3 \\ \infty & 5 & 1 & 3 & 0 \end{array} \qquad M = \begin{array}{ccccc} 1 & 2 & 3 & 4 & - \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ - & 2 & 3 & 4 & 5 \end{array}$$

$$k = 2$$

$$\begin{array}{ccccc} 0 & 1 & \mathbf{4} & \mathbf{2} & \mathbf{6} \\ 1 & 0 & 3 & 1 & 5 \\ \mathbf{4} & 3 & 0 & 1 & 1 \\ \mathbf{2} & 1 & 1 & 0 & 3 \\ \mathbf{6} & 5 & 1 & 3 & 0 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ 1 & 2 & 3 & 4 & 5 \\ \mathbf{2} & 2 & 3 & 4 & 5 \\ \mathbf{2} & 2 & 3 & 4 & 5 \\ \mathbf{2} & 2 & 3 & 4 & 5 \end{array}$$

$$k = 3$$

$$\begin{array}{ccccc} 0 & 1 & 4 & 2 & \mathbf{5} \\ 1 & 0 & 3 & 1 & \mathbf{4} \\ 4 & 3 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & \mathbf{2} \\ \mathbf{5} & \mathbf{4} & 1 & \mathbf{2} & 0 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & 2 & 2 & \mathbf{3} \\ 1 & 2 & 3 & 4 & \mathbf{3} \\ 2 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & \mathbf{3} \\ \mathbf{3} & \mathbf{3} & 3 & \mathbf{3} & 5 \end{array}$$

$$k = 4$$

$$\begin{array}{ccccc} 0 & 1 & \mathbf{3} & 2 & \mathbf{4} \\ 1 & 0 & \mathbf{2} & 1 & \mathbf{3} \\ \mathbf{3} & \mathbf{2} & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ \mathbf{4} & \mathbf{3} & 1 & 2 & 0 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & \mathbf{4} & 2 & \mathbf{4} \\ 1 & 2 & \mathbf{4} & 4 & \mathbf{4} \\ \mathbf{4} & \mathbf{4} & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 3 \\ \mathbf{4} & \mathbf{4} & 3 & 3 & 5 \end{array}$$

$$k = 5$$

$$\begin{array}{ccccc} 0 & 1 & 3 & 2 & 4 \\ 1 & 0 & 2 & 1 & 3 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 3 & 1 & 2 & 0 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & 4 & 2 & 4 \\ 1 & 2 & 4 & 4 & 4 \\ 4 & 4 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 3 \\ 4 & 4 & 3 & 3 & 5 \end{array}$$

Example

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 2 & 4 & 2 & 4 \\ 2 & 1 & 2 & 4 & 4 & 4 \\ 3 & 4 & 4 & 3 & 4 & 5 \\ 4 & 2 & 2 & 3 & 4 & 3 \\ 5 & 4 & 4 & 3 & 3 & 5 \end{array}$$

M

Overall

How to read this matrix M? Example path $1 \to 5$:

- Path $1 \to 5$ goes via node 4.
- Path $1 \to 4$ goes via node 2.
- Path $4 \to 5$ goes via node 3.
- Paths $1 \to 2$ and $2 \to 4$ are direct.
- Paths $4 \to 3$ and $3 \to 5$ are direct.

$$1 \xrightarrow{4} 5 \quad \Rightarrow \quad 1 \xrightarrow{2} 4 \xrightarrow{3} 5 \quad \Rightarrow \quad 1 \to 2 \to 4 \to 3 \to 5$$

Reconstruction via Recursion. Alternative? Store successor in the algorithm

Example

$k = 1$

D

0	1	5	3	∞
1	0	3	1	5
5	3	0	1	1
3	1	1	0	3
∞	5	1	3	0

S

1	2	3	4	-
1	2	3	4	5
1	2	3	4	5
1	2	3	4	5
-	2	3	4	5

$k = 2$

D

0	1	**4**	**2**	**6**
1	0	3	1	5
4	3	0	1	1
2	1	1	0	3
6	5	1	3	0

S

1	2	**2**	**2**	**2**
1	2	3	4	5
2	2	3	4	5
2	2	3	4	5
2	2	3	4	5

$k = 3$

D

0	1	4	2	**5**
1	0	3	1	**4**
4	3	0	1	1
2	1	1	0	**2**
5	**4**	1	**2**	0

S

1	2	2	2	**2**
1	2	3	4	**3**
2	2	3	4	5
2	2	3	4	**3**
3	**3**	3	**3**	5

$k = 4$

D

0	1	**3**	2	**4**
1	0	**2**	1	**3**
3	**2**	0	1	1
2	1	1	0	2
4	**3**	1	2	0

S

1	2	**2**	2	**2**
1	2	**4**	4	**4**
4	**4**	3	4	5
2	2	3	4	3
3	**3**	3	3	5

$k = 5$

D

0	1	3	2	4
1	0	2	1	3
3	2	0	1	1
2	1	1	0	2
4	3	1	2	0

S

1	2	2	2	2
1	2	4	4	4
4	4	3	4	5
2	2	3	4	3
3	3	3	3	5

Comparison of the approaches

Algorithm			Runtime
Dijkstra (Heap)	$c_v \geq 0$	1:n	$\mathcal{O}(\|E\| \log \|V\|)$
Dijkstra (Fibonacci-Heap)	$c_v \geq 0$	1:n	$\mathcal{O}(\|E\| + \|V\| \log \|V\|)$ *
Bellman-Ford		1:n	$\mathcal{O}(\|E\| \cdot \|V\|)$
Floyd-Warshall		n:n	$\Theta(\|V\|^3)$
Johnson		n:n	$\mathcal{O}(\|V\| \cdot \|E\| \cdot \log \|V\|)$
Johnson (Fibonacci-Heap)		n:n	$\mathcal{O}(\|V\|^2 \log \|V\| + \|V\| \cdot \|E\|)$ *

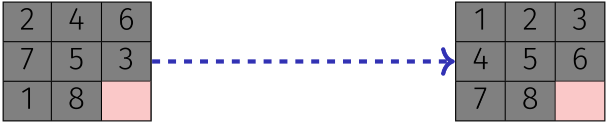
* amortized

Johnson (not explained this year) is better than Floyd-Warshall only for sparse graphs ($|E| \approx \Theta(|V|)$).

6. Code-Expert Example

Code-Example

'Sliding Puzzle' on Code-Expert



7. Outro

General Questions?

See you next time!

Have a nice week!