ETH zürich

Exercise Session 08 – Geometric Algorithms

Data Structures and Algorithms

These slides are based on those of the course, but were adapted and extended by the teaching assistant Adel Gavranović

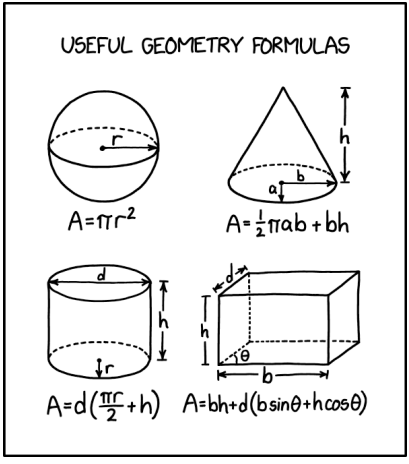
Today's Schedule

Intro
Follow-up
Feedback for **code** expert
    Closed Hashing
Learning Objectives
Geometric Algorithms
Convex hulls
Sweepline
Geometric Divide & Conquer: Closest Point Pair
Old Exam Question
Outro

n.ethz.ch/~agavranovic

- Exercise Session Material
- Adel's Webpage
- Mail to Adel

Comic of the Week



xkcd

1. Intro

Intro

Intro

- May 9th (Thu) is a public holiday
- We still want to provide a session for you
- Please indicate your availabilities:

Intro

- May 9th (Thu) is a public holiday
- We still want to provide a session for you
- Please indicate your availabilities:

when2meet

2. Follow-up

Functors, Lambdas and `const`

Functors, Lambdas and `const` (slide 24 ff. from ES07)

Functors, Lambdas and `const`

Functors, Lambdas and `const` (slide 24 ff. from ES07)

- Even though Functors, Lambdas and the keyword **`const`** are exam relevant (and important concepts!) the details of how exactly w.r.t. **`const`** a functor converts the arguments passed through its capture (`[]`) are *not exam relevant*

Functors, Lambdas and `const`

Functors, Lambdas and `const` (slide 24 ff. from ES07)

- Even though Functors, Lambdas and the keyword `const` are exam relevant (and important concepts!) the details of how exactly w.r.t. `const` a functor converts the arguments passed through its capture (`[]`) are *not exam relevant*
- The reason the function that was marked `const` was able to modify the referenced (`&`) `unsigned int` that was passed through its capture (`[&count]`), was that it's a reference to a (and not a "real") variable

Functors, Lambdas and `const`

Functors, Lambdas and `const` (slide 24 ff. from ES07)

- Even though Functors, Lambdas and the keyword `const` are exam relevant (and important concepts!) the details of how exactly w.r.t. `const` a functor converts the arguments passed through its capture (`[]`) are *not exam relevant*
- The reason the function that was marked `const` was able to modify the referenced (`&`) `unsigned int` that was passed through its capture (`[&count]`), was that it's a reference to a (and not a "real") variable
- The rabbit hole goes even deeper: C++ handles pointers in a very similar (maybe even same?) way

Functors, Lambdas and `const` – Code

```
#include <iostream>
#include <string>
#include <vector>
#include <algorithm>
#include <functional>

// the filter-helper
template <class Functor>
class Not
{
public:
    Not(Functor & f) : func(f) {}

    template <typename ArgType>
    bool operator()(ArgType & arg) {return !func(arg);}

private:
    Functor & func;
};

// ...
```

```
// ...

// the filter
template<typename T, typename B>
T filter(T list, B pred) {
    T ret;
    std::remove_copy_if(
        list.begin(),
        list.end(),
        std::back_insert_iterator<T>(ret),
        Not<B>(pred)
    );

    return ret;
}

// ...
```

Functors, Lambdas and `const` – Code

```
// ...

// the functor
class lambda1 {
  unsigned& count;
  int min;
public:
  lambda1(unsigned& c, int m):
    count(c), min(m) {}
  bool operator()(int e) const {
    ++count;
    return min <= e;
  }
};

// ...
```

```
// ...

int main() {

  unsigned count = 0;
  int min = 3;
  std::vector<int> data = {4,-2, 0, 10, 1, 2, 3, 5};
  data = filter(data, lambda1(count, min));

  for (auto datum : data){
    std::cout << datum << " ";
  }

  return 0;
}
```

3. Feedback for **code** expert

Task "Closed Hashing"

Task "Closed Hashing"

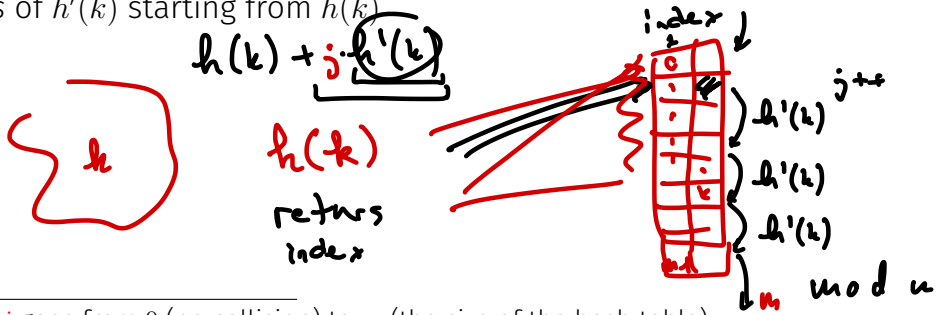
- Recap urgent and important, since only 6/24 people got 3/3!

Double Hashing (cf. Lecture Notes)

[1]where j goes from 0 (no collision) to m (the size of the hash table)

Double Hashing (cf. Lecture Notes)

For creating a probing sequence that does neither suffer from primary clustering nor from secondary clustering, we can use probe using *double hashing*. We use two hash functions $h(k)$ and $h'(k)$ and *probe along multiples* of $h'(k)$ starting from $h(k)$



[1]where j goes from 0 (no collision) to m (the size of the hash table)

Double Hashing (cf. Lecture Notes)

For creating a probing sequence that does neither suffer from primary clustering nor from secondary clustering, we can use probe using *double hashing*. We use two hash functions $h(k)$ and $h'(k)$ and *probe along multiples* of $h'(k)$ starting from $h(k)$, thus[1]

$$S(k) = \underbrace{(h(k) + 0h'(k),\ h(k) + 1h'(k),\ h(k) + 2h'(k),\ h(k) + 3h'(k), \ldots)}_{\text{Probing Sequence (but without the modulo)}} \bmod m$$

[1]where j goes from 0 (no collision) to m (the size of the hash table)

Closed Hashing from code expert

TASK Insert the keys

$$\cancel{17}, \cancel{6}, \cancel{7}, \cancel{8}, \cancel{11}, \cancel{28}, \cancel{21}, 20$$

in this order into an initially empty hash table of size 11. Use open addressing with the hash function

$$h(k) = k \bmod 11$$

and resolve the conflicts using *double hashing* with

$$h'(k) = 1 + (k \bmod 9)$$

$h(26) = 26 \,\%\, 11 = 9$ ✓

(6, 8, 10)

11		6	71			17	7	8	20	28
0	1	2	3	4	5	6	7	8	9	10

$h(17) = 17 \,\%\, 11 = 6$ ✓

$h(6) = 6 \,\%\, 11 = 6$ ↯

$h'(6) = 1 + (6 \,\%\, 9) = 7$

$h(6) + j\,h'(6) = 6 + 7 = 13 \,\%\, m$

$h(7) = 7 \,\%\, 11 = 7$ ✓

$h(8) = 8 \,\%\, 11 = 8$ ✓

$h(11) = 11 \,\%\, 11 = 0$

$h(28) = 28 \,\%\, 11 = 17 \,\%\, 11 = 6$ ↯

$h'(28) = 1 + (28 \,\%\, 9) = 2$

$6 + 2$ ↯

$6 + 4$

$h(21) = 21 \,\%\, 11 = 10 \,\%\, 11 = 10$ ↯

$h'(21) = 1 + 21 \,\%\, 9 = 1 + 3 = 4$

$14 \,\%\, 11 =$

Closed Hashing from code expert (Solution)

TASK Insert the keys

$$17, 6, 7, 8, 11, 28, 21, 20$$

in this order into an initially empty hash table of size 11. Use open addressing with the hash function

$$h(k) = k \bmod 11$$

and resolve the conflicts using *double hashing* with

$$h'(k) = 1 + (k \bmod 9)$$

11		6	21			17	7	8	20	28
0	1	2	3	4	5	6	7	8	9	10

Questions?

Questions regarding code expert from your side?

4. Learning Objectives

Learning Objectives

- ☐ Be able to apply simple *hashing methods* by hand
- ☐ Understand how the presented *geometric algorithms* work
- ☐ Understand why the presented *geometric algorithms* work

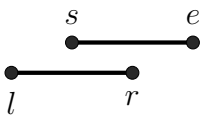
5. Geometric Algorithms

Overlaps of two intervals

Two intervals (l, r) and (s, e) overlap if

Overlaps of two intervals

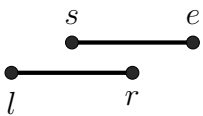
Two intervals (l, r) and (s, e) overlap if



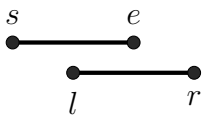
$$l \leq s < r$$

Overlaps of two intervals

Two intervals (l, r) and (s, e) overlap if



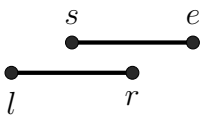
$l \leq s < r$



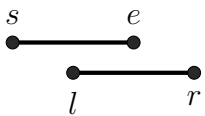
$l < e \leq r$

Overlaps of two intervals

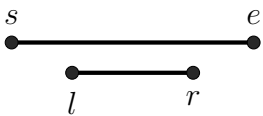
Two intervals (l, r) and (s, e) overlap if



$l \leq s < r$



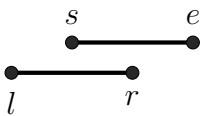
$l < e \leq r$



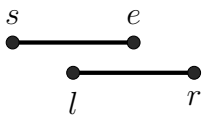
or $s \leq l \leq r \leq e$

Overlaps of two intervals

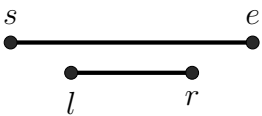
Two intervals (l, r) and (s, e) overlap if



$l \leq s < r$



$l < e \leq r$



or $s \leq l \leq r \leq e$

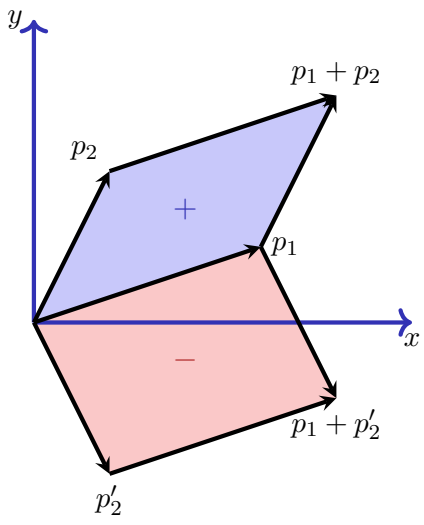
$\Rightarrow$ We can check in constant time whether two intervals intersect.

Properties of line segments

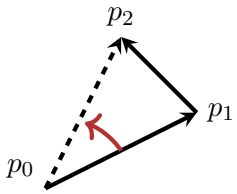
Cross-Product of two vectors $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ in the plane

$$p_1 \times p_2 = \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = x_1 y_2 - x_2 y_1$$

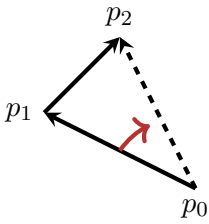
Signed area of the parallelogram



Turning direction



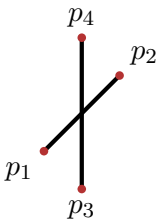
to the left:
$(p_1 - p_0) \times (p_2 - p_0) > 0$

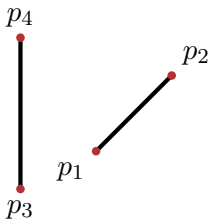


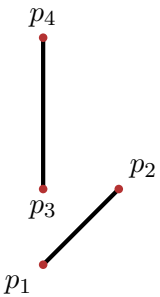
to the right:
$(p_1 - p_0) \times (p_2 - p_0) < 0$

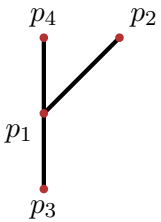
Intersection of two line segments

How to figure out whether two segments are intersecting without actually computing the intersection points (division!)?



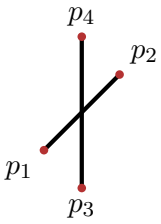




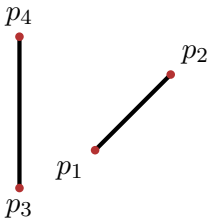


Intersection of two line segments

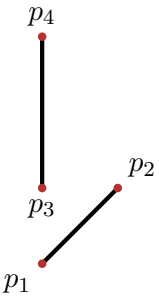
How to figure out whether two segments are intersecting without actually computing the intersection points (division!)?



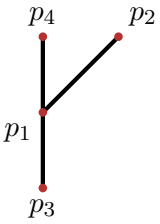
Intersection: p_1 and p_2 opposite w.r.t $\overline{p_3p_4}$ and p_3, p_4 opposite w.r.t. $\overline{p_1p_2}$



No intersection: p_1 and p_2 on the same side of $\overline{p_3p_4}$



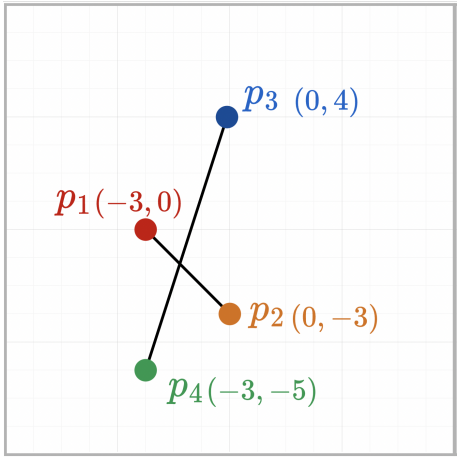
No intersection: p_3 and p_4 on the same side of $\overline{p_1p_2}$



Intersection: p_1 on $\overline{p_3p_4}$

Intersection of two line segments

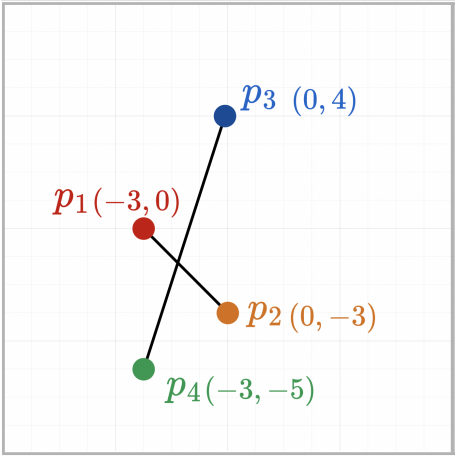
Part (a)



Intersection or no intersection?

Intersection of two line segments

Part (a)



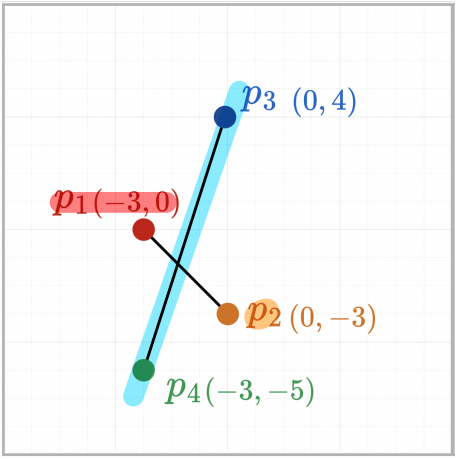
Intersection or no intersection?

Intersection

p_1, p_2 are opposite w.r.t $\overline{p_4 p_3}$,
and p_3, p_4 are opposite w.r.t. $\overline{p_1 p_2}$.

Intersection of two line segments

Part (a)

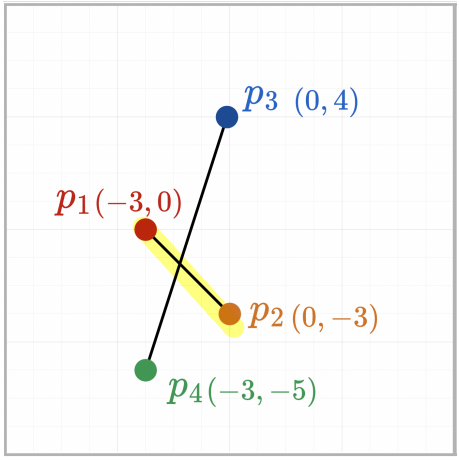


$$(p_3 - p_4) \times (p_1 - p_4) =$$
$$= ((0,4) - (-3,-5)) \times ((-3,0) - (-3,-5)) = (3,9) \times (0,5) = \det \begin{bmatrix} 3 & 0 \\ 9 & 5 \end{bmatrix}$$
$$= (3)(5) - (0)(9) = \mathbf{15} > \mathbf{0}.$$

$$(p_3 - p_4) \times (p_2 - p_4) =$$
$$= ((0,4) - (-3,-5)) \times ((0,-3) - (-3,-5))$$
$$= (3,9) \times (3,2) = \det \begin{bmatrix} 3 & 3 \\ 9 & 2 \end{bmatrix}$$
$$= (3)(2) - (3)(9) = -\mathbf{21} < \mathbf{0}.$$

Intersection of two line segments

Part (a)



and $(p_2 - p_1) \times (p_3 - p_1) =$

$= ((0,-3) - (-3,0)) \times ((0,4) - (-3,0))$

$= (3,-3) \times (3,4) = \det \begin{bmatrix} 3 & 3 \\ -3 & 4 \end{bmatrix}$

$= (3)(4) - (3)(-3) = \mathbf{21} > \mathbf{0}.$

$(p_2 - p_1) \times (p_4 - p_1) =$

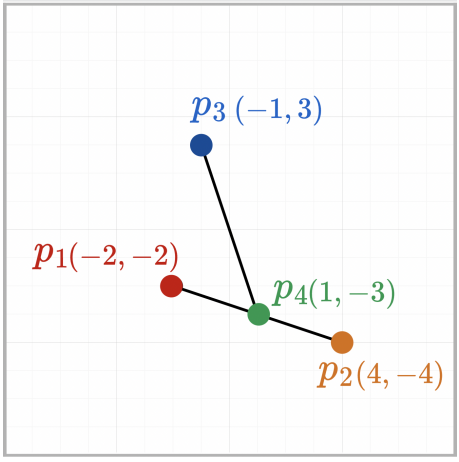
$= ((0,-3) - (-3,0)) \times ((-3,-5) - (-3,0))$

$= (3,-3) \times (0,-5) = \det \begin{bmatrix} 3 & 0 \\ -3 & -5 \end{bmatrix}$

$= (3)(-5) - (0)(-3) = \mathbf{-15} < \mathbf{0}.$

Intersection of two line segments

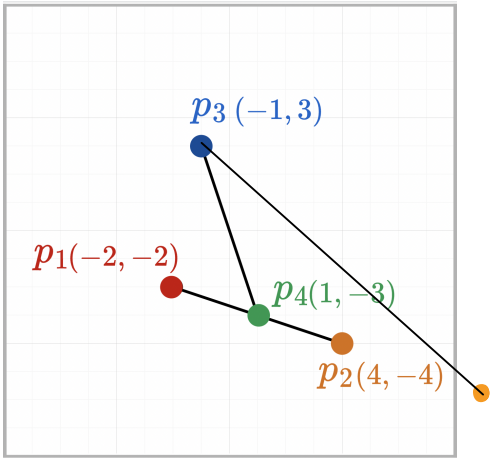
Part (b)



Intersection or no intersection?

Intersection of two line segments

Part (b)



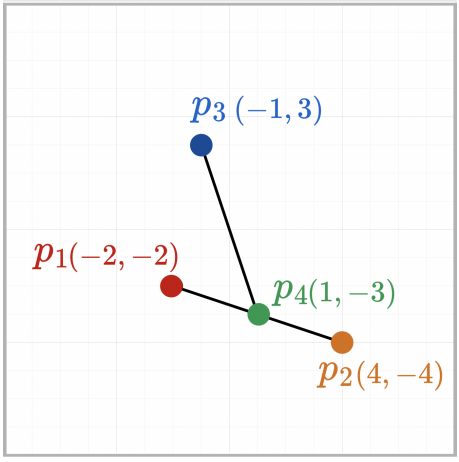
Intersection or no intersection?

Intersection

p_4 is on $\overline{p_1 p_2}$ for two reasons.

Intersection of two line segments

Part (b)

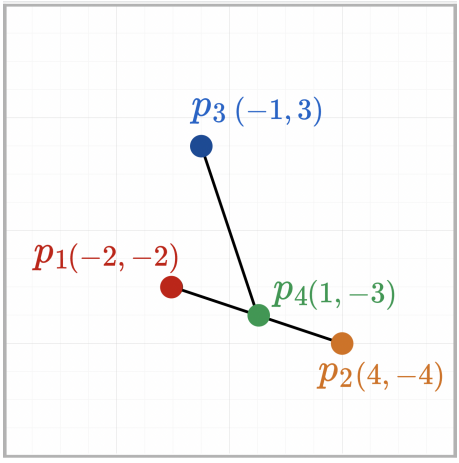


First,

$$(p_2 - p_1) \times (p_4 - p_1) =$$
$$= ((4,-4) - (-2,-2)) \times ((1,-3) - (-2,-2))$$
$$= (6,-2) \times (3,-1) = \det \begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}$$
$$= (6)(-1) - (3)(-2) = \mathbf{0}.$$

Intersection of two line segments

Part (b)

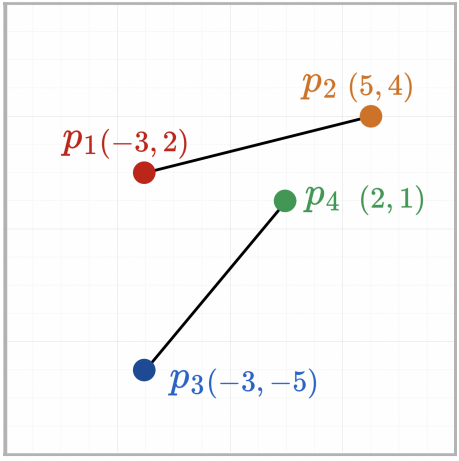


$p_4(\ ,\ )$

But this only shows that p_4 is in the line created by $\overline{p_1p_2}$.
To conclude that p_4 is in $\overline{p_1p_2}$, note that
$-2 = p_1[0] \leq 1 = p_4[0] \leq 4 = p_2[0]$
and
$-4 = p_2[1] \leq -3 = p_4[1] \leq -2 = p_1[1]$.

Intersection of two line segments

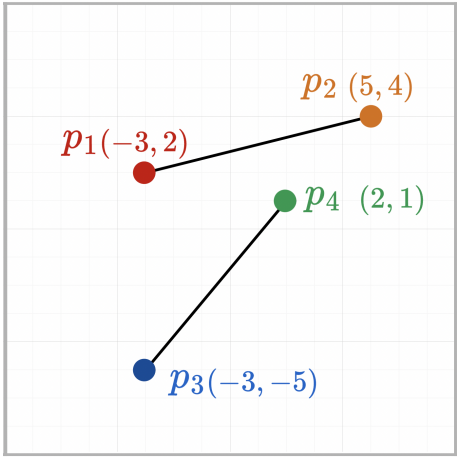
Part (c)



Intersection or no intersection?

Intersection of two line segments

Part (c)



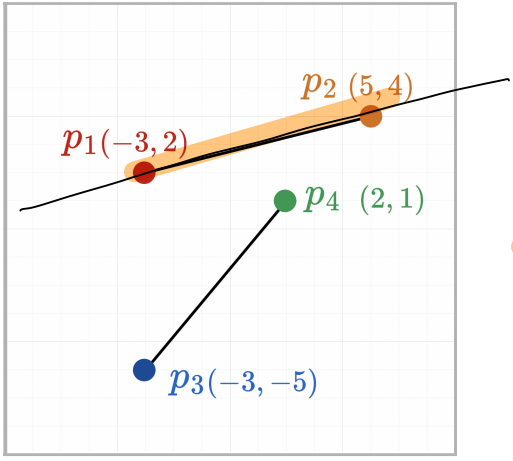
Intersection or no intersection?

No Intersection

p_3 and p_4 are on the same side of $\overline{p_1 p_2}$.

Intersection of two line segments

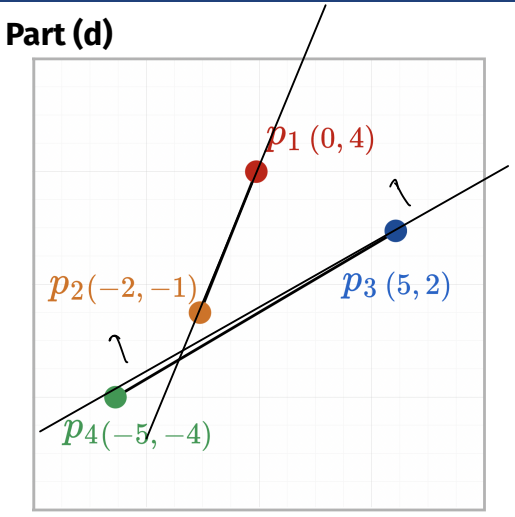
Part (c)



$$(p_2 - p_1) \times (p_3 - p_1) =$$
$$= ((5,4) - (-3,2)) \times ((-3,-5) - (-3,2))$$
$$= (8,2) \times (0,-7) = \det \begin{bmatrix} 8 & 0 \\ 2 & -7 \end{bmatrix}$$
$$= (8)(-7) - (0)(2) = \mathbf{-56} < \mathbf{0}.$$

$$(p_2 - p_1) \times (p_4 - p_1) =$$
$$= ((5,4) - (-3,2)) \times ((2,1) - (-3,2))$$
$$= (8,2) \times (5,-1) = \det \begin{bmatrix} 8 & 5 \\ 2 & -1 \end{bmatrix}$$
$$= (8)(-1) - (5)(2) = \mathbf{-18} < \mathbf{0}.$$

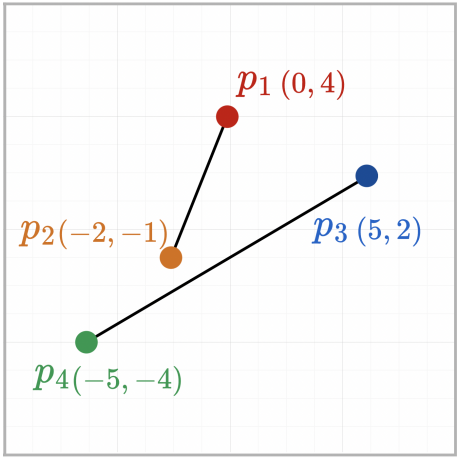
Intersection of two line segments



Intersection or no intersection?

Intersection of two line segments

Part (d)



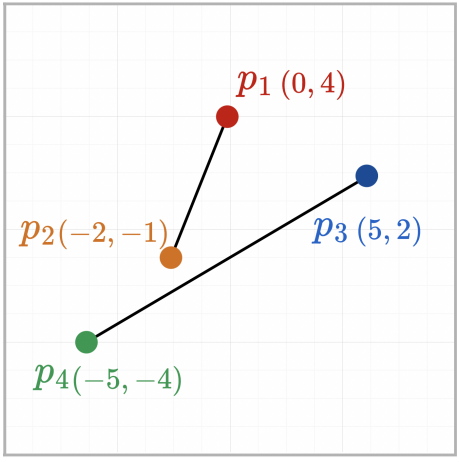
Intersection or no intersection?

No Intersection

p_1 and p_2 are on the same side of $\overline{p_4 p_3}$.

Intersection of two line segments

Part (d)



$$(p_3 - p_4) \times (p_1 - p_4) =$$
$$= ((5,2)-(-5,-4))\times((0,4)-(-5,-4))$$
$$= (10,6) \times (5,8) = \det\begin{bmatrix} 10 & 5 \\ 6 & 8 \end{bmatrix}$$
$$= (10)(8) - (5)(6) = \mathbf{60} > \mathbf{0}.$$

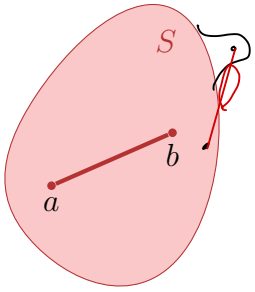
$$(p_3 - p_4) \times (p_2 - p_4) =$$
$$= ((5,2) - (-5,-4)) \times ((-2,-1) - (-5,-4))$$
$$= (10,6) \times (3,3) = \det\begin{bmatrix} 10 & 3 \\ 6 & 3 \end{bmatrix}$$
$$= (10)(3) - (6)(3) = \mathbf{12} > \mathbf{0}.$$

6. Convex hulls

Convex Hull

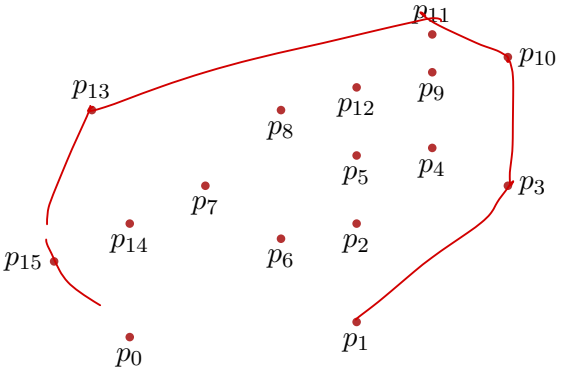
Subset S of a real vector space is called **convex**, if for all $a, b \in S$ and all $\lambda \in [0, 1]$:

$$\lambda a + (1 - \lambda) b \in S$$



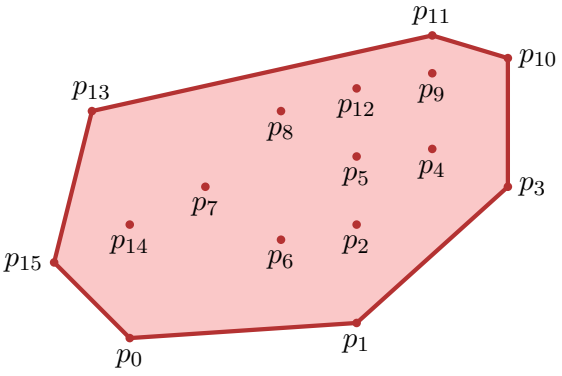
Convex Hull

Convex Hull $H(Q)$ of a set Q of points: smallest convex polygon P such that each point of Q is on P or in the interior of P.



Convex Hull

Convex Hull $H(Q)$ of a set Q of points: smallest convex polygon P such that each point of Q is on P or in the interior of P.



Jarvis Marsch / Gift Wrapping algorithm

1. Start with an extremal point (e.g. lowest point) $p = p_0$
2. Search point q, such that $\overline{pq}$ is a line to the right of all other points (or other points are on this line closer to p.
3. Continue with $p \leftarrow q$ at (2) until $p = p_0$.

Illustration Jarvis

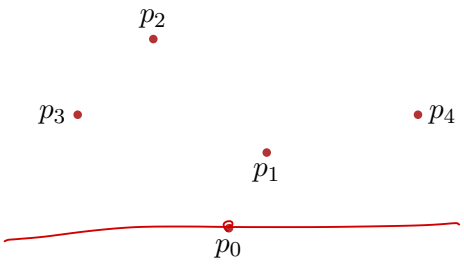
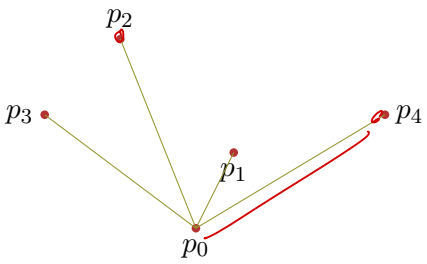
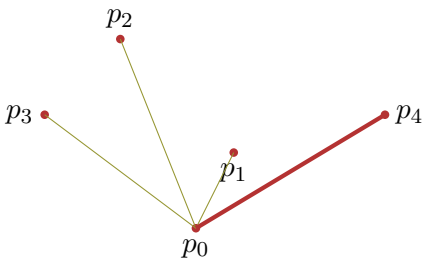


Illustration Jarvis



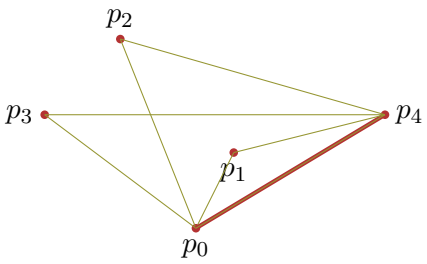
1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view

Illustration Jarvis



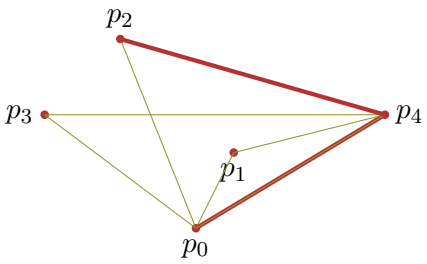
1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.

Illustration Jarvis



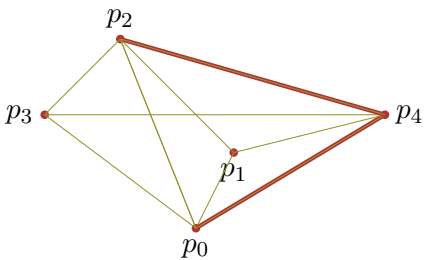
1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.
5. Set $q \leftarrow p$ and repeat from step 3 until q is the lowest point

Illustration Jarvis



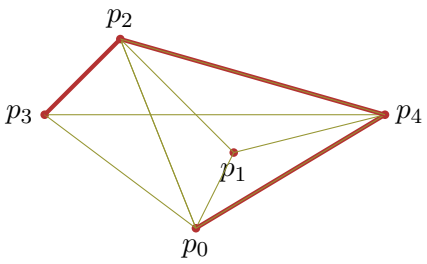
1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.
5. Set $q \leftarrow p$ and repeat from step 3 until q is the lowest point

Illustration Jarvis



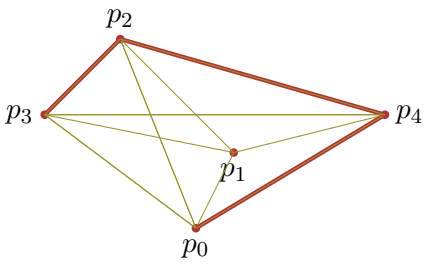
1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.
5. Set $q \leftarrow p$ and repeat from step 3 until q is the lowest point

Illustration Jarvis



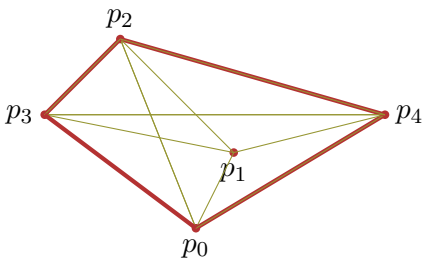
1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.
5. Set $q \leftarrow p$ and repeat from step 3 until q is the lowest point

Illustration Jarvis



1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.
5. Set $q \leftarrow p$ and repeat from step 3 until q is the lowest point

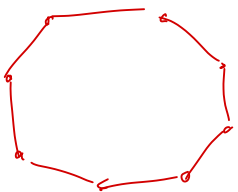
Illustration Jarvis



1. Set $H \rightarrow \emptyset$.
2. Find the lowest point q.
3. Find the rightmost point p, from q's point of view
4. Add p to H.
5. Set $q \leftarrow p$ and repeat from step 3 until q is the lowest point
6. H is the convex hull.

Graham Scan

- Graham Scan: Another algorithm that computes the convex hull
- See the implementation in the lecture slides
- Time complexity:
  - Jarvis March: $\mathcal{O}(h \cdot n)$ where h is the number of corner points on the convex hull
  - Graham Scan: $\mathcal{O}(n \log n)$
- **Question**: When does Jarvis March perform better?



$n = 8$

$J = 8 \cdot 8$

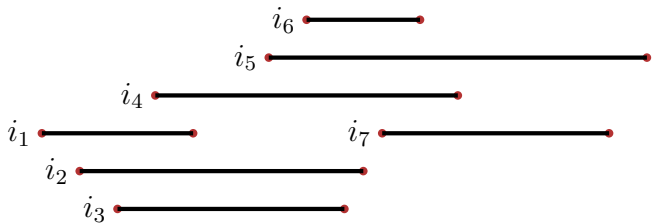
$G \quad 8 \log_2(8) < 8^2$

Graham Scan

- Graham Scan: Another algorithm that computes the convex hull
- See the implementation in the lecture slides
- Time complexity:
  - Jarvis March: $\mathcal{O}(h \cdot n)$ where h is the number of corner points on the convex hull
  - Graham Scan: $\mathcal{O}(n \log n)$
- **Question**: When does Jarvis March perform better?
- **Answer**: Jarvis March is better when h is small compared to n, as its time complexity depends on the number of corner points on the convex hull.
- Comment: Chan's algorithm improves on both, but is not taught in this course.

7. Sweepline

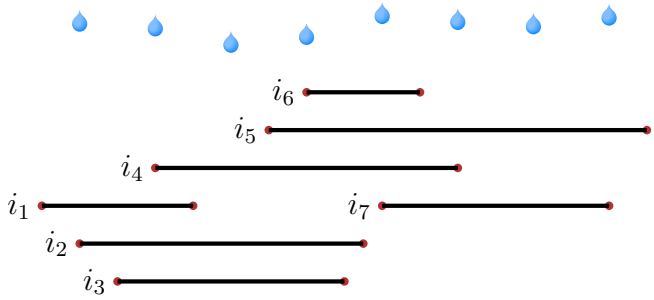
Preparation: Overlapping Intervals



Questions:

- How many intervals overlap maximally?

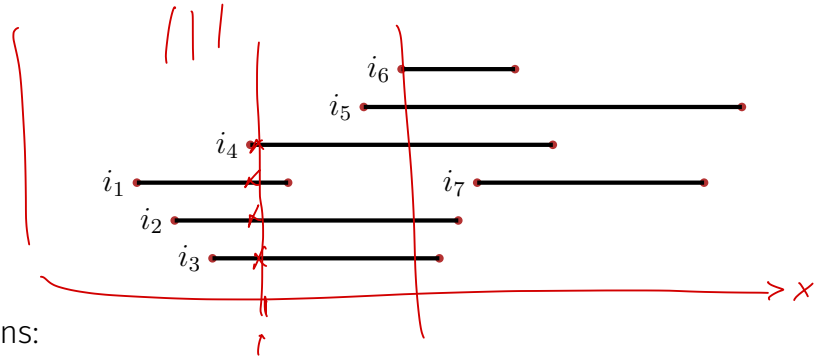
Preparation: Overlapping Intervals



Questions:

- How many intervals overlap maximally?
- Which intervals (don't) get wet?

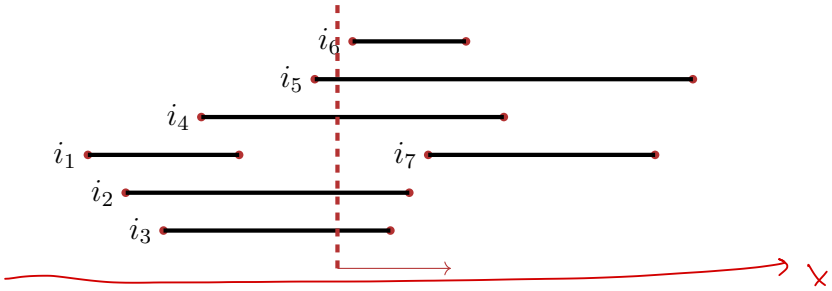
Preparation: Overlapping Intervals



Questions:

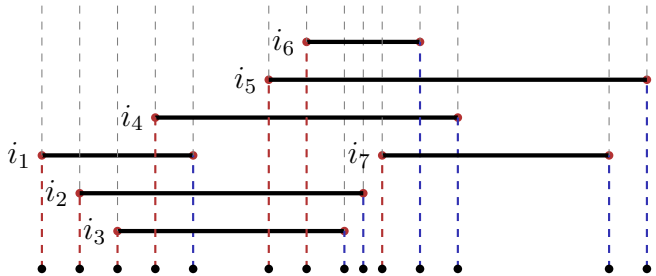
- How many intervals overlap maximally?
- Which intervals (don't) get wet?
- Which intervals are directly on top of each other?

Preparation: Overlapping Intervals



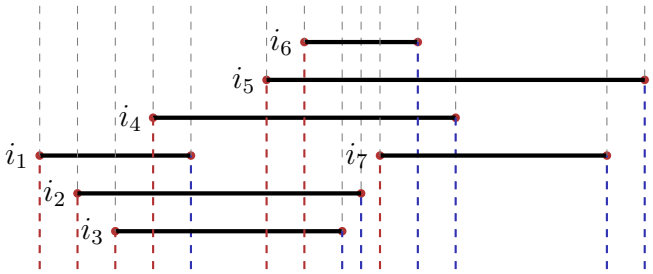
Idea of a sweep line: vertical line, moving in x-direction, observes the geometric objects.

Preparation: Overlapping Intervals



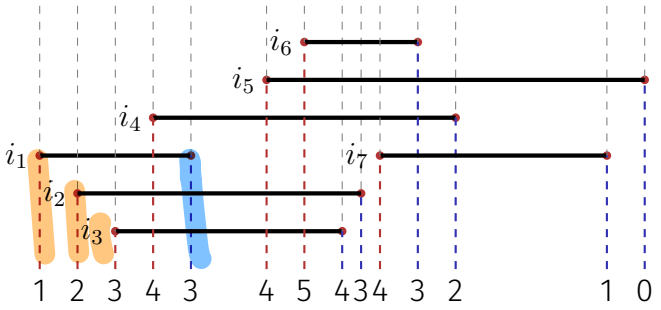
Event list: list of points where the state observed by the sweepline changes.

Preparation: Overlapping Intervals



Q: How many intervals overlap maximally?

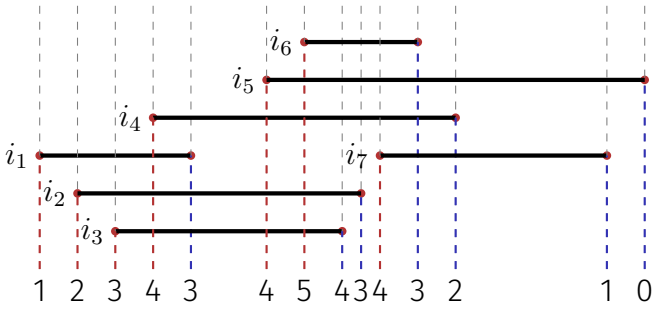
Preparation: Overlapping Intervals



Q: How many intervals overlap maximally?

Sweep line controls a counter that is incremented (decremented) at the left (right) end point of an interval.

Preparation: Overlapping Intervals

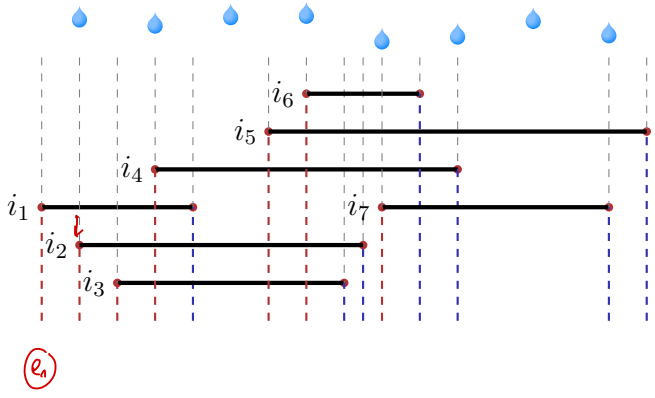


Q: How many intervals overlap maximally?

Sweep line controls a counter that is incremented (decremented) at the left (right) end point of an interval.

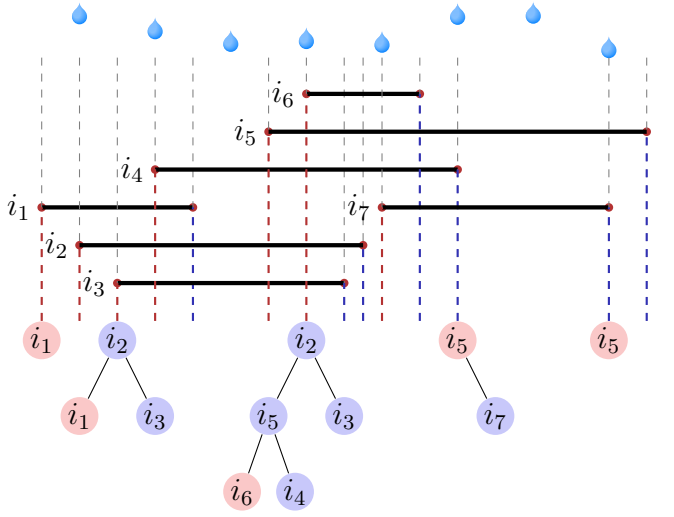
A: maximum counter value

Preparation: Overlapping Intervals



Q: Which intervals get wet?

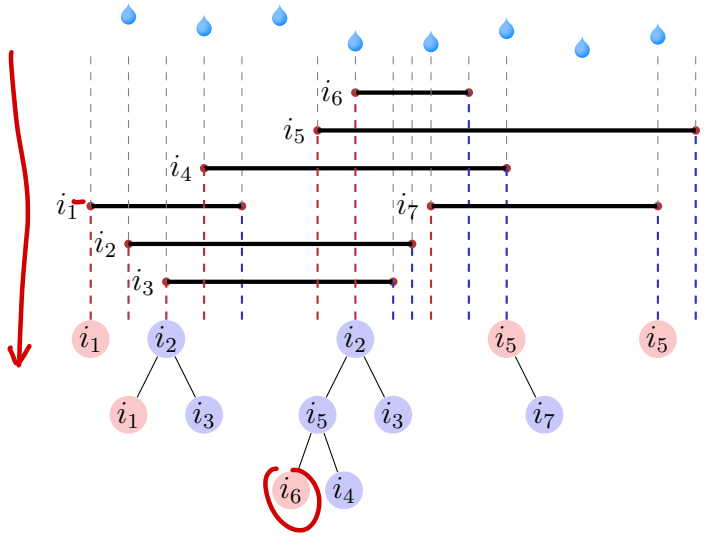
Preparation: Overlapping Intervals



Q: Which intervals get wet?

Sweep line controls a **binary search tree** that comprises the intervals according to their vertical ordering.

Preparation: Overlapping Intervals

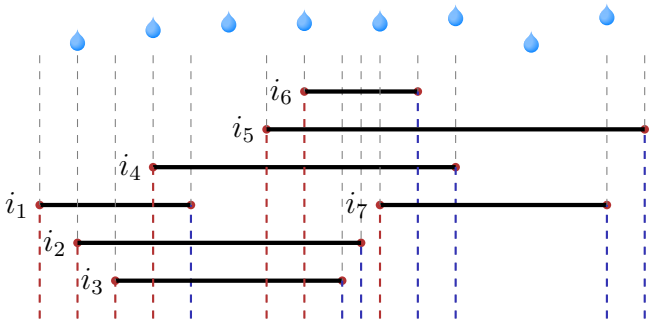


Q: Which intervals get wet?

Sweep line controls a **binary search tree** that comprises the intervals according to their vertical ordering.

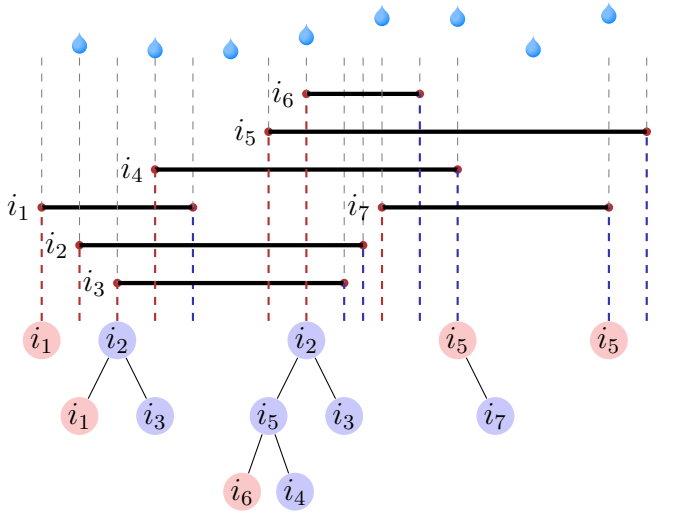
A: Intervals on the very left of the tree.

Preparation: Overlapping Intervals



Q: Why don't we use Max-Heap (instead of BST)?

Preparation: Overlapping Intervals

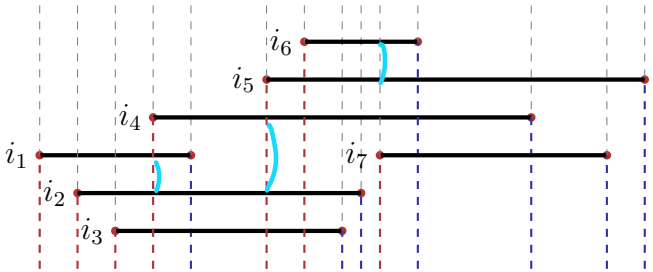


Q: Why don't we use Max-Heap (instead of BST)?

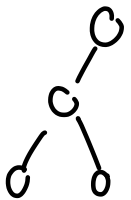
A: The deletion of an arbitrary element (not the maximum) from a heap is "not easy."

→ expensive

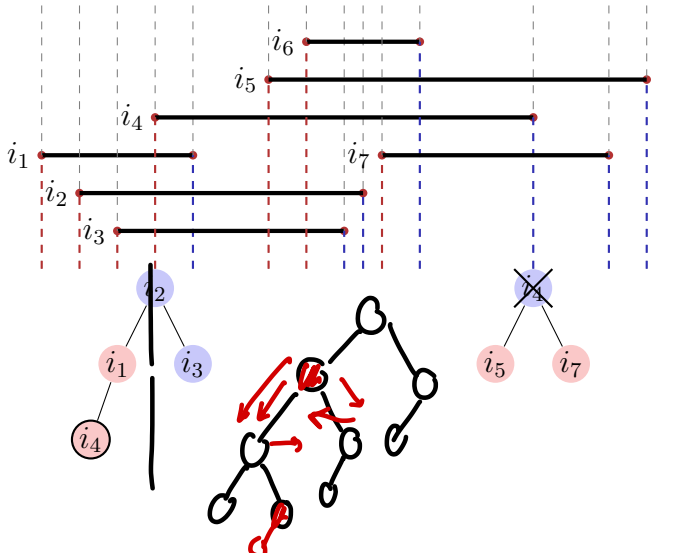
Preparation: Overlapping Intervals



Q: Which intervals are neighbours?



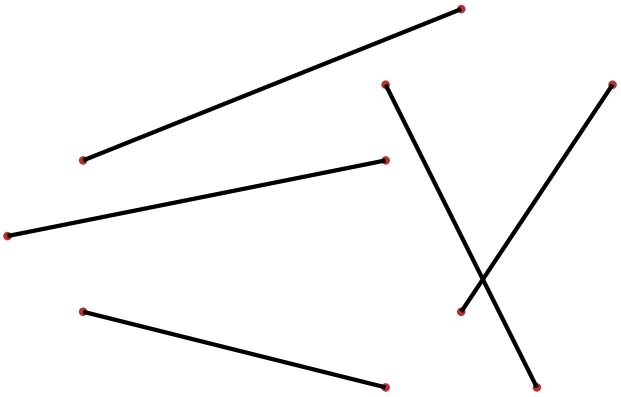
Preparation: Overlapping Intervals



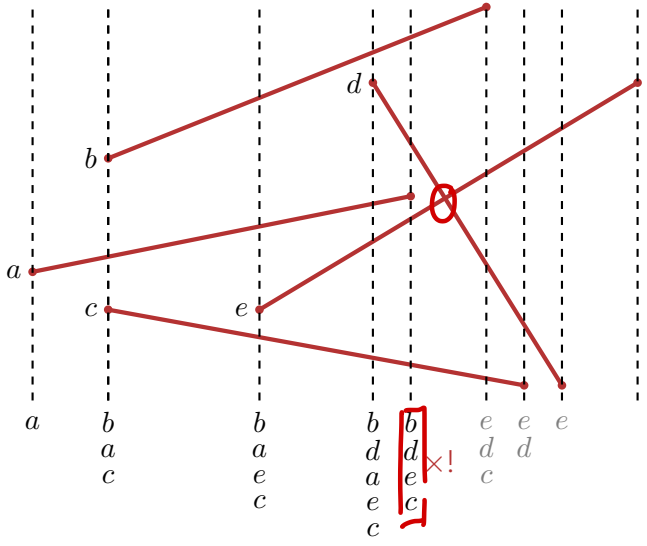
Q: Which intervals are neighbours?

A: If one is the symmetric predecessors or ancestor of the other in the tree.

Cutting many line segments



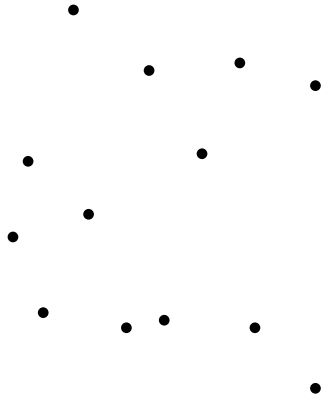
Intersection of line segments



8. Geometric Divide & Conquer: Closest Point Pair

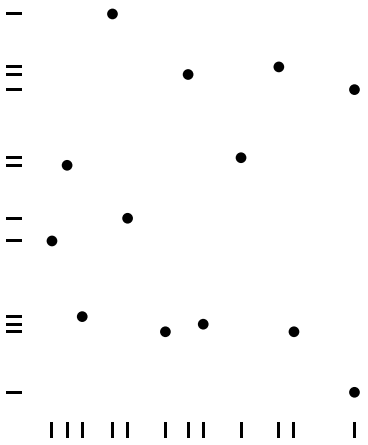
Divide And Conquer: Closest Point Pair

- Set of points P, starting with $P \leftarrow Q$



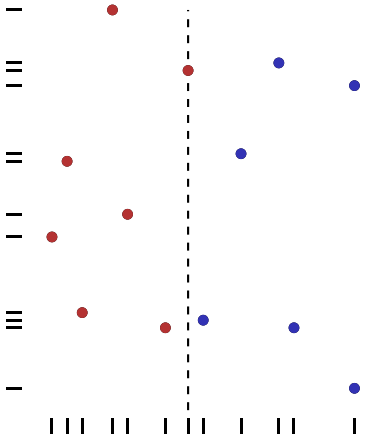
Divide And Conquer: Closest Point Pair

- Set of points P, starting with $P \leftarrow Q$
- Arrays X and Y, containing the elements of P , sorted by x- and y-coordinate, respectively.



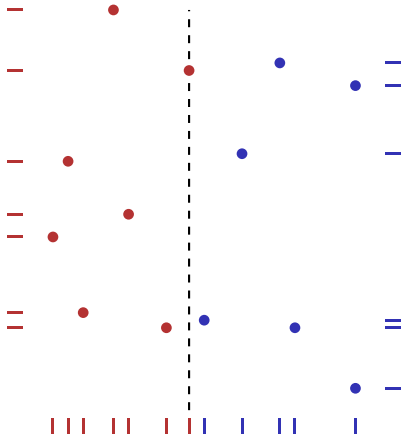
Divide And Conquer: Closest Point Pair

- Set of points P, starting with $P \leftarrow Q$
- Arrays X and Y, containing the elements of P , sorted by x- and y-coordinate, respectively.
- Partition point set into two (approximately) equally sized sets P_L and P_R, separated by a vertical line through a point of P.



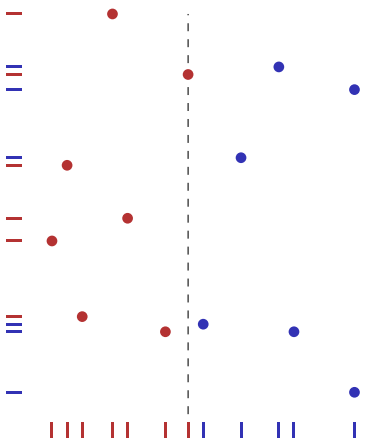
Divide And Conquer: Closest Point Pair

- Set of points P, starting with $P \leftarrow Q$
- Arrays X and Y, containing the elements of P , sorted by x- and y-coordinate, respectively.
- Partition point set into two (approximately) equally sized sets P_L and P_R, separated by a vertical line through a point of P.
- Split arrays X and Y accordingly in X_L, X_R. Y_L and Y_R.



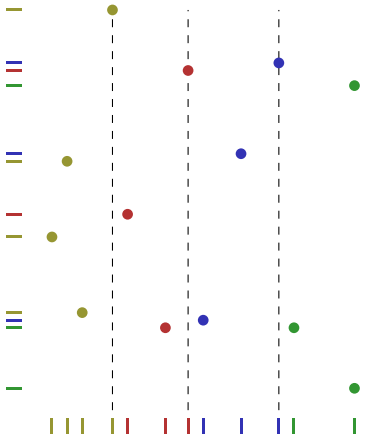
Divide And **Conquer**: Closest Point Pair

- Recursive call with P_L, X_L, Y_L and P_R, X_R, Y_R. Yields minimal distances δ_L, δ_R.



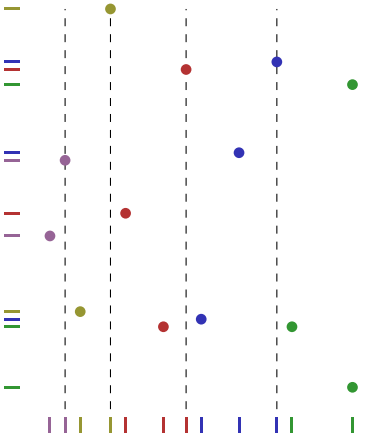
Divide And **Conquer**: Closest Point Pair

- Recursive call with P_L, X_L, Y_L and P_R, X_R, Y_R. Yields minimal distances δ_L, δ_R.



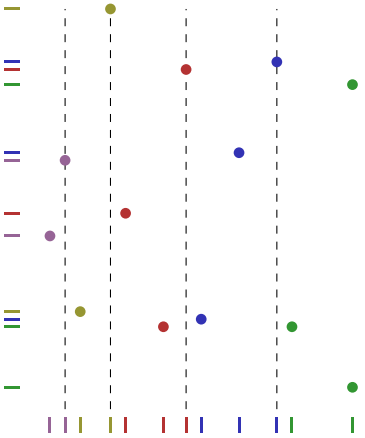
Divide And **Conquer**: Closest Point Pair

- Recursive call with P_L, X_L, Y_L and P_R, X_R, Y_R. Yields minimal distances δ_L, δ_R.
- (If only $k \leq 2$ points: compute the minimal distance directly)



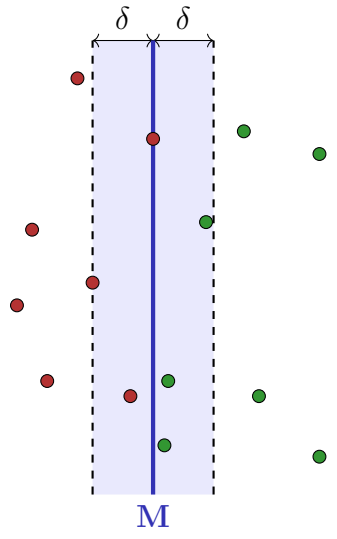
Divide And **Conquer**: Closest Point Pair

- Recursive call with P_L, X_L, Y_L and P_R, X_R, Y_R. Yields minimal distances δ_L, δ_R.
- (If only $k \leq 2$ points: compute the minimal distance directly)
- After recursive call $\delta = \min(\delta_L, \delta_R)$. Combine (next slides) and return best result.



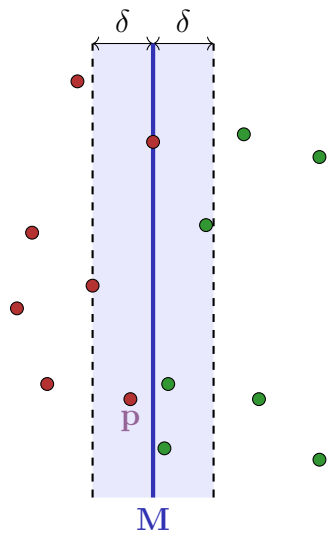
Minimum Distance across middle line: Observations

Minimum Distance across middle line: Observations



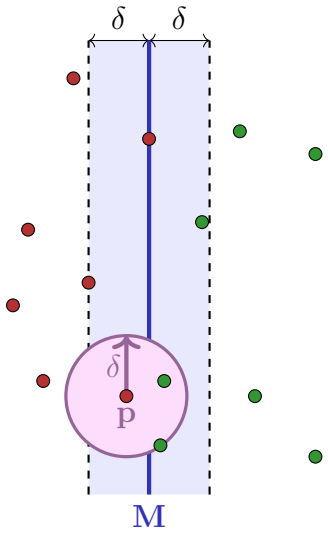
Minimum Distance across middle line: Observations

Which points are relevant for point p?



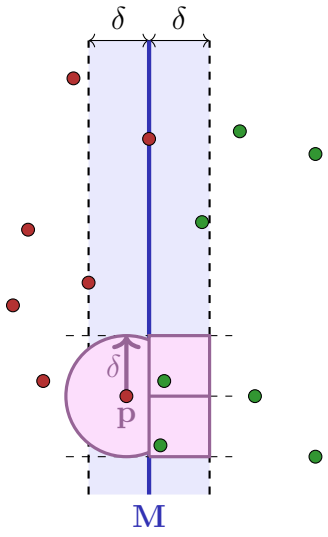
Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ



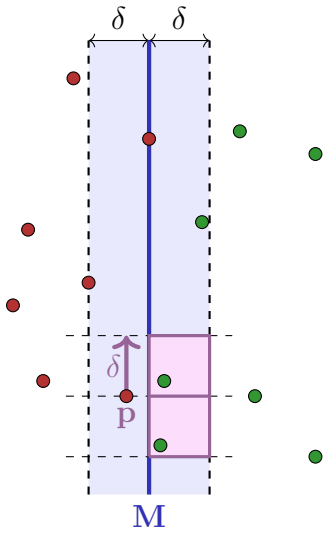
Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.



Minimum Distance across middle line: Observations

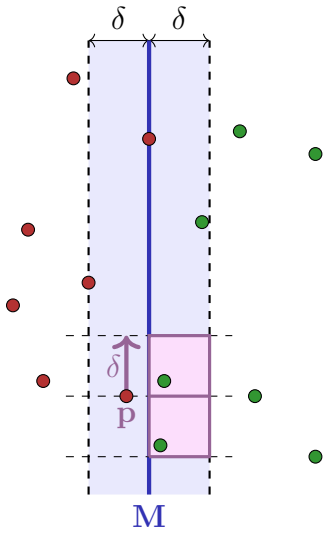
Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.



Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.

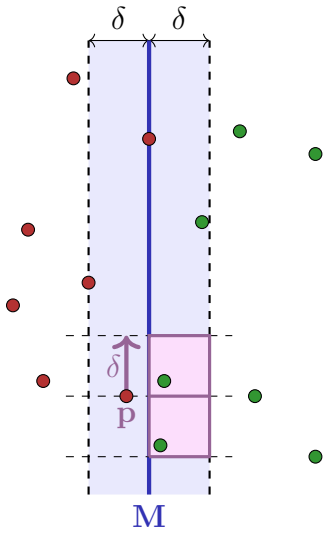
How many points are in these rectangles?



Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.

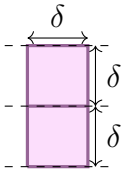
How many points are in these rectangles?
Observation 2: At most 8.

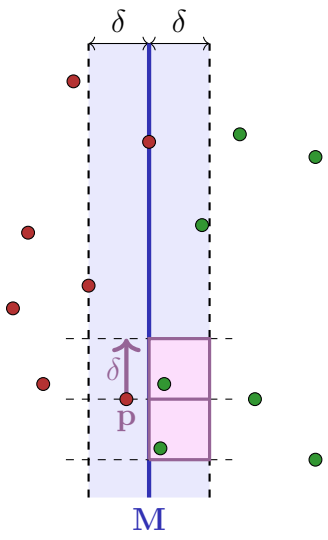


Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.

How many points are in these rectangles?
Observation 2: At most 8.

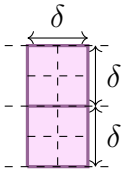


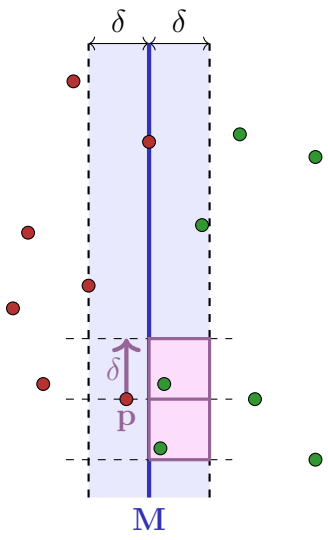


Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.

How many points are in these rectangles?
Observation 2: At most 8.

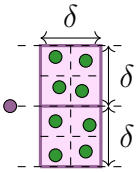




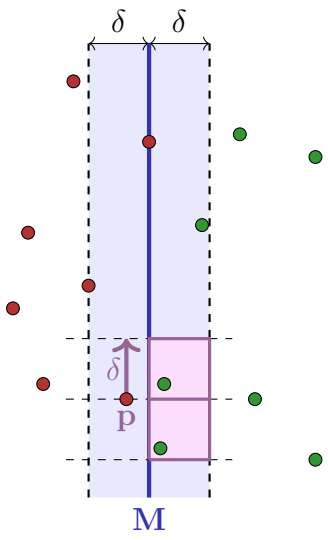
Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.

How many points are in these rectangles?
Observation 2: At most 8.



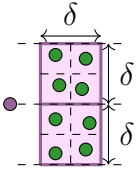
At most one point per $(\delta/2 \times \delta/2)$-rectangle,



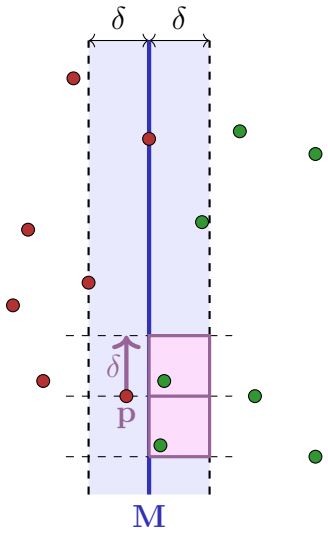
Minimum Distance across middle line: Observations

Which points are relevant for point p?
$\Rightarrow$ the ones in a circle around p with radius δ
Observation 1: The relevant points are contained in two $(\delta \times \delta)$-rectangles.

How many points are in these rectangles?
Observation 2: At most 8.

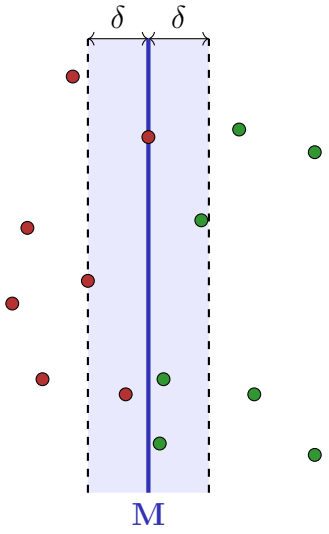


At most one point per $(\delta/2 \times \delta/2)$-rectangle, otherwise they have distance $\sqrt{2} \cdot \frac{\delta}{2} < \delta$.



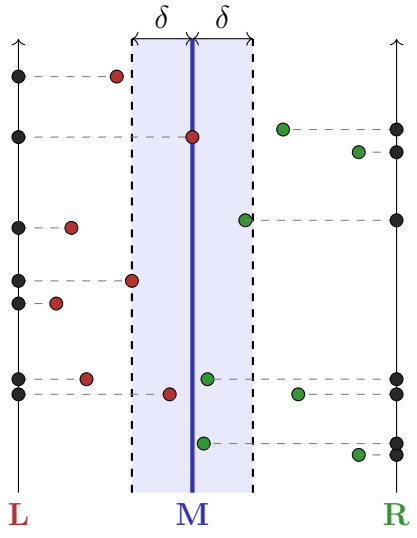
Minimum Distance across middle line: Algorithm

Minimum Distance across middle line: Algorithm



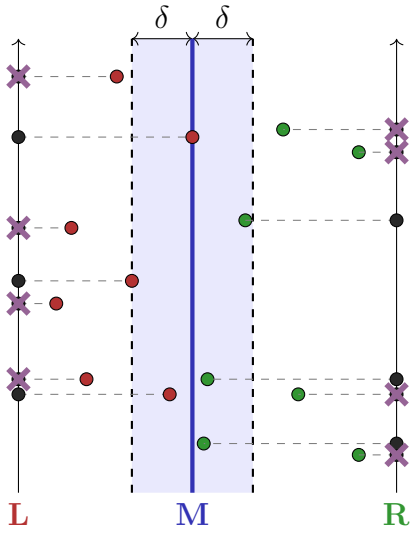
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates



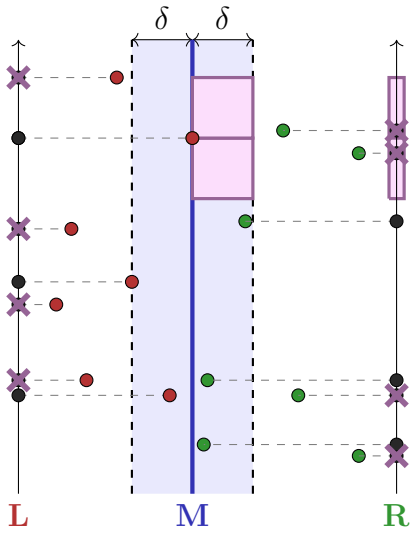
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M



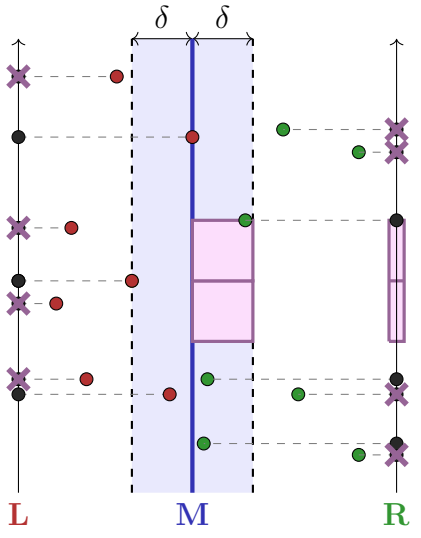
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$



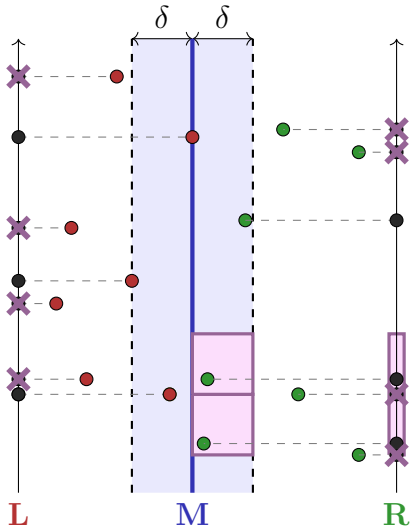
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$



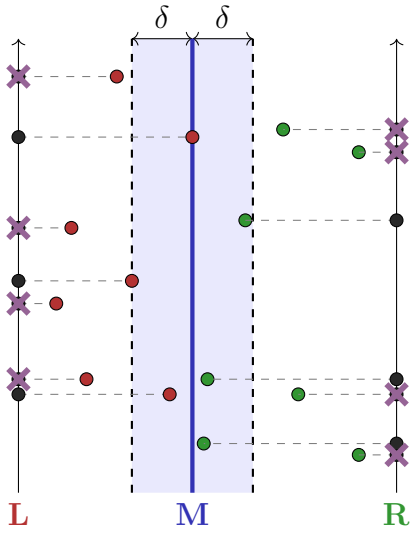
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$



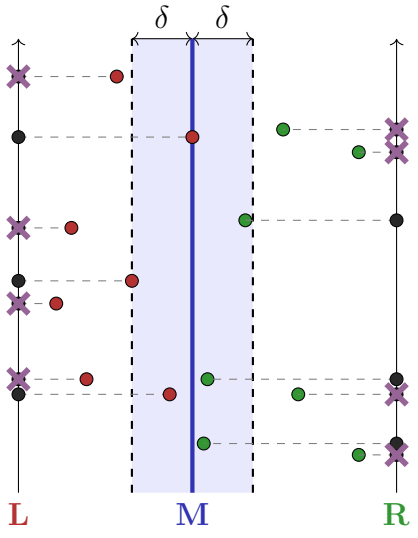
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$



Minimum Distance across middle line: Algorithm

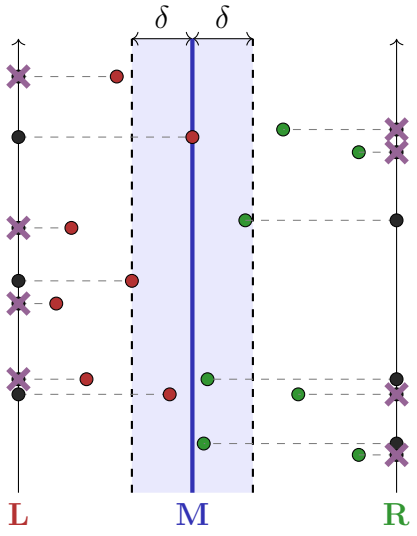
- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$



Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$

Running time:

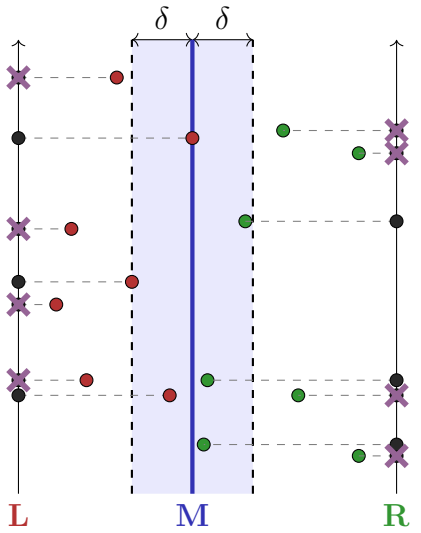


Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$

Running time:

- Sorting: $\Theta(n \log n)$

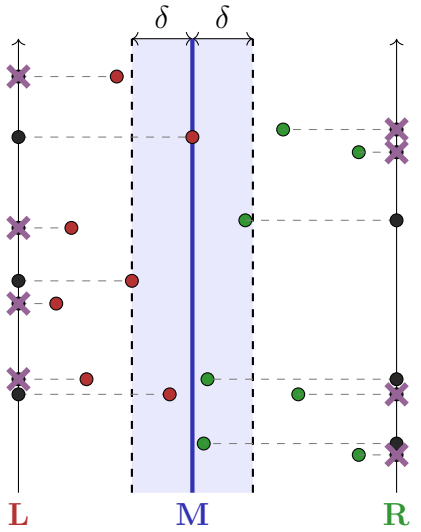


Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$
  $\Rightarrow$ at most 8 points

Running time:

- Sorting: $\Theta(n \log n)$
- Filtering: $\Theta(n)$

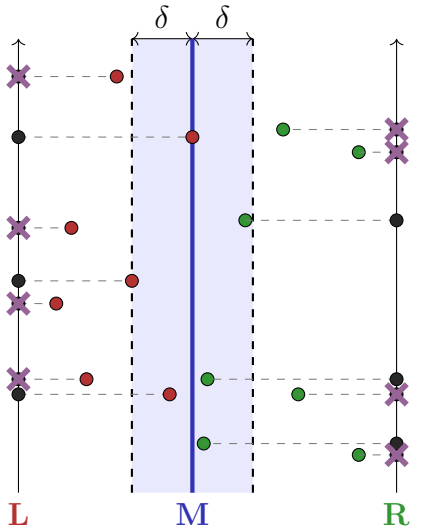


Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$
  $\Rightarrow$ at most 8 points

Running time:

- Sorting: $\Theta(n \log n)$
- Filtering: $\Theta(n)$
- compute the distances: $\Theta(n)$



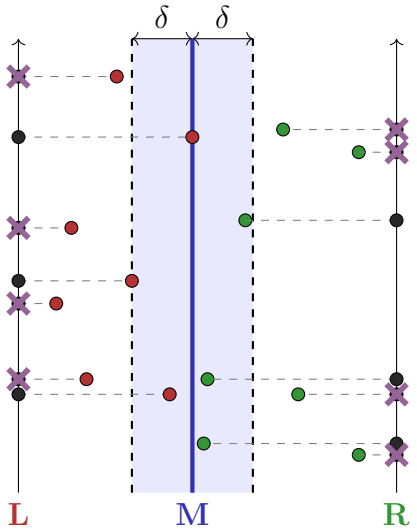
Minimum Distance across middle line: Algorithm

- sort L and R according to y-coordinates
- filter L and R according to band around M
- for every remaining point in L, compute distance to all points in R in the strip with y-distance $\leq \delta$
  $\Rightarrow$ at most 8 points

Running time:

- Sorting: $\Theta(n \log n)$
- Filtering: $\Theta(n)$
- compute the distances: $\Theta(n)$

$\Rightarrow \Theta(n \log n)$ per recursion step



Implementation

- Goal: recursion equation (runtime) $T(n) = 2 \cdot T(\frac{n}{2}) + \mathcal{O}(n)$.
- Non-trivial: only arrays Y and Y'
- Idea: merge reversed: run through Y that is presorted by y-coordinate. For each element follow the selection criterion of the x-coordinate and append the element either to Y_L or Y_R. Same procedure for Y'. Runtime $\mathcal{O}(|Y|)$.

Overall runtime: $\mathcal{O}(n \log n)$.

Questions

Questions

- How does the algorithm compare to a brute-force approach?

Questions

- How does the algorithm compare to a brute-force approach?
  - Divide and conquer reduces the problem size at each step, resulting in a time complexity of $\mathcal{O}(n)$, while a brute-force approach has a time complexity of $\mathcal{O}(n^2)$.

Questions

- How does the algorithm compare to a brute-force approach?
  - Divide and conquer reduces the problem size at each step, resulting in a time complexity of $\mathcal{O}(n)$, while a brute-force approach has a time complexity of $\mathcal{O}(n^2)$.
- Why do we avoid sorting at each step of the recursion?

Questions

- How does the algorithm compare to a brute-force approach?
  - Divide and conquer reduces the problem size at each step, resulting in a time complexity of $\mathcal{O}(n)$, while a brute-force approach has a time complexity of $\mathcal{O}(n^2)$.
- Why do we avoid sorting at each step of the recursion?
  - Sorting is $\mathcal{O}(n \log n)$ and the time complexity of conquer should be linear.

Questions?

9. Old Exam Question

Traversal

$r, r.L, r.R$

Die Hauptreihenfolgeausgabe eines binären Suchbaumes sei

Let the pre-order traversal output of a binary search tree be

$$15, 2, 1, 10, 5, 7, 14, 12.$$

Finden Sie die Nebenreihenfolge.

Find the post-order traversal.

Nebenreihenfolge / *post-order traversal*:

Traversal – Solution

Die Hauptreihenfolgeausgabe eines binären Suchbaumes sei

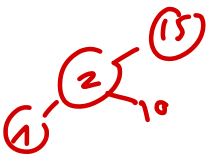
Let the pre-order traversal output of a binary search tree be

$$15, 2, 1, 10, 5, 7, 14, 12.$$

Finden Sie die Nebenreihenfolge.

Find the post-order traversal.

Nebenreihenfolge / *post-order traversal*: `1, 7, 5, 12, 14, 10, 2, 15`



10. Outro

General Questions?

See you next time

Have a nice week!