Algorithms and Data Structures

Exercise Session 2

https://n.ethz.ch/~ahmala/and

Quiz

- Every question 0.2 bonus points worth
- 5 minutes

252-0026-00L Algorithmen und Datenstrukturen HS2024 / Week 3 (30.09.2024 - 06.10.2024) / Moodle Quiz Week 3

Moodle Quiz Week 3

Quiz   Results   More

Opens: Monday, 30 September 2024, 9:10 AM
Closes: Monday, 30 September 2024, 9:30 AM

PASSWORD: **********

PASSWORD: induction5

O-Notation

- Upper Bound
- useful in the analysis of algorithms

Definition 1 (O-Notation)**.** For $f : N \to \mathbb{R}^+$,

$$O(f) \coloneqq \{g : N \to \mathbb{R}^+ \mid \exists C > 0\, \forall n \in N\; g(n) \leq C \cdot f(n)\}.$$

O-Notation

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Definition 1 (O-Notation). For $f : N \to \mathbb{R}^+$,

$$O(f) \coloneqq \{g : N \to \mathbb{R}^+ \mid \exists C > 0\ \forall n \in N\ g(n) \leq C \cdot f(n)\}.$$

$$\mathcal{O}(f) = \{g : \mathbb{N} \mapsto \mathbb{R}^+ \mid \exists C > 0\ \forall n \in N\ g(n) \leq C \cdot f(n)\}$$

$$\mathcal{O}(n^2) = \{n, log(n), \sqrt{n}, log^2(n), 1, 100n^2, \cdots\}$$

O-Notation

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Definition 1 (O-Notation). For $f : N \to \mathbb{R}^+$,

$$O(f) \coloneqq \{g : N \to \mathbb{R}^+ \mid \exists C > 0\, \forall n \in N\ g(n) \leq C \cdot f(n)\}.$$

$$n \leq \mathcal{O}(n^2) \tag{1}$$
$$n \leq \mathcal{O}(n) \tag{2}$$
$$1 \leq \mathcal{O}(n) \tag{3}$$
$$log(n) \leq \mathcal{O}(\sqrt{n}) \tag{4}$$
$$100n^2 \leq \mathcal{O}(n^2) \tag{5}$$

Theorem 1. *Let N be an infinite subset of $\mathbb{N}$ and $f : N \to \mathbb{R}^+$ and $g : N \to \mathbb{R}^+$.*

- *If* $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$, *then* $f \leq O(g)$ *and* $g \nleq O(f)$.
- *If* $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, *then* $f \leq O(g)$ *and* $g \leq O(f)$.
- *If* $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, *then* $f \nleq O(g)$ *and* $g \leq O(f)$.

Theorem 2. *Let* $f, g, h : N \to \mathbb{R}^+$. *If* $f \leq O(h)$ *and* $g \leq O(h)$, *then*

1. *For every constant* $c > 0, c \cdot f \leq O(h)$.
2. $f + g \leq O(h)$.

Corrections

- You receive your corrections in a week
- Will be uploaded to Moodle

Which one grows asymptotically faster?

I) $\log_2 n$

II) $\log_{10} n$

Which one grows asymptotically faster?

I) $\log_2 n$

II) $\log_{10} n$

Any two logarithms differ by a constant factor, the base makes no difference.

Exercise 1.1 *Mathematical induction* **(2 points)**.

(a) Prove by mathematical induction that for every integer $n \geq 0$,

$$1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

Solution:

Base Case.
Let $n = 0$. Then,

$$1 = (0+1)^2,$$

so the property holds for $n = 0$.

Induction Hypothesis.
Assume that the property holds for some integer $k \geq 0$, that is,

$$1 + 3 + 5 + \ldots + (2k+1) = (k+1)^2.$$

Induction Step.
We must show that the property also holds for $k+1$. Let us add $2(k+1)+1$ to both sides of the induction hypothesis. We get

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2k+1) + (2(k+1)+1) &= (k+1)^2 + (2(k+1)+1) \\ &= (k^2 + 2k + 1) + (2k+3) \\ &= k^2 + 4k + 4 \\ &= (k+2)^2 = ((k+1)+1)^2. \end{aligned}$$

By the principle of mathematical induction, $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$ is true for any integer $n \geq 0$.

(b) Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 6a_n - 2$ for $n \geq 1$. Determine the smallest positive integer m such that $a_m > 2^{2m}$. Then, prove by induction that $a_n \geq 2^{2n}$ for all integers $n \geq m$. (If you are unable to determine m, use $m = 10$. You may assume that $a_{10} \geq 2^{20}$). In your solution, you should address the base case, the induction hypothesis and the induction step.

Solution:

Writing out the first few terms of the recursive formula, we get $a_1 = 2, a_2 = 10, a_3 = 58, a_4 = 346$. On the other hand, $2^{2\cdot 1} = 4, 2^{2\cdot 2} = 16, 2^{2\cdot 3} = 64, 2^{2\cdot 4} = 256$. So $m = 4$ is the smallest positive integer where $a_m \geq 2^{2m}$.

We prove that $a_n \geq 2^{2n}$ for all integers $n \geq m = 4$ using induction.

Base Case.
Our base case is $n = m = 4$. As we have seen above,

$$a_4 = 346 \geq 256 = 2^{2\cdot 4}.$$

So the inequality holds for $n = 4$.

Induction Hypothesis.
We now assume that it is true for $n = k$, i.e., $a_k \geq 2^{2k}$.

Induction Step.
We want to prove that it is also true for $n = k + 1$. Using the recursive formula, and the induction hypothesis, we get

$$a_{k+1} \overset{\text{RF}}{=} 6a_k - 2 \overset{\text{IH}}{\geq} 6 \cdot 2^{2k} - 2 = 4 \cdot 2^{2k} + 2 \cdot 2^{2k} - 2 \geq 4 \cdot 2^{2k} = 2^2 \cdot 2^{2k} = 2^{2(k+1)}.$$

Hence, it is true for $n = k + 1$.

By the principle of mathematical induction, we conclude that $a_n \geq 2^{2n}$ is true for any integer $n \geq m = 4$.

Exercise 1.2 *Sums of powers of integers.*

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \leq n^4$.

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \geq \frac{1}{2^4} \cdot n^4$.

Hint: *Consider the second half of the sum, i.e.,* $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3$. *How many terms are there in this sum? How small can they be?*

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \geq \frac{1}{2^4} \cdot n^4$.

Hint: *Consider the second half of the sum, i.e.,* $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3$. *How many terms are there in this sum? How small can they be?*

Solution:

We have

$$\sum_{i=1}^{n} i^3 \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3 \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^3 = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^3.$$

By definition of $\lceil \cdot \rceil$, we have $\lceil \frac{n}{2} \rceil - 1 \leq \frac{n}{2}$, hence, $n - \lceil \frac{n}{2} \rceil + 1 \geq \frac{n}{2}$. Hence,

$$\sum_{i=1}^{n} i^3 \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^3 = \frac{1}{2^4} \cdot n^4.$$

(c)* Show that parts (a) and (b) generalise to an arbitrary $k \geq 4$, i.e., show that $\sum_{i=1}^{n} i^k \leq n^{k+1}$ and that $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$ holds for any $n \in \mathbb{N}_0$.

Solution:

Similar to part (a), to show that $\sum_{i=1}^{n} i^k \leq n^{k+1}$ we note that all terms in the sum are at most n^k. Thus, we get that

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}.$$

To show that $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$, we consider again the second half of the sum, i.e. only $i \geq \lceil \frac{n}{2} \rceil$. We get that

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^k.$$

As before, we have that $\lceil \frac{n}{2} \rceil - 1 \leq \frac{n}{2}$ and $n - \lceil \frac{n}{2} \rceil + 1 \geq \frac{n}{2}$. Hence, we can conclude that

$$\sum_{i=1}^{n} i^k \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} \cdot n^{k+1}.$$

Exercise 1.3 *Asymptotic growth* **(1 point).**

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements with a computation.

(a) $f(m) = 10m^5 + 90m^4$ grows asymptotically slower than $g(m) = 100m^5$.

(b) $f(m) = 80 \cdot 2^m \log(m) - 2^m$ grows asymptotically slower than $g(m) = 5 \cdot 2^m \log(m)^2$.

(c) $f(m) = \log(m^3)$ grows asymptotically slower than $g(m) = \log(m)^3$.

(d) $f(m) = 4^{(m^2+m+1)}$ grows asymptotically slower than $g(m) = 2^{(3m^2)}$.

(e)* If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any* $a, b : \mathbb{N} \to \mathbb{R}^+$, *if* $\lim_{m\to\infty} a(m) = A$ *and* $\lim_{m\to\infty} b(m) = B$, *then* $\lim_{m\to\infty} a(m)b(m) = AB$.

(f)* If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

(e)* If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any $a, b : \mathbb{N} \to \mathbb{R}^+$, if $\lim_{m\to\infty} a(m) = A$ and $\lim_{m\to\infty} b(m) = B$, then $\lim_{m\to\infty} a(m)b(m) = AB$.*

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty}\frac{f(m)}{h(m)} &= \lim_{m\to\infty}\frac{f(m)g(m)}{h(m)g(m)}\\ &= \lim_{m\to\infty}\frac{f(m)}{g(m)}\cdot\frac{g(m)}{h(m)}\\ &= \lim_{m\to\infty}\frac{f(m)}{g(m)}\cdot\lim_{m\to\infty}\frac{g(m)}{h(m)} = 0.\end{aligned}$$

(f)* If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Solution:

False, consider $f(m) = 1/m^2$, $g(m) = 1/m$ and $h(m) = m^3$. They satisfy the conditions, but f does not grow slower than $g(h(m)) = 1/m^3$.

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n}\leq\frac{1}{\sqrt{3n+1}},\qquad n\geq 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

(b)* Replace $3n+1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction.

Solution:

Base Case.
For $n = 1$ we have

$$\frac{1}{2} \leq \frac{1}{\sqrt{4}},$$

which is even an equality.

Induction Hypothesis.
Now we assume that it is true for $n = k$, i.e.,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}.$$

Induction Step.
We will prove that it is also true for $n = k+1$, that is we want to show that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}.$$

Plugging in the induction hypothesis, it is sufficient to prove.

$$\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}} \iff \frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}.$$

We can simplify this further as follows:

$$\begin{aligned}
\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k+1}{3k+4}} &\iff \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4} \\
&\iff (4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1) \\
&\iff 12k^3+28k^2+19k+4 \leq 12k^3+28k^2+20k+4 \\
&\iff 0 \leq k
\end{aligned}$$

This last statement is true since $k \in \mathbb{N}$. Hence, the inequality is also true for $n = k+1$.

Note that for the first equivalence we need that all terms involved are positive, which is true since we have $k \in \mathbb{N}$.

By the principle of mathematical induction, we conclude that the inequality is true for any positive integer n.

Important Series

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

$$\sum_{k=0}^{n} q^k = \frac{q^{n+1} - 1}{q - 1} \tag{3}$$

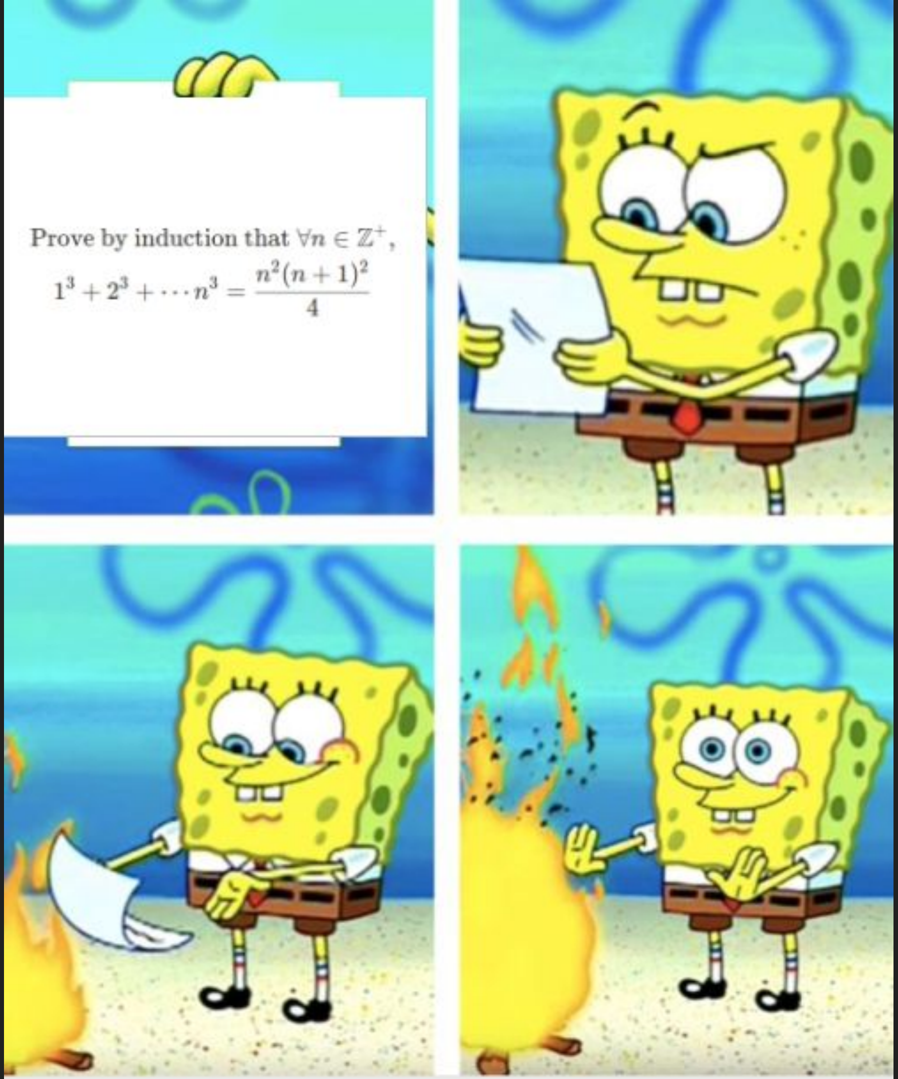
For now assume q > 1

$$\sum_{k=0}^{3} 3^k = 3^0 + 3^1 + 3^2 + 3^3 = \frac{3^4 - 1}{3 - 1} = 40 \tag{4}$$

IN-CLASS EXERCISE

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$



Prove by induction that $\forall n \in \mathbb{Z}^+$,
$1^3 + 2^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$

IN-CLASS EXERCISE

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

- **Base Case.**
  Let $n = 1$. Then,
  $$1^3 = 1 = \frac{1^2 \cdot (1+1)^2}{4},$$
  so the property holds for $n = 1$.
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer k, that is,
  $$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

- **Induction Step.**
  We must show that the property also holds for $k+1$. Let us add $(k+1)^3$ to both sides of the induction hypothesis. We get

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2 + ((k+1)+1)^2}{4}.
\end{aligned}$$

  By the principle of mathematical induction, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ is true for any positive integer n.

Peer Grading

- Deadline is (today) Monday 23:59
- 1 point every week
- One exercise chosen every week
- This week Exercise 1.1
- Find mistakes, note that on paper/digital, use a distinctive color
- If everything is correct, just write that it is correct
- The document to be peer graded is on Moodle
- Upload it on Moodle