Exercise 4.1.

(a)

$$T(n) \leq 4 \cdot T(n/2) + 100n$$

$a = 4, \quad b = 1$

$b < \log_2 a \Rightarrow T(n) \leq O(n^{\log_2(b)})$

$\Rightarrow T(n) \leq O(n^2)$

(b)

$$T(n) = T(n/2) + \frac{3}{2}n$$

$a = 1, \quad b = 1$

$b > \log_2(a) \Rightarrow T(n) \leq O(n^b)$

$\Rightarrow T(n) \leq O(n)$

(c)

$$T(n) = 4 \cdot T(\frac{n}{2}) + \frac{7}{2}n^2$$

$a = 4, \quad b = 2$

$b = \log_2(a) \Rightarrow T(n) \leq O(n^{\log_2(a)} \cdot \log_2 n)$

$\Rightarrow T(n) \leq O(n^2 \cdot \log n)$

Exercise 4.2.a

- $\frac{n}{\log n} > \frac{n}{\sqrt{n}} \geq \Omega(\sqrt{n})$, so false

- $\log(n!) = \sum_{i=1}^{n} \log(i) \leq \sum_{i=1}^{n} \log(n) \lesssim O(n\log n)$, so false

- $n^k \geq \Omega(k^n)$, consider $k=2 \Rightarrow n^2 \not\geq \Omega(2^n)$, so false

- $\log_3 n^4 = \Theta(\log_7 n^8)$, true

Exercise 4.2.b

- $\frac{n}{\log n} \geq \frac{n}{\sqrt{n}} = \sqrt{n} = n^{1/2} \geq \Omega(n^{1/2})$, so true

- $\log_3(n^{\sqrt{n}}) = \sqrt{n} \cdot \log_3(n) = \Theta(\sqrt{n} \cdot \log n)$

  $\log_7(n^8) = 8 \cdot \log_7(n) = \Theta(\log n)$, so false

- $3n^4 + n^2 + n \geq n^2 \geq \Omega(n^2)$, so true

- Assume $n \to \infty$ for convenience

  : $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n \geq \lceil \frac{n}{10} \rceil \cdot \dots \cdot n \geq \left(\frac{n}{10}\right)^{0,9n}$

  We prove the given expression is false, if we prove $\left(\frac{n}{10}\right)^{0,9n}$ is asymptotically growing faster than $n^{n/2}$.

$$\lim_{n \to \infty} \frac{\left(\frac{n}{10}\right)^{0,9n}}{n^{n/2}} = \frac{n^{0,9n - 0,5n}}{10^{0,9n}} = \frac{n^{0,4n}}{10^{0,9n}} = \left(\frac{n}{10^{4/9}}\right)^{0,4n} = \infty$$

So $\left(\frac{n}{10}\right)^{0,9n}$ grows asymptotically faster than $n^{n/2}$.

Thus $n! \leq O(n^{n/2})$ is wrong.

Exercise 4.3

Prove $I(J)$ by a mathematical induction on J.

Base Case ($J=1$): Assume the largest element in the array is in index X. Until $i=X$, the place of largest element doesn't change. But in each iteration of i for $i \geq X$ $A[i]$ and $A[i+1]$ are going to be swapped since $A[i]$ will be the largest element. This way the largest element $A[X]$ at its initial position lands up to $A[n]$, since i becomes $n-1$ at the end. Hence, after 1 iteration of J, the largest element of the array is at the correct place.

I. H.: Assume for an arbitrary $k \in \mathbb{N}, k<N$; after k iterations the largest k elements are at the correct place.

Induction Step ($k \to k+1$): Using induction hypothesis we know that k largest elements are at the correct place. Hence $A[n-k+1]$, $A[n-k+1]$.... $A[n]$ are at their correct place. Assume that $(k+1)$-th largest element is at index X. Until $i=X$, the place of $(k+1)$-th largest element is not going to change. But in each iteration of i for $n-k-1 \geq i \geq X$ $A[i]$ and $A[i+1]$ are going to be swapped. Hence, at the end of $(k+1)$-th iteration of J, the $(k+1)$-th largest element lands at position $(n-k)$. This means that $(k+1)$-largest elements are at the correct place.

Exercise 4.4

finding min T satisfying $f(T) \geq N$

a)

```
x ← 1
while ( f(x) < N ):
    x ← 2·x
return x
```

Correctness and Time Complexity to be added

b)

```
r ← 1
while ( f(r) < N ):
    r ← 2·r
l ← r/2
while (l < r):
    mid ← (l+r)/2
    if ( f(mid) ≥ N ):
        r ← mid
    else if ( f(mid) < N ):
        l ← mid+1
return r
```

Correctness and Time Complexity to be added

Exercise 4.5

a) • f will be $\left(\sum_{i=1}^{\lfloor \log n \rfloor} i\right)$ - times called.

$$\sum_{i=1}^{\lfloor \log n \rfloor} i \leq \sum_{i=1}^{\lfloor \log n \rfloor} \lfloor \log n \rfloor \leq O(\log^2(n))$$

$$\sum_{i=1}^{\lfloor \log n \rfloor} i \geq \sum_{\left\lceil \frac{\lfloor \frac{\log n}{2} \rfloor}{2} \right\rceil}^{\lfloor \log n \rfloor} i \geq \sum_{\left\lceil \frac{\lfloor \frac{\log n}{2} \rfloor}{2} \right\rceil}^{\lfloor \log n \rfloor} \left\lfloor \frac{\log n}{2} \right\rfloor / 2 \geq \frac{\log(n)}{2} \cdot \left\lfloor \frac{\log(n)}{2} \right\rfloor / 2 \geq \Omega(\log^2(n))$$

Thus, $\Theta(\log^2(n))$.

: f will be called $\left(\sum_{i=1}^{n} \sum_{j=i}^{\lfloor \log_2 n \rfloor} 1\right)$ - times, while $2^i \leq n$; $i \leq \lfloor \log_2 n \rfloor$.

if $i > \lfloor \log_2 n \rfloor$, f won't be called.

So, let's calulate $\sum_{i=1}^{\lfloor \log n \rfloor} \sum_{j=i}^{\lfloor \log n \rfloor} 1$.

$$\sum_{i=1}^{\lfloor \log n \rfloor} \sum_{j=i}^{\lfloor \log n \rfloor} = \sum_{i=1}^{\lfloor \log n \rfloor} \left(\lfloor \log n \rfloor - i + 1\right)$$

$$= \lfloor \log n \rfloor \cdot \lfloor \log n \rfloor - \frac{\lfloor \log n \rfloor \cdot (\lfloor \log n \rfloor + 1)}{2} + \lfloor \log n \rfloor = \Theta(\log^2(n))$$

b) $T(n) = 8 \cdot T\left(\frac{n}{2}\right) + n^2 \cdot 2n = 8 \cdot T\left(\frac{n}{2}\right) + 2n^3$

Using Master Theorem: $3 = \log(8)$

$\Rightarrow T(n) = \Theta(n^3 \cdot \log n)$

c)

$T(n) = 8 \cdot T(\frac{n}{2}) + 2n^3$

Base Case: Show $T(2) \geq T(1)$

Induction Hypothesis: Assume for some $k \in \mathbb{N}$ $T(k'+1) \geq T(k')$ for all $k' \in \mathbb{N}$ with $k' \leq k$.

Inductive Step: Show $T(k+2) \geq T(k+1)$