Exercise 7.1

2D DP-Table.

$DP[a][s]$ is true iff s can be written as a sum $\sum_{i \in I} c_i \cdot A[i]$ where $I \subseteq \{i : 1 \leq i \leq a\}$ and $c_i \in \{1,3\}$ for each $i \in I$.

Base Case
$$\begin{cases} DP[0][s] = \text{false} & \text{for } 1 \leq s \leq b \\ DP[a][0] = \text{true} & \text{for } 0 \leq a \leq b \end{cases}$$

Computation: (do all three lines) for $1 \leq a \leq n$ && $1 \leq s \leq b$

$DP[a][s] = DP[a-1][s]$

if $(s - A[a] \geq 0)$ $DP[a][s] = DP[a][s]$ or $DP[s - A[a]]$

if $(s - 3A[a] \geq 0)$ $DP[a][s] = DP[a][s]$ or $DP[s - 3 \cdot A[a]]$

Compute first by order of increasing a then by increasing order of s.

Solution found in $DP[n][b]$.

$(n+1)(b+1)$ entries $\Rightarrow \Theta(nb)$ as a running time.

Exercise 7.5

The idea here is to work with smaller numbers to decrease the running time effected by W.

Example $(n=4)$:

	I_1	I_2	I_3	I_4
Weights	4	8	5	3
Values	5	12	8	1

Given Weight Limit $W = 12$, we find opt in $O(4 \cdot 12)$ $O(n \cdot W)$.

opt is 17 with items I_1 and I_2.

Assume $k = 3$.

Weights: 3, 6, 3, 3

Values: 5, 12, 8, 1

opt can be found in $O(4 \cdot \frac{12}{3})$.

opt is 25 with I_1, I_2, I_3.

Let's check the weights. In this modified version we have weight $= 4+8+5 = 17$

$$17 \leq (1+\varepsilon) \cdot 12$$

$$\Rightarrow \varepsilon = 0{,}42$$

$I = \{1,2,3\}$ is a $0{,}42$- feasible.

7.5.a

We know weights are multiples of k.

So we compute each dp[i][j], where j is a multiple of k. So, the running time becomes $O(n \cdot \frac{W}{k})$.

Note that $1 \le i \le n$ and $1 \le j \le W$.

7.5.b)

We have $\tilde{w}_i \le w_i$. So the weights get decreased while the weight limit stayed the same. So $opt_k \ge opt$, since any set I of items with $w(I) \le W$ also satisfies $\sum_{i \in I} \tilde{w}_i \le W$.

7.5.c)

Show $\varepsilon = (nk) / w_{max} \Rightarrow w(I_k) \le (1+\varepsilon) \cdot W$

$$\tilde{w}_i = k \cdot \left\lfloor \frac{w_i}{k} \right\rfloor \Rightarrow \tilde{w}_i \ge k \cdot \left(\frac{w_i}{k} - 1 \right) = k \cdot \left(\frac{w_i - k}{k} \right) = w_i - k$$

$$w(I_k) = \sum_{i \in I_k} w_i \le \sum_{i \in I_k} (\tilde{w}_i + k)$$

$$= \sum_{i \in I_k} \tilde{w}_i + \sum_{i \in I_k} k$$

$$\le W + n \cdot k$$

$$\le W + \frac{W}{w_{max}} \cdot nk = W \left(1 + \frac{nk}{w_{max}} \right)$$

7.5.d)

$k = (w_{max} \cdot \varepsilon)/n$

$$n \cdot \frac{W}{k} = n \cdot \frac{W}{w_{max}\,\varepsilon} \cdot n = \frac{W}{w_{max} \cdot \varepsilon} \cdot n^2 = \frac{n \cdot w_{max}}{w_{max} \cdot \varepsilon} \cdot n^2 \leq O\left(\frac{n^3}{\varepsilon}\right)$$

We know $opt_k \geq opt \Rightarrow v(I_k) \geq opt$ (I)

$k = (w_{max} \cdot \varepsilon)/n \Rightarrow \varepsilon = \frac{k \cdot n}{w_{max}} \longrightarrow w(I_k) \leq (1+\varepsilon) \cdot W$

7.5.e)

$v_1 = v_2 = w_1 = w_2 = 101$ and $W = 200$

Exercise 7.2)

0 km	4 km	7 km	12 km	15 km	19 km
C_0	C_1	C_2	C_3	C_4	C_5
	1 possibilities	2 poss.	3 poss.	5 poss.	8 poss.

$d = 8$, we don't want do drive more than d kilometers without making a stop.

$$DP[0] = 1, \quad DP[i] = \sum_{\substack{0 \leq j < i \\ k_i \leq k_j + d}} DP[j] \quad \text{for } 1 \leq i \leq n$$

7.2.b)

$$DP[0] = 1 \quad , \quad DP[i] = \sum_{\substack{0 < j < i \\ k_i \leq k_j + d}} DP[j] \qquad \text{for } 1 \leq i \leq n$$

there are at most 10 terms

Exercise 7.3

$DP[1 \ldots N][1 \ldots M]$

$DP[i][j]$ counts the number of distinct safe pawn lines on an $N \times j$ chessboard with the pawn in the last column located in row i.

$$\begin{array}{|c|} \hline 1 \\ 1 \\ 1 \\ 1 \\ \hline \end{array} \cdots\cdots \begin{array}{|c|} \hline 1 \\ 2 \\ 2 \\ 1 \\ \hline \end{array} \cdots\cdots \begin{array}{|c|} \hline 2 \\ 3 \\ 3 \\ 2 \\ \hline \end{array} \cdots\cdots$$

Exercise 7.4

$n = 7 \qquad k = 3 \qquad \Rightarrow$

1111000
1110110
1110011
1101110
1100111
1011110
1001111

0101111 —

0011110 —

0001111 —

$$\boxed{DP[i][j][0] = DP[i-1][j][0] + DP[i-1][j][1]}$$

$$\boxed{DP[i][j][1] = DP[i-1][j-1][1] + DP[i-1][j][0]}$$