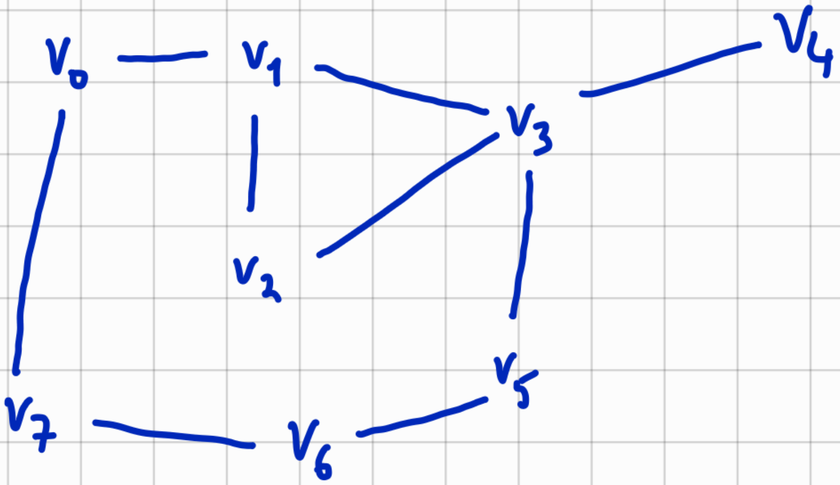
No multi graphs and self loops in AnD.



(v_0, v_1, v_2) is a walk.

$(v_0, v_1, v_3, v_4, v_3, v_1, v_0)$ is a closed walk.

$(v_4, v_3, v_5, v_6, v_7, v_0, v_1, v_2, v_3, v_1)$ is an Eulerian Walk

every edge appears exactly once

$(v_4, v_3, v_5, v_6, v_7, v_7, v_0, v_1, v_2)$ is a Hamiltonian Path.

every vertex appears once

This graph doesn't have any Hamiltonian Cycle.

cycle that contains every vertex

Above graph is connected.

Above graph is not a tree.

Exercise 8.1.

a) Let our graph be $G = (V, E)$. Prove the statement

a) Let our graph be $G = (V, E)$. Prove the statement by case distinction.

i) $\exists v \in V \ \deg(v) \geq 3$: Consider three neighbors n_1, n_2, n_3 of v.

i.i.) At least one of the edges $(n_1, n_2), (n_2, n_3), (n_1, n_3)$ are present in G. Then the vertices in this present edge and v are all pairwise adjacent.

i.ii) None of the edges $(n_1, n_2), (n_2, n_3), (n_1, n_3)$ are present in G. Then the vertices n_1, n_2 and n_3 are all pairwise not adjacent.

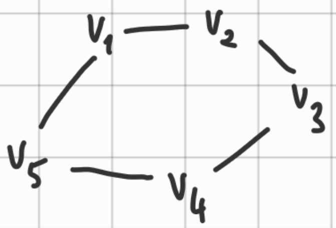
ii) $\forall v \in V \ \deg(v) \leq 2$. Choose arbitrarily $v' \in V$. Consider three non-neighbors n_1, n_2, n_3 of v'.

ii.i.) At least one of the edges $(n_1, n_2), (n_2, n_3), (n_1, n_3)$ is not present in G. Then the vertices in this non-present edge and v are all pairwise not adjacent.

ii.ii) All of the edges $(n_1, n_2), (n_2, n_3), (n_1, n_3)$ are present in G. Then the vertices n_1, n_2 and n_3 are all pairwise adjacent.

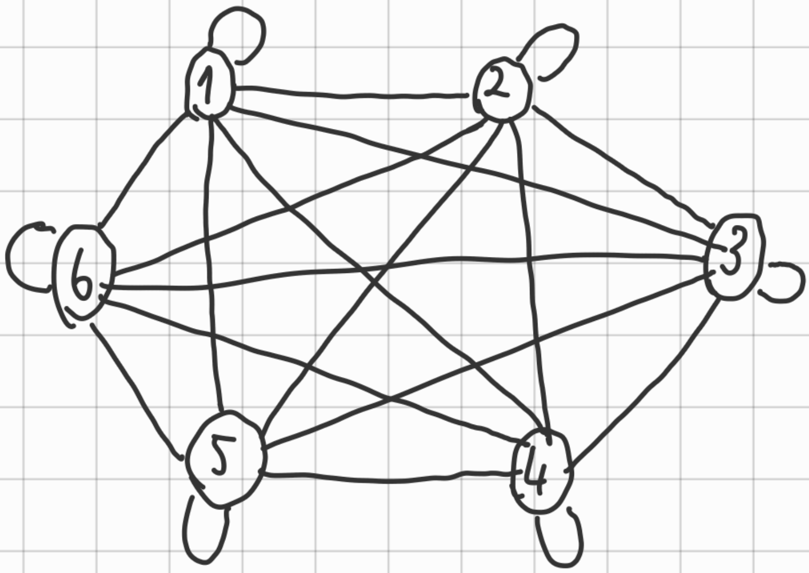
b)

Counter Example :



Exercise 8.2.

Describe it as a graph. For example we get the following graph for $n = 6$.



It is a complete graph with self loops. If there exists an Eulerian Walk we can line up all different tiles along a line.

Exercise 8.3

```
while countNeighbors() ≠ n−1 do
    setFlag()
    for i=1 to countNeighbors() do
        moveTo(i)
        if isSet() then
            undo()
        else
            break
```

in each iteration of the while loop a new previously unmarked vertex is explored.

If the current vertex has no unmarked neighbors we loop forever on this vertex. But this won't happen. For this case to occur, vertex star must have been previously marked. But the algorithm always terminates when reaching the star.

Cost of each iteration is $O(1) + O(\deg v)$, so in total $O(|V|) + O(\sum_{v \in V} \deg v) = O(|V|) + O(2 \cdot |E|) = O(|V| + |E|)$

Exercise 8.4

Exercise 8.4.

a) Consider the longest path $P = (v_0, v_1, \dots, v_{k-1}, v_k)$ in G. Assume v_0 is not a leaf. Thus it has a degree min 2. Hence, there exists a neighbor $x \neq v_1$ of v_0. Consider the path $P' = (x, v_0, v_1, \dots, v_{k-1}, v_k)$. P' is longer than P, but this is a contradiction. So, v_0 must be a leaf.

b) We know that a tree is connected and has no cycles.

Base Case: $n=1$, 0 edges ; $n=2$, 1 edge

I.H. : Assume hypothesis is true for every tree with $n \geq 2$ vertices i.e. it contains $n-1$ edges.

Induction Step: Show that property holds for every tree $G = (V, E)$ with $|V| = n+1$ vertices.
Let u be a leaf in G. Let v be u's only neighbor in the tree $G = (V, E)$. Consider the graph $G' = (V \setminus \{u\}, E \setminus \{u, v\})$. If we prove G' is a tree then we are done.

- G' is connected : Let $a, b \in V \setminus \{u\}$. There exists a path between a and b in G. u is not in this path. Hence, there exits a path between a and b in G'.

- G' has no cycles : Assume P is a cycle in G'. But then it is also a cycle in G.

contradiction.

c) n vertices & $n-1$ edges & connected $\longrightarrow$ tree (connected & no cycles)

B.C. : $n=1$ and $n=2$

I.H. : Assume the hypothesis, connected $n-1$ edges $\rightarrow$ tree

I.S. : Show that property holds for $n+1$.

Average degree: $\frac{2\cdot|E|}{|V|} = \frac{2n}{n+1} < 2$

So there must exist a vertex v with degree 1.

So v is a leaf.

Consider $G' = (V \setminus \{v\}, E \setminus \{u,v\})$.

If G' is connected it is a tree since it has $n-1$ edges. Then G' has no cycles. So G has no cycles and it is connected.

↳ because if there is a cycle in G, it must be fully contained in G' (since it cannot contain a leaf), this is impossible since G' is a tree.

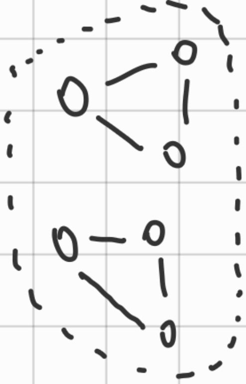
8.5.

a) Consider the shortest walk between v and w with length k.

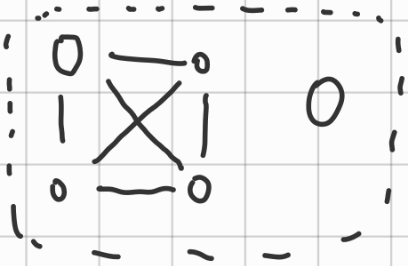
$W: v=v_0, v_1, \dots, v_k = w$

If W is not a path then there are $v_i = v_j$ with $0 \leq i < j \leq k$. But then $v_0, \dots, v_i = v_j, \dots, v_k$ is a walk between v and w with less length. Contradiction.

b)

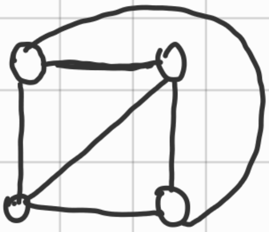


$6 \geq 6$



$6 \geq 5$

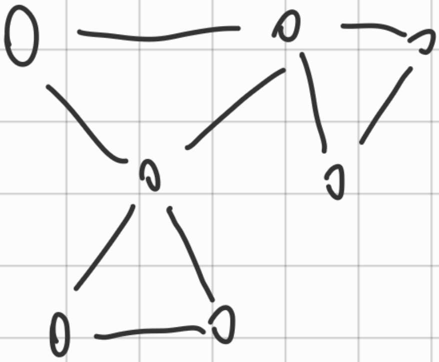
c)

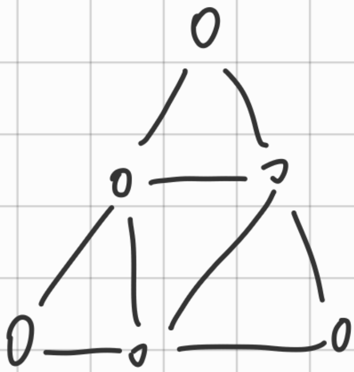


Each vertex has degree 3. No Eulerian Walk Possible.

d) i. Choose a connected component c of G. Assume it has n vertices. So, n-1 edges are present in c. Since each vertex has degree min 2, impossible to have n-1 edges present.

e)





f)

Assume G has a Hamiltonian cycle $v_1, v_2, v_3, \ldots, v_n, v_1$. After removing v_k (random node chosen), graph is still connected.