Algorithms and Data Structures

Exercise Session 6

https://n.ethz.ch/~ahmala/and

Tree Terminology-1

- Node
- Root
- Edge
- Size
- Depth
- Height

Tree Terminology-2

- Ancestor
- Descendant
- Degree
- Subtree
- Rank

https://en.wikipedia.org/wiki/Tree_(data_structure)#Terminology

Exercise 6.1 *Finding the i-th smallest key in an AVL tree* **(1 point)**.

Let A be an AVL tree (as described in the lecture) with n nodes. Let $k_1 < k_2 < \ldots < k_n$ be the keys of A, in ascending order. For a given $1 \leq i \leq n$, our goal is to find k_i, the i-th smallest key of A.

(a) Suppose $i = 1$. Describe an algorithm that finds k_1 in $O(\log n)$ time.

Hint: *An AVL tree is a BST (binary search tree).*

(b) Describe an algorithm that finds k_i in $O(i \cdot \log n)$ time.

Hint: *You are allowed to make changes to A while executing your algorithm.*

Exercise 6.1 *Finding the i-th smallest key in an AVL tree* **(1 point)**.

Let A be an AVL tree (as described in the lecture) with n nodes. Let $k_1 < k_2 < \ldots < k_n$ be the keys of A, in ascending order. For a given $1 \leq i \leq n$, our goal is to find k_i, the i-th smallest key of A.

(a) Suppose $i = 1$. Describe an algorithm that finds k_1 in $O(\log n)$ time.

Hint: *An AVL tree is a BST (binary search tree).*

(b) Describe an algorithm that finds k_i in $O(i \cdot \log n)$ time.

Hint: *You are allowed to make changes to A while executing your algorithm.*

Exercise 6.1 *Finding the i-th smallest key in an AVL tree* **(1 point).**

Let A be an AVL tree (as described in the lecture) with n nodes. Let $k_1 < k_2 < \ldots < k_n$ be the keys of A, in ascending order. For a given $1 \leq i \leq n$, our goal is to find k_i, the i-th smallest key of A.

(a) Suppose $i = 1$. Describe an algorithm that finds k_1 in $O(\log n)$ time.

Hint: *An AVL tree is a BST (binary search tree).*

(b) Describe an algorithm that finds k_i in $O(i \cdot \log n)$ time.

Hint: *You are allowed to make changes to A while executing your algorithm.*

It turns out that we can find k_i in time $O(\log n)$, if we modify the definition of an AVL tree a bit.

(c) Modify the definition of an AVL tree by storing two additional integers $s_l(v), s_r(v) \in \mathbb{N}$ in each node v. Assuming now that A satisfies your modified definition, describe an algorithm that finds k_i in $O(\log n)$ time.

Remark. Your modified definition should still allow for the search, insert and remove operations to be performed in $O(\log n)$ time, but you are not required to prove that this is the case.

(c) Modify the definition of an AVL tree by storing two additional integers $s_l(v), s_r(v) \in \mathbb{N}$ in each node v. Assuming now that A satisfies your modified definition, describe an algorithm that finds k_i in $O(\log n)$ time.

Remark. Your modified definition should still allow for the search, insert and remove operations to be performed in $O(\log n)$ time, but you are not required to prove that this is the case.

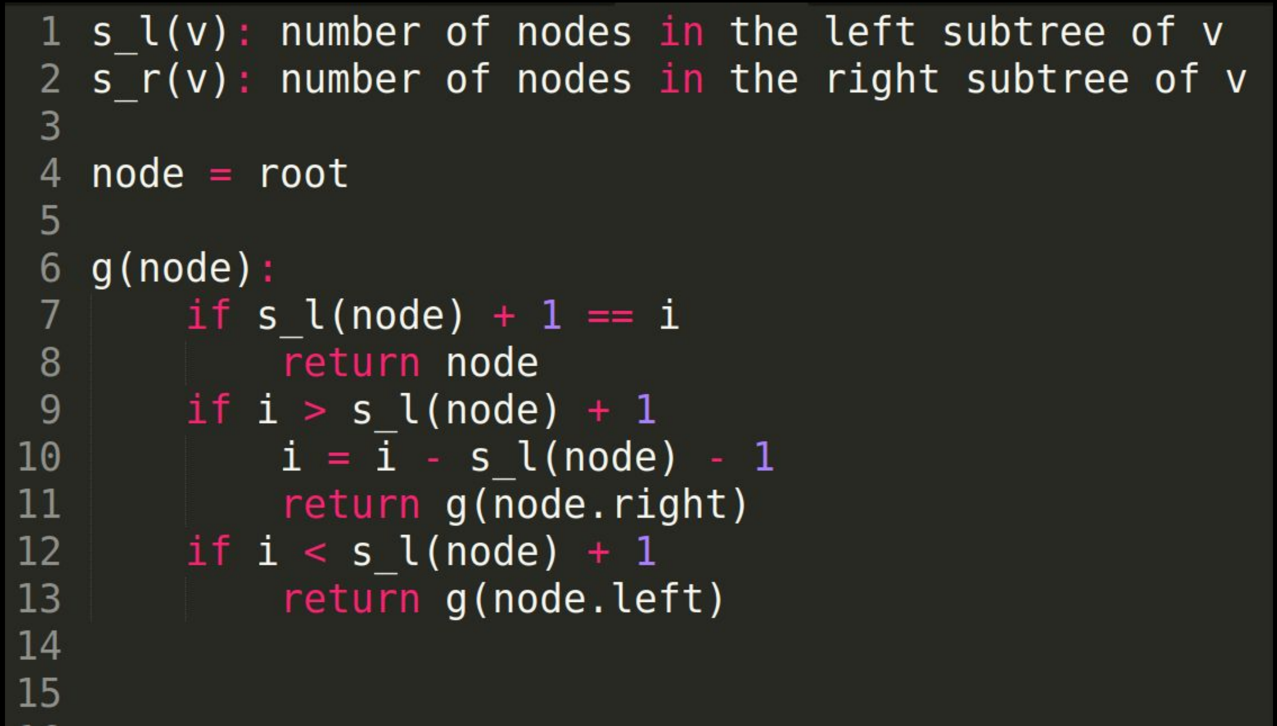
(c) Modify the definition of an AVL tree by storing two additional integers $s_l(v), s_r(v) \in \mathbb{N}$ in each node v. Assuming now that A satisfies your modified definition, describe an algorithm that finds k_i in $O(\log n)$ time.

Remark. Your modified definition should still allow for the search, insert and remove operations to be performed in $O(\log n)$ time, but you are not required to prove that this is the case.

```
s_l(v): number of nodes in the left subtree of v
s_r(v): number of nodes in the right subtree of v

node = root

g(node):
    if s_l(node) + 1 == i
        return node
    if i > s_l(node) + 1
        return g(node.right)
    if i < s_l(node) + 1
        return g(node.left)

```

(c) Modify the definition of an AVL tree by storing two additional integers $s_l(v), s_r(v) \in \mathbb{N}$ in each node v. Assuming now that A satisfies your modified definition, describe an algorithm that finds k_i in $O(\log n)$ time.

Remark. Your modified definition should still allow for the search, insert and remove operations to be performed in $O(\log n)$ time, but you are not required to prove that this is the case.



```
s_l(v): number of nodes in the left subtree of v
s_r(v): number of nodes in the right subtree of v

node = root

g(node):
    if s_l(node) + 1 == i
        return node
    if i > s_l(node) + 1
        i = i - s_l(node) - 1
        return g(node.right)
    if i < s_l(node) + 1
        return g(node.left)


```

Exercise 6.2 *Round and square brackets.*

A string of characters on the alphabet $\{\texttt{A}, \ldots, \texttt{Z}, \texttt{(}, \texttt{)}, \texttt{[}, \texttt{]}\}$ is called *well-formed* if either

1. It does not contain any round or square brackets, <u>or</u>
2. It can be obtained from an empty string by performing a sequence of the following operations, in any order and with an arbitrary number of repetitions:
   (a) Take two non-empty well-formed strings a and b and concatenate them to obtain ab,
   (b) Take a well-formed string a and add a pair of round brackets around it to obtain $\texttt{(}a\texttt{)}$,
   (c) Take a well-formed string a and add a pair of square brackets around it to obtain $\texttt{[}a\texttt{]}$.

The above reflects the intuitive definition that all brackets in the string are 'matched' by a bracket of the same type. For example, $s = \texttt{FOO(BAR[A])}$, is well-formed, since it is the concatenation of $s_1 = \texttt{FOO}$, which is well-formed by 1., and $s_2 = \texttt{(BAR[A])}$, which is also well-formed. String s_2 is well-formed because it is obtained by operation 2(b) from $s_3 = \texttt{BAR[A]}$, which is well-formed as the concatenation of well-formed strings $s_4 = \texttt{BAR}$ (by 1.) and $s_5 = \texttt{[A]}$ (by 2(c) and 1.). String $t = \texttt{FOO[(BAR])}$ is not well-formed, since there is no way to obtain it from the above rules. Indeed, to be able to insert the only pair of square brackets according to the rules, its content $t_1 = \texttt{(BAR}$ must be well-formed, but this is impossible since t_1 contains only one bracket.

Provide an algorithm that determines whether a string of characters is well-formed. Justify briefly why your algorithm is correct, and provide a precise analysis of its complexity.

Hint: *Use a data structure from the last exercise sheet.*

Algorithm 1 Detecting well-formed strings

```
function IsWellFormed($s$)
    $S \leftarrow$ emptyStack()
    for $i \in \{0, \ldots, |s| - 1\}$ do
        if $s[i] =$ "(" then
            $S$.push("(")
        else if $s[i] =$ "[" then
            $S$.push("[")
        else if $s[i] =$ ")" then
            if $S$.pop() $\neq$ "(" then
                return False
        else if $s[i] =$ "]" then
            if $S$.pop() $\neq$ "[" then
                return False
    return $S$.isEmpty()
```

Exercise 6.3 *Introduction to dynamic programming* **(1 point)**.

Consider the recurrence

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 2 \\ A_3 &= 3 \\ A_4 &= 4 \\ A_n &= A_{n-1} + A_{n-3} + 2A_{n-4} \text{ for } n \geq 5. \end{aligned}$$

(a) Provide a recursive function (using pseudo code) that computes A_n for $n \in \mathbb{N}$. You do not have to argue correctness.

(b) Lower bound the run time of your recursion from (a) by $\Omega(C^n)$ for some constant $C > 1$.

Exercise 6.3 *Introduction to dynamic programming* **(1 point)**.

Consider the recurrence

$$\begin{aligned}
A_1 &= 1\\
A_2 &= 2\\
A_3 &= 3\\
A_4 &= 4\\
A_n &= A_{n-1} + A_{n-3} + 2A_{n-4} \text{ for } n \geq 5.
\end{aligned}$$

(a) Provide a recursive function (using pseudo code) that computes A_n for $n \in \mathbb{N}$. You do not have to argue correctness.

Algorithm 2 $A(n)$

```
if $n \leq 4$ then
    return $n$
else
    return $A(n-1) + A(n-3) + 2A(n-4)$
```

Exercise 6.3 *Introduction to dynamic programming* **(1 point)**.

Consider the recurrence

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 2 \\ A_3 &= 3 \\ A_4 &= 4 \\ A_n &= A_{n-1} + A_{n-3} + 2A_{n-4} \text{ for } n \geq 5. \end{aligned}$$

(a) Provide a recursive function (using pseudo code) that computes A_n for $n \in \mathbb{N}$. You do not have to argue correctness.

(b) Lower bound the run time of your recursion from (a) by $\Omega(C^n)$ for some constant $C > 1$.

We now continue with the proof and want to show $T(n) \geq \frac{1}{3} \cdot 3^{n/4}$ by induction.

- **Base Case.**
  For $n \in \mathbb{N}$ with $n \leq 4$, we have $T(n) = 1 \geq \frac{1}{3} \cdot 3^{n/4}$ since $n/4 \leq 1$ implies $3^{n/4} \leq 3$.
- **Induction Hypothesis.**
  Assume that for some integer $k \geq 5$ the statement holds for all $k' < k$.

- **Induction Step.**
  We compute

$$\begin{aligned}
T(k) &= T(k-1) + T(k-3) + T(k-4) + d \\
&\geq \frac{1}{3} \cdot 3^{(k-1)/4} + \frac{1}{3} \cdot 3^{(k-3)/4} + \frac{1}{3} \cdot 3^{(k-4)/4} \\
&\geq \frac{1}{3} \cdot \left(3^{(k-4)/4} + 3^{(k-4)/4} + 3^{(k-4)/4}\right) \\
&= \frac{1}{3} \cdot 3 \cdot 3^{k/4-1} \\
&= \frac{1}{3} \cdot 3^{k/4}.
\end{aligned}$$

  Thus, the statement also holds for k.
  By the principle of mathematical induction, $T(n) \geq \frac{1}{3} \cdot 3^{n/4}$ holds for every $n \in \mathbb{N}$.

Hence, the run time of the algorithm in (a) is $T(n) \geq \frac{1}{3} \cdot 3^{n/4} \geq \Omega(C^n)$ for $C = 3^{1/4} > 1$.
Remark: With a bit more care, it can be shown by induction that $T(n) = \Theta(\phi^n)$, *where* $\phi \approx 1.618$ *is the unique positive solution of* $x^4 = x^3 + x + 1$.

Algorithm 2 $A(n)$

```
if $n \leq 4$ then
    return $n$
else
    return $A(n-1) + A(n-3) + 2A(n-4)$
```

(c) Improve the run time of your algorithm using memoization. Provide pseudo code of the improved algorithm and analyze its run time.

Algorithm 2 $A(n)$

```
if $n \leq 4$ then
    return $n$
else
    return $A(n-1) + A(n-3) + 2A(n-4)$
```

(c) Improve the run time of your algorithm using memoization. Provide pseudo code of the improved algorithm and analyze its run time.

Algorithm 3 Compute A_n using memoization

```
memory ← $n$-dimensional array filled with $(-1)$s
function A_MEM(n)
    if memory[n] $\neq -1$ then                                ▷ If $A_n$ is already computed.
        return memory[n]
    if $n \leq 4$ then
        memory[n] ← $n$
        return $n$
    else
        $A_n \leftarrow$ A_Mem$(n-1)$ + A_Mem$(n-3)$ + 2A_Mem$(n-4)$
        memory[n] ← $A_n$
        return $A_n$
```

Consider the recurrence

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 2 \\ A_3 &= 3 \\ A_4 &= 4 \\ A_n &= A_{n-1} + A_{n-3} + 2A_{n-4} \text{ for } n \geq 5. \end{aligned}$$

(d) Compute A_n using bottom-up dynamic programming and state the run time of your algorithm. Address the following aspects in your solution:

(1) *Definition of the DP table:* What are the dimensions of the table $DP[\ldots]$? What is the meaning of each entry?

(2) *Computation of an entry*: How can an entry be computed from the values of other entries? Specify the base cases, i.e., the entries that do not depend on others.

(3) *Calculation order*: In which order can entries be computed so that values needed for each entry have been determined in previous steps?

(4) *Extracting the solution*: How can the final solution be extracted once the table has been filled?

(5) *Run time*: What is the run time of your solution?

Exercise 6.4 *Jumping game* **(1 point)**.

We consider the jumping game from the lecture for the following array of length $n = 10$:

$$A[1..n] = [2, 4, 2, 2, 1, 1, 1, 1, 5, 2].$$

We start at position 1. From our current position i, we may jump a distance of at most $A[i]$ forwards. Our goal is to reach the end of the array in as few jumps as possible. Recall the dynamic programming solution given for the problem in the lecture, revolving around the numbers:

$$M[k] := \text{`largest position reachable in at most } k \text{ jumps'}.$$

In this exercise, we compare two different methods for computing the $M[k]$.

(a) Consider the recursive relation:

$$\begin{aligned} M[0] &= 1, \\ M[k] &= \text{the maximum element of the array } R_k, \end{aligned} \tag{R}$$

where R_k is the array with indices i in the range $1 \leq i \leq M[k-1]$ and $R_k[i] := A[i] + i$. Compute $M[k]$ for $k = 1, 2, \ldots, K$ using relation (R), where K is the smallest integer for which $M[K] \geq n = 10$. For each $1 \leq k \leq K$, write down the array R_k used in the recursion. Finally, compute $\sum_{k=1}^{K} |R_k|$.

We consider the jumping game from the lecture for the following array of length $n = 10$:

$$A[1..n] = [2, 4, 2, 2, 1, 1, 1, 1, 5, 2].$$

We start at position 1. From our current position i, we may jump a distance of at most $A[i]$ forwards. Our goal is to reach the end of the array in as few jumps as possible. Recall the dynamic programming solution given for the problem in the lecture, revolving around the numbers:

$$M[k] := \text{‘largest position reachable in at most } k \text{ jumps’}.$$

In this exercise, we compare two different methods for computing the $M[k]$.

(b) Now consider the recursive relation:

$$\begin{aligned} M'[0] &= 1, \\ M'[1] &= 1 + A[1], \\ M'[k] &= \text{the maximum element of the array } R'_k, \end{aligned} \tag{R'}$$

were R'_k is the array with indices i in the range $M'[k-2] < i \leq M'[k-1]$ and $R'_k[i] := A[i] + i$. Compute $M'[k]$ for $k = 1, 2, \ldots, K$ using relation (R’), where K is the smallest integer for which $M'[K] \geq n = 10$. For each $2 \leq k \leq K$, write down the array R'_k used in the recursion. Finally, compute $\sum_{k=1}^{K} |R'_k|$.

(c*) Now let A be an arbitrary array of size $n \geq 2$ containing positive, non-repeating[1] integers. Let $M[k], M'[k]$ be the numbers computed using relations (R) and (R'), respectively. Prove that $M[k] = M'[k]$ for all $k \geq 0$.

Hint: *Use induction. First show that $M[0] = M'[0]$ and that $M[1] = M'[1]$. Then, use the induction hypothesis '$M[k-2] = M'[k-2]$ and $M[k-1] = M'[k-1]$' to show that $\max R_k = \max R'_k$ for all $k \geq 2$.*

(a) Given are the two arrays

$$A = [7, 6, 3, 2, 8, 4, 5, 1]$$

and

$$B = [3, 9, 10, 8, 7, 1, 2, 6, 4, 5].$$

Use the dynamic programming algorithm from the lecture to find the length of a longest common subsequence and the subsequence itself. Show all necessary tables and information you used to obtain the solution.

Dynamic Programming on Trees

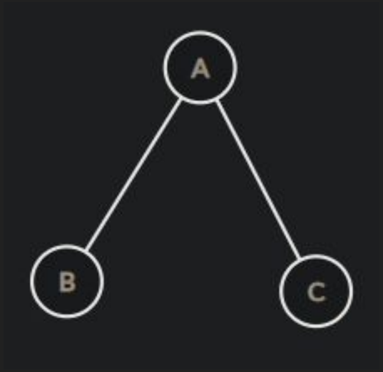
https://atcoder.jp/contests/dp/tasks/dp_p

Tree Coloring

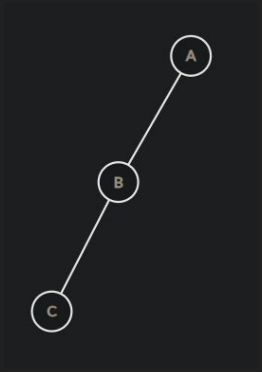
Given a rooted tree with n nodes.

We paint each node in white or black. Here, it is not allowed to paint two adjacent node both in black.

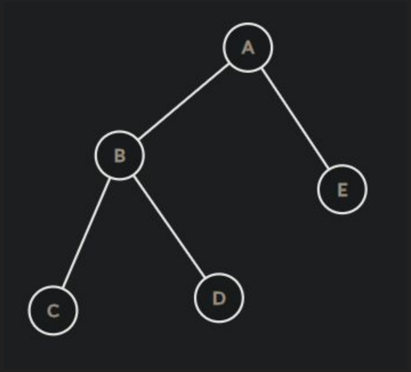
Find the number of ways in which the nodes can be painted



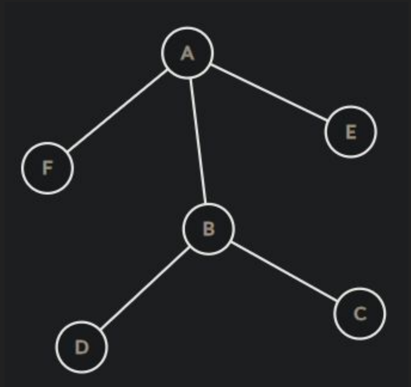
Answer: 5



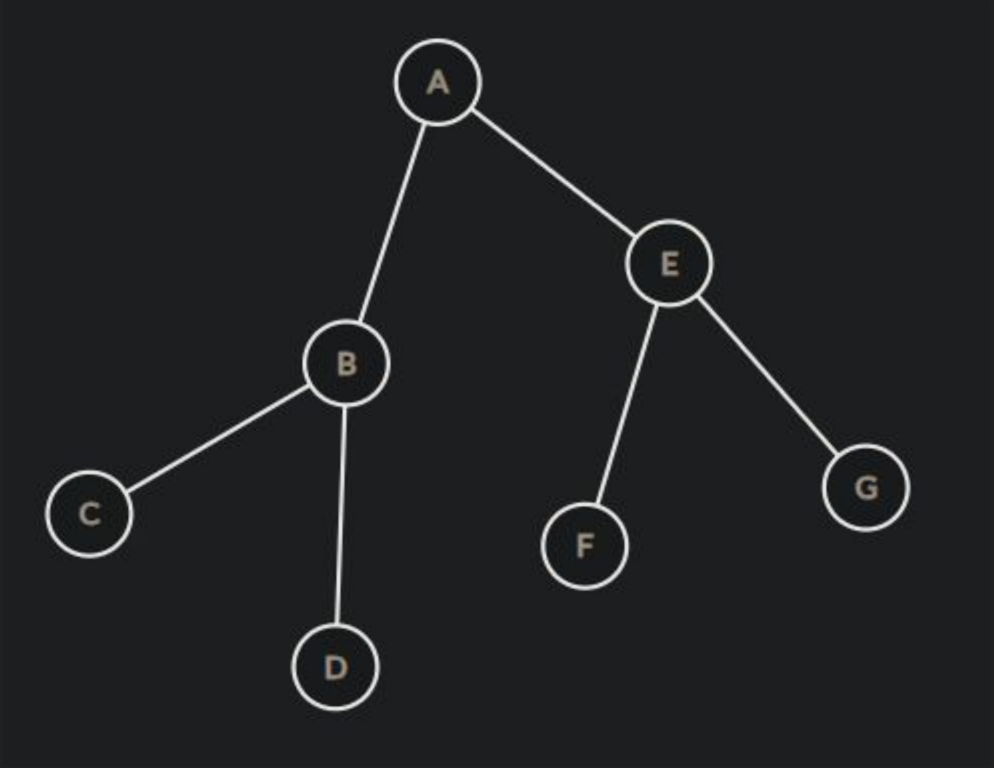
Answer: 5



Answer: 14



Answer: 24



Answer: 41

Peer Grading(with the former groups)

Exercise 6.1

While sending to me please include the group you received their work in cc.

https://docs.google.com/spreadsheets/d/1owPsJsd9THBWInwFcVjKCc0f_r6n4pGwwDKMDdwaCjM/edit?usp=sharing

New groups will be published later on Monday(today)