1) G must have an even number of vertices of odd degree. Add an artificial x vertex to G and connect it to all vertices that have odd degree in G.

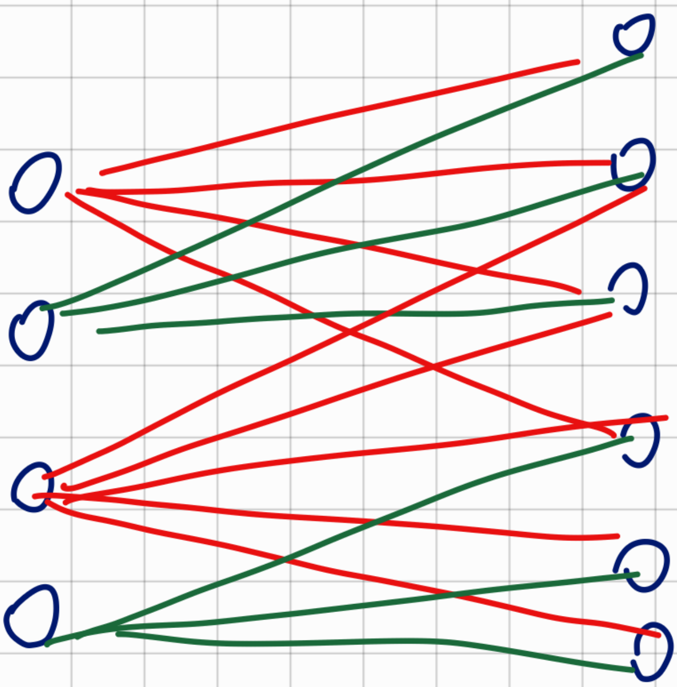
2) Euler cycle : $v_1 \underset{e_1}{-} v_2 \underset{e_2}{-} v_3 - \cdots\cdots v_n \underset{e_m}{-} v_1$ in G

$e_1 - e_2 - \cdots - e_m$ goes over all vertices in G' and has the same start- and end-point.

$\forall v \in G'$ $\deg(v)$ is even $\longrightarrow$ G' is Eulerian

$v \underset{e}{\text{———}} u$: $(\deg(v) - 1) + (\deg(u) - 1)$ is even

3)



$X \subseteq A$

$|N(X)| \geq |X|$

Prove $|N(X)| \geq |X|$ for $\forall X \subseteq A$ (Hall)

$|N(X)| \cdot k \geq$ # edges between X and $N(X) \geq |X| \cdot k$

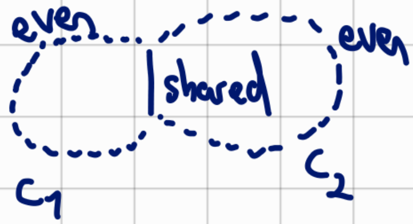
4)

4) Assign colors $v_1, \dots, v_n$ sequentially.

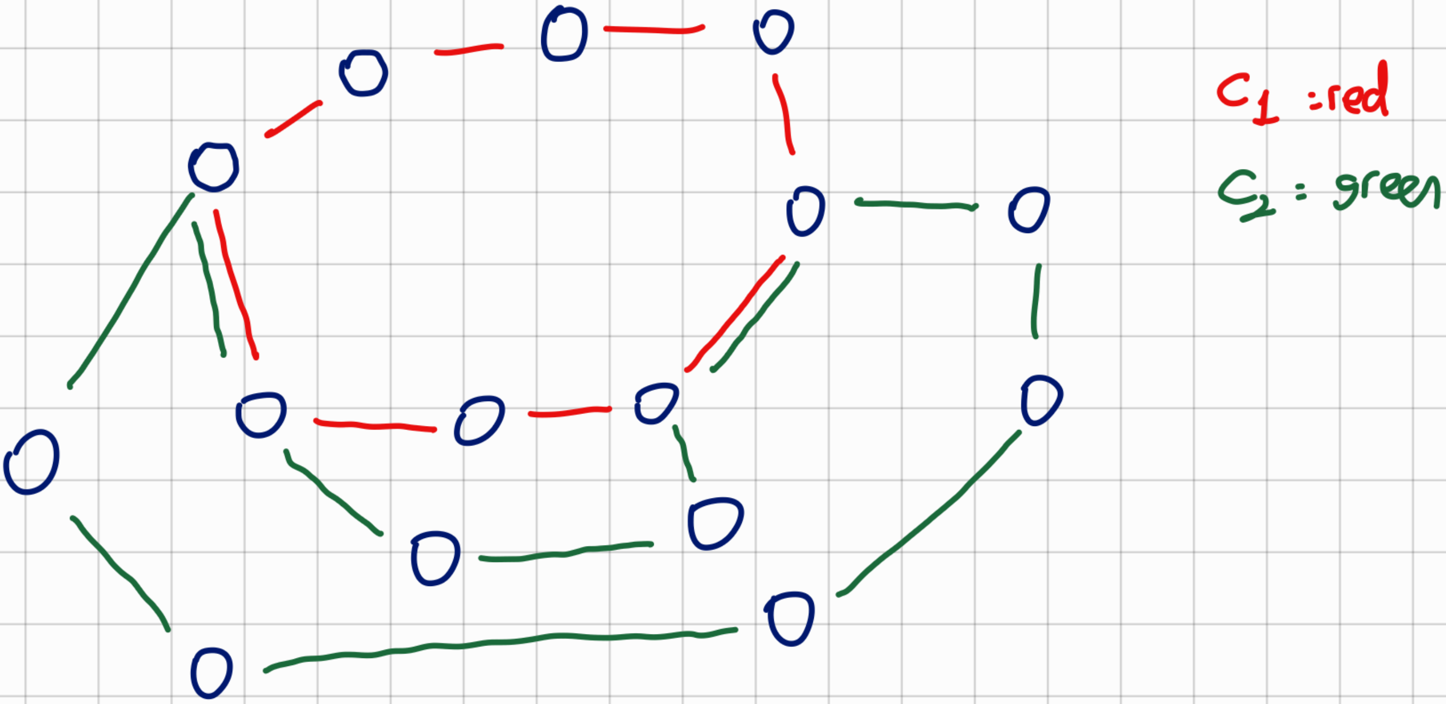
When assigning color v_i, it needs to be different from all the colors of its neighbors. There are at most 5 such neighbors so we can use at most 6 colors.

If v_i and v_j are neighbors and $i < j$ wlog then we surely assigned the color of v_j different from v_i.

5) Let c_1 and c_2 odd length cycles.



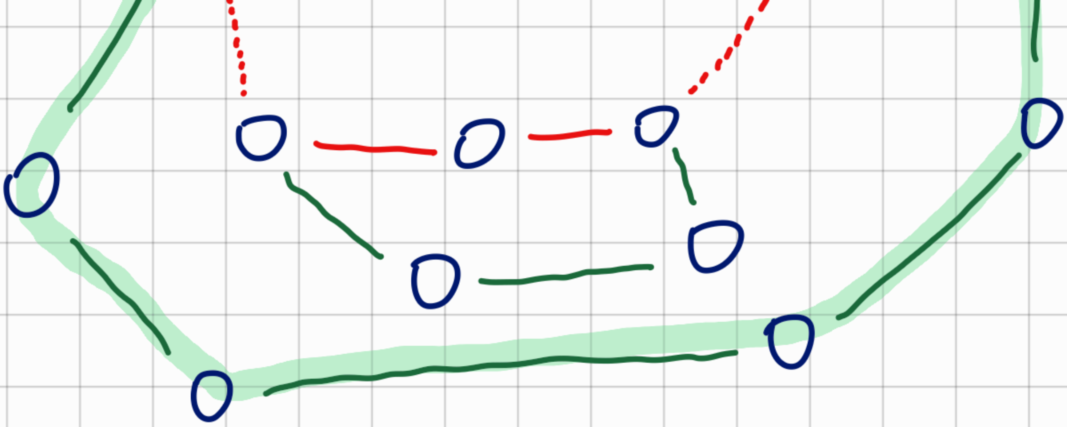
Since c_1 and c_2 aren't the same cycle there exists an edge of c_2 that is not an edge of c_1 wlog.



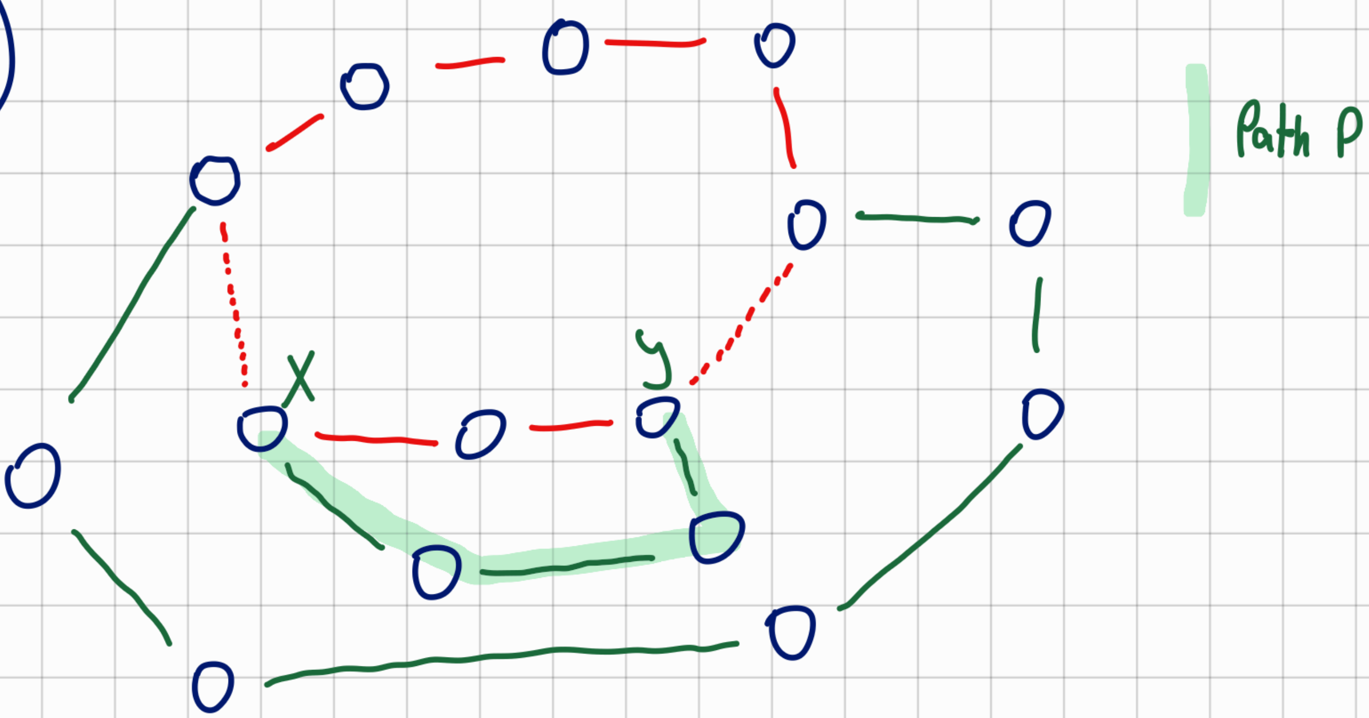
Remove from c_2 all the edges that are also in c_1.

What remains of c_2 is union of paths. Take one such path P. P is between x and y of c_1. x and y divide c_1 into two paths one of which has even length and one has odd length. Both of this parts create a cycle together with P and one of these cycles will have even length and the other one odd length.

i)



ii)



6)

Let x describe the number of leaves the tree has.

Let $T = (V, E)$

$$|V| = x + 10$$

$$2 \cdot |E| = \sum_{v \in E} \deg(v) = 100 + x$$

We know $|E| = |V| - 1 = x + 9$

Solve $(100+x)\cdot\frac{1}{2} = x+9$ and get $x = 82$.

7) i) Let X describe the number of snowboards used.

$X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

X_i is the indicator random variable describing if i-th snowboard was used.

$Pr[\text{Snowboard } i \text{ never used}] = \left(\frac{5}{6}\right)^6 \approx 0.3349$

$Pr[\text{Snowboard } i \text{ was used}] = 1 - \left(\frac{5}{6}\right)^6 = 0.6651$

$E(X) = 6 \cdot E(X_i) = 3{,}9906$

On average Rachel sharpens less than 4 snowboards.

ii)

$$y_i = \begin{cases} 1 & \text{if Rachel used a new snowboard on day } i \\ 0 & \text{if Rachel reused a snowboard on day } i \end{cases}$$

$E[y_i] = Pr[y_i = 1] = \left(\frac{5}{6}\right)^{i-1}$

Because Rachel had a chance of not picking the new snowboard in the past $(i-1)$ days.

$E[X] = E[y_1] + E[y_2] + \dots + E[y_6]$

$$= 1 + \frac{5}{6} + \left(\frac{5}{6}\right) + \dots + \left(\frac{5}{6}\right)$$

$$= \frac{1 - \left(\frac{5}{6}\right)^6}{1 - \frac{5}{6}} = 3{,}9906$$