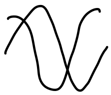
Analysis I

10.1 Wiederspruch: $\arccos(1) = 0 \qquad \arcsin(1) = \frac{\pi}{2}$

sei $x = \frac{\pi}{2} - \arcsin(y)$

$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) = \sin(\arcsin(y)) = y$



also $\arccos(y) = x = \frac{\pi}{2} - \arcsin(y)$

zu beweisen

$\frac{\pi}{2} = \arccos y + \arcsin y$

$\frac{\pi}{2} - \arcsin y = \arccos y$

$\cos\left(\underbrace{\frac{\pi}{2} - \arcsin y}_{x}\right) = y$

Also nehmen wir $x = \frac{\pi}{2} - \arcsin y$

10.3

(e) $\lim\limits_{x \to \infty} (e^x + x)^{1/x}$

$(e^x + x)^{\frac{1}{x}} = \exp\left(\frac{1}{x} \ln(e^x + x)\right)$

$= \lim\limits_{x \to \infty} \exp\left(\frac{\ln(e^x + x)}{x}\right)$

exp st.

$= \exp\left(\underbrace{\lim\limits_{x \to \infty}\left(\frac{\ln(e^x + x)}{x}\right)}_{\text{Form } \text{"}\frac{\infty}{\infty}\text{"}}\right)$

$= \lim \dfrac{\frac{1}{e^x + x}(e^x + 1)}{1}$

$= \lim \dfrac{e^x + 1}{e^x + x} \quad \text{"}\frac{\infty}{\infty}\text{"}$

$= \lim \dfrac{e^x}{e^x + 1}$

$\cdot/. e^x$

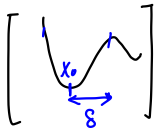
$= \lim \dfrac{\frac{e^x}{e^x}}{\frac{e^x}{e^x} + \frac{1}{e^x}} = \lim \dfrac{1}{1 + \frac{1}{e^x}} \longrightarrow 0$

$= 1$

$\Rightarrow \lim = e^1 = e$

10.5

Sei x_0 ein lok. Min und nehmen wir an (per Wiederspruch), dass $\exists x_1 \in \mathbb{R}$ s.d $f(x_1) < f(x_0)$

[Lokales Min: $\exists \delta > 0 : f(x) \geq f(x_0) \ \forall x \in (x_0 - \delta, x_0 + \delta)$ 

[f konvex: $\forall x, y, \lambda \in [0,1] \quad f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$

Insb. sei $x = x_0$, $y = x_1$

$$\Rightarrow f(\lambda(x_0) + (1-\lambda) x_1) \leq \lambda f(x_0) + (1-\lambda) f(x_1)$$

$$\overset{f(x_1) < f(x_0)}{<} \lambda f(x_0) + (1-\lambda) f(x_0)$$

$$= f(x_0)$$

$$\Rightarrow f(\lambda x_0 + (1-\lambda) x_1) < f(x_0)$$

$$\lim_{\lambda \to 1^-} \lambda x_0 + (1-\lambda) x_1 = x_0$$

d.h. $\exists \lambda'$ s.d $\lambda' x_0 + (1-\lambda') x_1 \in (x_0 - \delta, x_1 + \delta)$

aber wobei $f(\lambda' x_0 + (1-\lambda') x_1) < f(x_0)$ ↯ zu $f(x) > f(x_0)$

10.6 Sei $p > 0$

(a) $\frac{1}{p} + \frac{1}{q} = 1$ try to express $q = f(p)$

$$\frac{1}{q} = 1 - \frac{1}{p}$$

$$\frac{1}{q} = \frac{p-1}{p}$$

$$q = \frac{p}{p-1}$$ eindeutig da $p > 1$: immer wohl def.

da $p > 1 \Rightarrow \frac{p}{p-1} > 1$ $\quad$ ($1 > 0 \overset{p>0}{\Rightarrow} p + 1 > p$

$\Rightarrow q > 1$ $\quad$ $\Rightarrow p > p - 1$

$\Rightarrow \frac{p}{p-1} > 1$)

(b)

$$(-\ln x)'' = \left(-\frac{1}{x}\right)' = \frac{1}{x^2} > 0 \Rightarrow \text{strikt konvex in } (0,\infty)$$

$$\Rightarrow -\ln(\lambda w + (1-\lambda)z) \leq -\lambda \ln(w) - (1-\lambda)\ln(z) \quad \forall w,z \in (0,\infty),\ \lambda \in [0,1]$$

Sei $w = x^p \quad \lambda = \frac{1}{p}$

$z = y^q \quad 1-\lambda = 1 - \frac{1}{p} = \frac{1}{q}$

$$\Rightarrow -\ln\left(\frac{1}{p}x^p + \frac{1}{q}x^q\right) \leq -\frac{1}{p}\ln(x^p) - \frac{1}{q}\ln(x^q)$$

$$= -\ln(x) - \ln(y)$$

Sei $x, y \geq 0$

Fall $x=0$, $y=0$ klar.

Fall $x>0$, $y>0$

$$xy = \exp(\ln(xy))$$

$$= \exp(\ln(x) + \ln(y))$$

$$\leq \exp\left(\ln\left(\frac{x^p}{p} + \frac{y^q}{q}\right)\right) = \frac{x^p}{p} + \frac{y^q}{q}$$

Wiederholung

$f' > 0 \longrightarrow$ streng monoton wachsend

$f' < 0 \longrightarrow$ streng monoton fallend

$f' \leq 0 \longrightarrow$ monoton fallend

$f' \geq 0 \xrightarrow{\Leftrightarrow}$ monoton wachsend

$f'' \geq 0 \xrightarrow{\Leftrightarrow}$ konvex

$f'' > 0 \longrightarrow$ strikt konvex.

aber strikt konvex $\Rightarrow f'' \underline{\underline{\geq}}\, 0$

(Beispiel x^4)

Def Höhere Ableitungen

1) n-mal diff. falls $f^{(n-1)}$ in D existiert und diff ist.

$f^{(n)} = (f^{(n-1)})'$ d.h n-mal diff $\Rightarrow$ n-k diff

2) n-mal stetig - diff. $\begin{cases} f \text{ n-mal diff.} \\ f^{(n)} \text{ stetig} \end{cases}$

3) Glatt: n-mal diff $\forall n$ (exp, cos, sin, ln, x^k)

n-mal diff $\Rightarrow$ (n-1)-mal stetig diff.

Satz f, g n-mal diff. / glatt.

1) f+g auch

2) $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$

3) $\frac{f}{g}$ auch falls $g \neq 0$

4) $(g \circ f)^{(n)} = \sum_{k=0}^{n} A_{n,k}(x) (g^{(k)} \circ f)(x)$

$A_{n,k}(x)$ ist ein Polynom in $f', f^{(2)}, .., f^{(n+1-k)}$

Potenzreihen & Taylorapproximationen

Satz 4.4.1

f_n stetig diff. gl. konv. $\lim f_n = f$

f_n' gl. konv $\lim f_n' = p$

$\Rightarrow f' = p$

Satz 4.4.2

$\sum c_k x^k$ mit $\rho > 0$. (also konv in $(-\rho, \rho)$)

$f(x) = \sum c_k (x-x_0)^k$ diff. für $x \in (x_0 - \rho, x_0 + \rho)$

$$f'(x) = \sum_{k=1}^{\infty} c_k \cdot k (x-x_0)^{k-1}$$

$$\Rightarrow f^{(j)}(x) = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!} (x-x_0)^{k-j}$$

mit $c_j = \dfrac{f^{(j)}(x_0)}{j!}$

da $f^{(j)}(x_0) = \underbrace{c_j \cdot \dfrac{j!}{0!}}_{\text{only non-zero term}} = c_j \cdot j! \quad \Rightarrow \quad c_j = \dfrac{f^{(j)}(x_0)}{j!}$

4.4.5 f stetig, und (n+1)-mal diff in (a,b)

$\exists \xi \in (a,x)\ \forall x$ s.d.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

wobei $R_n^a(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ $\quad \lim\limits_{x \to a} \dfrac{R_n^a(x)}{(x-a)^n} = 0$

Taylor polynom e^x an $x = 0$

immer an eine Stelle

$$e^x \simeq f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \dots$$

$$= e^0 + e^0 x + \frac{e^0}{2} x^2 + \dots$$

$$= 1 + x + \frac{x^2}{2} + \dots + \frac{x^m}{m!}$$

Reminder $\exp(x) = \sum \frac{x^k}{k!}$

- cos
- sin

$\Big\}$ similarly their Taylor approximations are also a part of their "Reihenentwicklung"

- $(1+x)^\alpha$, $x = 0$. never insert 0 before deriving

$f(x) = (1+x)^\alpha \quad \to 1$

$f'(x) = \alpha(1+x)^{\alpha-1} \quad \to \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-1} \quad \to \alpha(\alpha-1)$

$x=0$

$$= \sum_{k=0}^{\infty} \frac{\alpha!}{(\alpha-k)!} \cdot \frac{x^k}{k!}$$

- $\sin^2(x) \to 2\sin x \cos x \to 2\cos^2 x - 2\sin^2 x \to -4\cos x \sin x - 4\sin x \cos x$

$$= -8\cos x \sin x$$

$$= -4\sin(2x)$$

$$\to -8\cos(2x)$$

$$|R_3^0(x)| = \left|\frac{f^{(4)}(\xi)}{4!}(x-0)^4\right|$$

$$= \left|\frac{-8\cos(2\xi)}{4!}x^4\right|$$

$$= \frac{|\cos(2\xi)|}{3}|x^4|$$

$\ln(1+x)$ an der Stelle $x = 0$:

$$f(0) = \ln(1) = 0$$

$$f'(0) = \frac{1}{1+x}\Big|_{x=0} = 1$$

$$f''(0) = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1$$

$$f'''(0) = \frac{1}{(1+x)^4} 2(1+x)\Big|_{x=0} = 2$$

$$T_3^0(x) = 0 + 1\cdot(x-0) - \frac{1}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3}$$

Taylor Reihe ist also

$$\sum_{k=0}^{n} \frac{(-1)^{n+1}}{n} x^n$$

Substitution trick

$T_n^a(x)$ von $\ln(\cos(x))$ an $x = 0$

1. Was sind bekannte $T_n^a(x)$ von ln und cos?

$$\cos \rightarrow \cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\ln \rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

2. Substitution mit Taylor Reihe von cos

$$\ln(\cos(x)) = \ln\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right)$$

3. Pattern matching mit $1 + x$

$$x = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

4. Taylor Entwicklung von $\ln(1+x)$ verwenden mit $x = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} = \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - \frac{\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^2}{2} + \dots$$

→ vereinfachen