Korrektur Serie 11

11.2 n-te Ableitung

c) $h(x) = \sin^2(x)$

selon closed form: $\sum_{k=0}^{n} \binom{n}{k} \sin(x)^{(k)} \sin(x)^{(n-k)}$

$h'(x) = 2\sin(x)\cos(x) = \sin(2x)$

$\sin'(2x) = 2\cos(2x)$

$(2\cos(2x))' = -4\sin(2x)$

$$h^{(n)}(x) = \begin{cases} 2^{n-1}\sin(2x) & 4k+1 = n \\ 2^{n-1}\cos(2x) & n = 4k+2 \\ -2^{n-1}\sin(2x) & \\ -2^{n-1}\cos(2x) & \end{cases} \quad n \in \mathbb{N}^*$$

11.4

(a) $f'(x) = -\frac{2x}{x^4} \cdot x\exp\left(\frac{1}{x^2}\right) + \exp\left(\frac{1}{x^2}\right)$

$= \exp\left(\frac{1}{x^2}\right)\left(1 - \frac{2}{x^2}\right) = 0$

$\Leftrightarrow 1 - \frac{2}{x^2} = 0 \Rightarrow x = \pm\sqrt{2}$

$\Rightarrow x = \sqrt{2}$ da $x > 0$.

$f''(x) = -\frac{2x}{x^4}\exp\left(\frac{1}{x^2}\right)\left(1 - \frac{2}{x^2}\right) + \exp\left(\frac{1}{x^2}\right)\frac{4x}{x^4}$

$= \exp\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3} + \frac{4}{x^6} + \frac{4}{x^3}\right)$

$= \exp\left(\frac{1}{x^2}\right)\left(\frac{2}{x^3} + \frac{4}{x^6}\right)$

$= \exp\left(\frac{1}{x^2}\right)\frac{2}{x^3}\left(1 + \frac{2}{x^2}\right) > 0 \quad \forall x > 0.$

$\Rightarrow f''(x) > 0 \quad \forall x > 0$

in $x = \sqrt{2}$ $f''(\sqrt{2}) > 0$

Also f hat ein Minimum in $x = \sqrt{2}$

11.5

$\ln(1+x)$ in $x = 0$ $\quad x - \frac{x^2}{2} + \frac{x^3}{3}$

$x = (1+x)^2$ so $x = -1$

$$(1+x)^2 - \frac{(1+x)^4}{2} + \frac{(1+x)^6}{3}$$

(b) $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\qquad \tanh(0) = 0.$

$$\frac{\cosh^2 - \sinh^2}{\cosh^2} = \frac{1}{\cosh^2}\Big|_{x=0} = 1$$

$$f'' = -\frac{1}{\cosh^4} 2\cosh\sinh$$

$$= \frac{-2\sinh}{\cosh^3}\Big|_{x=0} = 0$$

$$T_0^3(x) = x - \frac{2}{3!}x^3$$

$$= x - \frac{1}{3}x^3$$

$$f''' = \frac{-2\cosh^4 + 6\sinh^2\cosh^2}{\cosh^6}$$

$$= \frac{-2\cosh^2 + 6\sinh^2}{\cosh^4}$$

$$= \frac{-2\cosh^2 + 2\sinh^2 + 4\sinh^2}{\cosh^4} = \frac{4\sinh^2 - 2}{\cosh^4} = \frac{2(2\sinh^2 - 1)}{\cosh^4}\Big|_{x=0} = -2$$

Das Riemann Integral

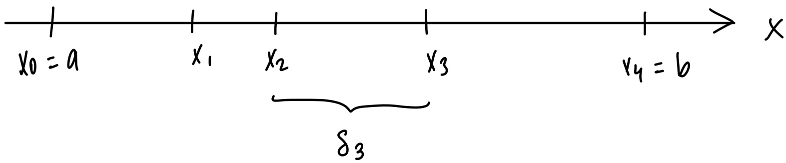
Partition P von $[a,b]$, $P \subsetneq [a,b] \wedge \{a,b\} \in P$ mit eine Bijektion $\{0,..,n\} \longrightarrow P$

$i \longmapsto x_i$

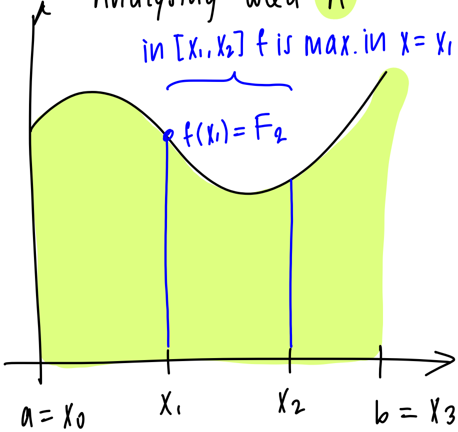
z.B $\{x_0 = a, x_1, x_2, ..., x_n = b\}$

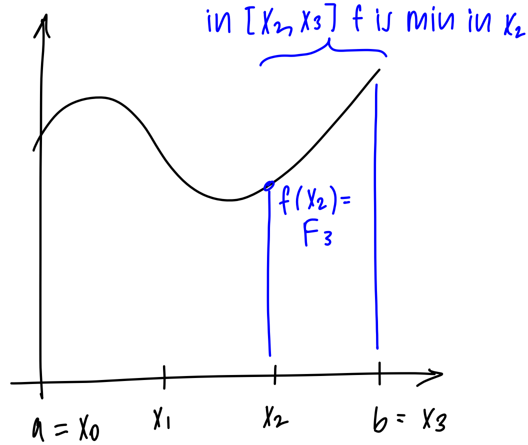
wobei $n = \text{card } P$

und $i<j \Rightarrow x_i < x_j$ also sei $\delta_i = x_i - x_{i-1}$, $i \in \{1,..,n\}$



Analysing area A





Untere Summe $s(f,P)$

$$s(f,P) = \sum_{i=1}^{n} \delta_i \cdot f_i$$

w/ $f_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$

→ undershoots (ALWAYS)

obere Summe $S(f,P)$

depends on function you are approx.

depends on how you partition x-axis

$$S(f,P) = \sum_{i=1}^{n} \delta_i F_i \quad \text{w/}$$

$F_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$

→ overshoots

- Sei P' ein Verfeinerung von P also $P \subset P'$ und $\operatorname{card} P' > \operatorname{card} P$

Dann gilt $\underbrace{s(f,P) \leq s(f,P') \leq S(f,P') \leq S(f,P)}$

e.g. $f(x) = c$ then $s(f,P) = A = S(f,P) \quad \forall P$

- $\forall P_1, P_2 \quad s(f,P_1) \leq S(f,P_2)$

Def: $\mathcal{P}(I)$ all partitions of I.

$$s(f) = \sup_{P \in \mathcal{P}(I)} s(f,P)$$ "grosste untere Summe"

$$S(f) = \inf_{P \in \mathcal{P}(I)} S(f,P)$$ "kleinste obere Summe"

$s(f) \leq S(f)$

Approximiert es immer zu die gross-möglichste kleins-möglichste Konstante Funktion.

Def - Riemannintegrierbar

Falls $s(f) = S(f) = \int_a^b f(x)\,dx$

$\Leftrightarrow \forall \varepsilon > 0 \ \exists P \in \mathcal{P}(I): S(f,P) - s(f,P) < \varepsilon$

"Desto kleiner δ_i wird desto näher S und s werden."

i.e. look at $\lim_{n \to \infty} S(f, P_n)$

Gegenbeispiel zum Riemmanintegrierbarkeit

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

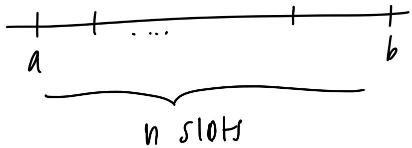
$\forall P$ we have $\left.\begin{array}{l} S(f,P) = 1 \\ s(f,P) = 0 \end{array}\right\}$ does not depend on n...

Beispiel :

$f(x) = x \quad x \in [a,b]$

$P_n = \{ a + ih \mid i = 0, \ldots n \}$

$h = \dfrac{b-a}{n}$



so $\delta_i = \dfrac{b-a}{n}$

$$s(f, P_n) = \sum \inf f(x) \cdot \delta_i$$

$$= h \sum \inf x$$

$$= \frac{b-a}{n} \sum_{i=1} x_{i-1}$$

$$= \frac{b-a}{n} \sum_{i=0} x_i \qquad x_i = a + ih$$

$$= \frac{b-a}{n} \left(n \cdot a + h \frac{(n-1)n}{2} \right)$$

$$= \frac{(b-a)\cancel{n}a}{\cancel{n}} + \frac{(b-a)^2(n-1)}{2n}$$

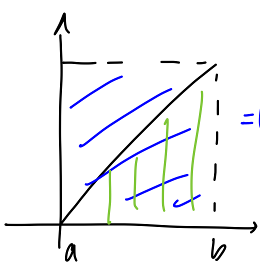
$$= (b-a) \cdot a + \frac{(b-a)^2}{2} \cancel{\frac{(n-1)}{n}} \to 1$$

$$= (b-a)\left(a + \frac{b-a}{2}\right)$$

$$= (b-a)\left(\frac{2a+b-a}{2}\right)$$

$$= \frac{(b-a)(b+a)}{2}$$

$$= \frac{(b-a)^2}{2}$$



$= \dfrac{(b-a)^2}{2}$

Satz 5.1.8

beschränkte Funktion $f: [a,b) \to \mathbb{R}$ integrierbar $\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta : \forall P \in \mathcal{P}_\delta(I)$

$S(f,P) - s(f,P) < \varepsilon$

$\delta \geq \max_i \delta_i$

Korollar 5.1.9

beschränkte.

f ist g.d integrierbar mit $A = \int f(x)$ falls

$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall P \in \mathcal{P}_\delta(I)$ und $\xi_i \in [x_{i-1}, x_i)$

$$\Big| A - \underbrace{\sum f(\xi_i)(x_i - x_{i-1})}_{\text{nicht obere oder untere Riemann Summe.}} \Big| < \varepsilon$$

5.2 Integrierbare Funktionen.

f, g beschr. & int.

$\Rightarrow \lambda f, |f|$

$\Rightarrow f+g, f\cdot g$

max, min

Das Integral ist linear.

Kor. 5.2.3

P ist integrierbar

$\frac{P}{Q}$ auch falls $Q \neq 0$ auf I.

Note

$\int fg \neq \int f \int g$

Gleichmässig stetig

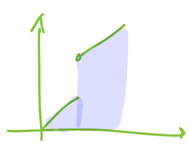
$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall x, y \in D \; |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

stetig + kompakt $\Rightarrow$ stetig gl.

Satz 5.2.7

stetig + kompakt $\Rightarrow$ Integrierbar

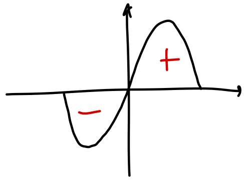
kann auch andere Funktionen geben. z.B. monotone + kompakt



Note :

- Ungerade Funktionen. haben immer $I = 0$ auf ein symmetrisches Interval über 0.

z.B $\displaystyle\int_{-\pi}^{\pi} \sin x \, dx = 0.$



5.3 Ungleichungen & der Mittelwertsatz

5.3.1 Monotonie des Integrals

f besch, int, $f \leq g \Rightarrow \int_a^b f(x) \leq \int_a^b g(x)$

Kor 5.3.2

besch, int, $\left|\int f\right| \leq \int |f|$

5.3.3 CSB

$$\left|\int fg\right| \leq \sqrt{\int f^2}\sqrt{\int g^2}$$

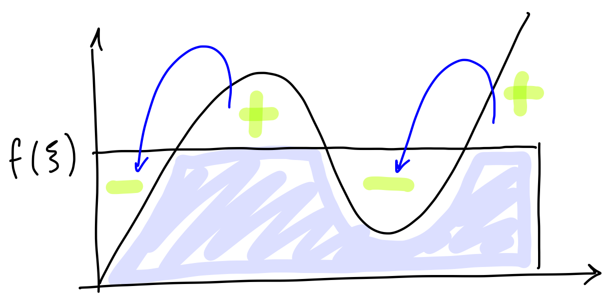
5.3.4 Mittelwertsatz

stetig. $\exists \xi \in [a,b]$ $\int f(x) = f(\xi)(b-a)$

5.3.6

$$\int fg = f(\xi)\int g(x)$$

Find a rectangle where above and below it cancels itself out.



Fundamentalsatz der Diff. Rechnung

$f : [a, b]$ stetig

$$F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a,b]$$

important to rename

$F(x)$ ist <u>stetig</u> diff

$F'(x) = f(x)$

stetig

F Stammfunktion

Fundamentalsatz

nur in den Grenzen

$$\int_a^b f(x) = F(b) - F(a) = F(x)\Big|_a^b$$

$F \simeq \pm c$

Partielle Integration

$$\int f(x) g'(x) = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x)\, dx$$

Substitution

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\,\phi'(t)\,dt$$

Integral tricks

$$\int f(g(x)) \cdot g'(x) = F(g(x))$$

$$\int \frac{1}{\log x \cdot x}\, dx = \log(\log x)) + \underline{\underline{C}} \qquad (\log(\log x))' = \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\int \frac{1}{e^x+1}\, e^x\, dx = \log(e^x+1) + \underline{\underline{C}}$$

$$\int \tan x = \int \frac{\sin x}{\cos x} = -\int \frac{-\sin x}{\cos x} = -\log(\cos x)$$

Partielle Integration.

1. To reduce: polynomials.

2. To cycle through: sin, cos

$$\underbrace{I. = f(I)}$$

eq. to solve for I.

$$\int f' g = f \cdot g - \int f g'$$

can also be $1 \cdot g$

reduce via integration
1 (falls log)

reduce by integration
g: x^n, log, arcsin, arccos

Egal: e^x, $\sin x$, $\cos x$

- $\int x e^x$ $\quad f' = e^x \quad f = e^x \quad g = x \quad g' = 1 \quad = e^x x - \int e^x\, dx = e^x x - e^x + C$

• $I = \int \cos^2 x \, dx = \cos x \sin x + \int \sin^2 x$

$= \cos x \sin x + \int 1 - \cos^2 x$

$= \cos x \sin + x - I$

$\Rightarrow 2I = \cos x \sin x + x$

$I = \dfrac{\cos x \sin x + x}{2}$

• $\int \log(x)$ $\quad f' = 1 \quad f = x$

$g = \log \quad g' = \dfrac{1}{x}$

$= x \log(x) - \int x \cdot \frac{1}{x} \, dx$

$= x \log(x) - \int 1 \, dx$

$= x(\log x - 1)$