Analysis I - Übungsstunde 13

Correction Serie 12

$$I(p,q) = \int_0^1 x^p (1-x)^q \, dx$$

wollen x^{p+1} zu x^p reduzieren also $f(x) = x^{p+1}$

$$I(p+1,q) = \int_0^1 x^{p+1} (1-x)^q \, dx$$

$f(x) = x^{p+1}$ $\qquad f'(x) = (p+1)x^p$

$g'(x) = (1-x)^q$ $\qquad g(x) = - \frac{1}{q+1}(1-x)^{q+1}$

$$= -x^{p+1} \frac{(1-x)^{q+1}}{q+1} \Big|_0^1 + \int_0^1 (p+1) x^p \frac{(1-x)^{q+1}}{q+1} \, dx$$

$$= -0 + 0 + \frac{(p+1)}{(q+1)} \underbrace{\int_0^1 x^p (1-x)^{q+1} \, dx}_{I(p,q+1)}$$

$$I(p+1,q) = \frac{(p+1)}{(q+1)} I(p,q+1)$$

(b)
$$I(p,q) \overset{(a)}{=} \frac{p}{q+1} I(p-1,q+1)$$

$$= \frac{p}{q+1} \cdot \frac{p-1}{q+2} I(p-2,q+2)$$

$$\dots = \frac{p!\,q!}{(q+p)!} I(0,q+p)$$

$$= \frac{p!\,q!}{(q+p)!} \int_0^1 (1-x)^{q+p} \, dx$$

$$= \frac{p!\,q!}{(q+p)!} \left[-\frac{(1-x)^{q+p+1}}{q+p+1} \right]_0^1$$

$$= \frac{p!\,q!}{(q+p)!} \left(\frac{1}{q+p+1} \right)$$

$$= \frac{p!\,q!}{(q+p+1)!}$$

12.5)

$$\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} -f(-x)\,dx$$

let $y = -x$

$$= \int_{a}^{0} f(y)\,dy = -\int_{0}^{a} f(y)\,dy$$

$\frac{dy}{dx} = -1 \quad dy = -dx$

12.4)

$$\int_{a}^{b} f(x) - g(x)\,dx$$

b) Red parabola: $k(x-2)(x+2) = f(x)$

with $f(0) = 4$ so $k \cdot (-2) \cdot 2 = 4$

$\Rightarrow -4k = 4 \Rightarrow k = -1$

$\Rightarrow f(x) = -(x-2)(x+2)$

$= -x^2 + 4$

Blue line: $g(x) = ax + b$

$a = \frac{0-(-1)}{2-0} = \frac{1}{2}$

$\frac{1}{2} \cdot 0 + b = -1 \Rightarrow b = -1$

$g(x) = \frac{1}{2}x - 1$

Find a, b: $\frac{1}{2}x - 1 = -x^2 + 4$

$x^2 - \frac{1}{2}x - 5 = 0$

$\Delta = \frac{1}{4} - 4 \cdot (-5) = 20.25$

12.3)

a) $\varphi'(\phi(x))\phi'(x)$

Function when you derive it gives this

$\varphi(\phi(x))$

D.h wenn wir $\varphi(\phi(x))\cdot\phi'(x)$

ist $F = \Phi(\phi(x))$ wobei $\Phi'(x) = \varphi(x)$

Substitution

$$\int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\phi(x))\,dx$$

$$= \int_{\phi(a)}^{\phi(b)} f(t)\,dt \qquad (t = \phi(x))$$

sei $t = \phi(x)$

$$\frac{dt}{dx} = \frac{d}{dx}\phi(x) = \phi'(x)$$

$\Rightarrow dt = \phi'(x)\,dx$

a,b: wir werten a,b im Substitutionsbereich ab.

Tip: $\int_a^b f(\phi(x))\,\phi'(x)\,dx$ gut sichtbar ist also gilt

$$= F(\phi(x))\Big|_a^b$$

$\longrightarrow$ Beispiel mit implizit + explizit Substitution

$$\int_{-1}^{1} x\, f(x^2)\,dx \qquad \phi(x) = x^2 \qquad \phi(1) = 1$$

$$\phi'(x) = \underline{\underline{2}}x \qquad \phi(-1) = 1$$

fehlt

NOTE: $\int_a^a f(x)\,dx = 0$

Tip: introduce the missing parts of $\phi'(x)$ by $x\,\frac{\phi'(x)}{\phi'(x)}$

$$= \int_{-1}^{1} \frac{2}{2}\, x\, f(x^2)\,dx$$

$$= \frac{1}{2}\int_{\phi(-1)}^{\phi(1)} f(t)\,dt = 0.$$

Explizit: Recognize $f(\phi(x))\cdot\phi'(x)$

What is $F(x)$? $F(x) =$

Substitution, practically speaking

Choose $y = f(x)$ or $x = f(y)$

$\downarrow$ $y = x^2$ $\downarrow$ z.B $x = r\sin y$

Adjust bounds: current bounds x_1, x_2 are x in *

so here we want to find y_1, y_2

z.B. $\begin{cases} y_1 = f(x_1) \text{ in case 1} & \text{z.B } y_1 = x_1^2 \\ x_1 = f(y_1) \text{ in case 2} & \text{z.B } \dfrac{\arcsin(x_1)}{r} = y_1 \end{cases}$

Change dx to dy:

case $y = f(x)$ do $\dfrac{dy}{dx} = f'(x)$

$\Rightarrow dy = f'(x)\,dx$

case $x = f(y)$ do $\dfrac{dx}{dy} = f'(y) \Rightarrow dx = f'(y)\,dy$

you can always invert & choose the one that is more practical.

$$\int_a^b \frac{1}{1-x^2} = \int \frac{-1}{\sin t}\,dt$$

$x = \cos t$

$\dfrac{dx}{} = -\sin t\,dt$

$a = \cos(t)$

$a = \cos t$

$\arccos(a) = t$

Substitution: !

Bestimmte: As long as you adjust the bounds, no need to plug x back in.

Unbestimmte: need to plug x back in

Beispiele:

Substitution mit $\ln(x)$: $\int f(\ln(x))$

$t = \ln(x)$

Trick: invertierbar $x = e^t$ $\quad \frac{dx}{dt} = e^t$

$dx = e^t \, dt$

$$\boxed{\int f(t)\, e^t \, dt}$$

- $\int \cos(\ln(x))\, dx = \int \cos(t)\, e^t \, dt$ $\quad f(t) = \cos t \quad f'(t) = -\sin t$, $\quad g'(t) = e^t \quad g(t) = e^t$

$= \cos t\, e^t + \int \sin t\, e^t \, dt$ $\quad f(t) = \sin t \quad f' = \cos t$, $\quad g' = e^t \quad g = e^t$

$= \cos t\, e^t + \sin t\, e^t - \int \cos t\, e^t \, dt$

$$I = \frac{e^t(\cos t + \sin t)}{2} + C$$

$$\xrightarrow{x} I = \frac{x(\cos(\ln(x)) + \sin(\ln(x)))}{2} + C$$

Beispiel: $\frac{1}{x^2+1}$

$x = \tan t = \frac{\sin t}{\cos t}$

$dx = \frac{dt}{\cos^2 t}$

$$\frac{1}{1 + \frac{\sin^2 t}{\cos^2 t}} = \frac{1}{\frac{\cos^2 t + \sin^2 t}{\cos^2 t}} = \cos^2 t$$

$\int 1\, dt = t = \arctan(x)$ $\quad$ Indeed $\arctan'(x) = \frac{1}{1+x^2}$

Partielle Integration

Reminder:

both Stammfunktionen

$$\int_a^b f(x)\,g'(x)\,dx = \overbrace{f(x)\,g(x)}\Big|_a^b - \underbrace{\int_a^b f'(x)\,g(x)\,dx}_{\text{opposite combo.}}$$

leicht ableiten und die "einfacher" wird mit Ableitung

2 tricks:

1. Rekursiv bis I einfach ist (also $f'(x)$ "verschwindet")

2. sei $I = I$, rekursiv bis I nochmals vorkommt (beim cos, sin z.B.)

Tip: Immer die Tabelle

$f(x) =$	$f'(x) =$
$g'(x) =$	$g(x) =$

aufschreiben

Tip: Mann kann $g'(x)$ auch als 1 wählen wenn $I = \int_a^b f(x)\,dx$

Wahl von f Priority 1: $\ln(x)$
2. Monomial

cos, sin can be either.

Partialbruchzerlegung

$\int f(x)\,dx$ mit $f(x) = \frac{P(x)}{Q(x)}$

mit $\deg(Q) > \deg(P)$ (sonst dividieren)

Q als $\prod (x-\gamma_i)^{n_i}$ schreiben z.B $x^2 + x = (x+1)(x+0) = (x+1)x$

$\frac{P}{Q}$ als $\sum_i \sum_{j=1}^{n_i} \frac{c_i}{(x-\gamma_i)^j}$ schreiben z.B $\frac{x}{x^2+x} = \frac{A}{x+1} + \frac{B}{x}$

Beispiel:

$$\frac{3x+1}{x^2+2x} = \frac{3x+1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

wobei $A(x+2) + Bx = 3x+1$

$\Rightarrow Ax + Bx = 3x \Rightarrow A+B = 3 \Rightarrow B = \frac{5}{2}$

$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$\Rightarrow \frac{1}{2x} + \frac{5}{2(x+2)}$

$$\int \frac{1}{2x} + \int \frac{5}{2(x+2)}$$

$$= \frac{1}{2}\ln|x| + \frac{5}{2}\ln|x+2| + C$$

Alles wird ln!

Polynomdivision: $\deg(Q) \leq \deg(P)$

$P(x) = x^4 + 2x + 3$

$Q(x) = x^2 + x + 1$

$$\begin{array}{l|l}
\overset{P}{x^4 + 2x + 3} & x^2 + x + 1 \quad Q \\
\hline
-(x^4 + x^3 + x^2) & x^2 - x \quad q \\
\hline
-x^3 - x^2 + 2x + 3 & \\
-(-x^3 - x^2 - x) & \\
\hline
\underset{R}{3\underset{=}{x} + 3} \text{ remainder} &
\end{array}$$

$\deg < 2$
$\Rightarrow$ done ✓

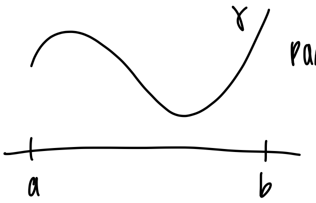
$P = Qq + R$

$\Rightarrow \frac{P}{Q} = q + \frac{R}{Q}$

$$\frac{P}{Q} = x^2 - x + \underbrace{\frac{3x+3}{x^2+x+1}}_{\text{PBZ.}}$$

Theorie :

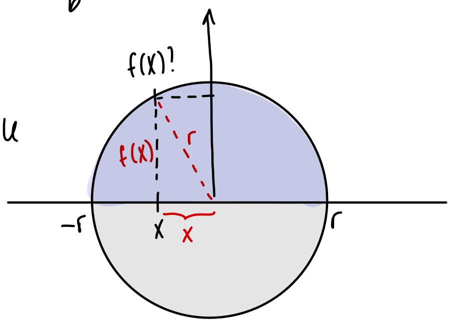
Parametrisierung : Fancy word for given I and a curve γ. Given $x \in I$ what is $\gamma(x)$



Parametrisierung of γ

Halbkreis:

γ is the half-circle



For $x \in [-r, r]$, what is $f(x)$?   Pythagoras: $a^2 + b^2 = c^2$

$$x^2 + f(x)^2 = r^2$$

$$\Rightarrow f(x) = \sqrt{r^2 - x^2}$$

$$\text{Area } A = \int_{-r}^{r} f(x)\,dx$$

$$= \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

When we talk about circles: it is useful to parametrize x as sin and y as cos

$x = r\sin t$, Be careful $\phi(x)$ is the other direction.

$$A = \int_{t_1}^{t_2} \sqrt{r^2 - r^2\sin^2 t}\; r\cos t\,dt$$

Here we are not recognizing a pattern $\phi(x)$ but inserting a function

1. Adjust bounds, $-r, r$ are in x domain so $r = r\sin(t_2)$ find $t_2 \Rightarrow \sin(t_2) = 1$ $t_2 = \frac{\pi}{2}$
   $-r = r\sin(t_1)$ find $t_1 \Rightarrow \sin(t_1) = -1$ $t_1 = -\frac{\pi}{2}$
2. Find dt $\frac{dx}{dt} = (r\sin t)' = r\cos t \Rightarrow dx = r\cos t\,dt$
3. Insert substitution

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2(1-\sin^2 t)}\, r\cos t \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (r\cos t)^2 \, dt$$

$$= r^2 \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt}_{I} \qquad \begin{array}{ll} f = \cos & f' = -\sin \\ g' = \cos & g = \sin \end{array}$$

$$I = \underbrace{\cos t \cdot \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{=0} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sin^2 t}_{1-\cos^2 t} \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dt - I$$

$$\Rightarrow 2I = t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} + \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{2}$$

$A = r^2 \cdot \frac{\pi}{2}$ We know from high school: Full circle area $= \pi r^2$

Unbestimmte Integrale

Fundamentalsatz: $F(x) = \int_a^x f(x)\,dx$

$$F(x) = \int f(x)\,dx + C$$

d.h. $F'(x) = \left(\int f(x)\,dx + C\right)'$

$$= f(x) + C'$$

$$= f(x)$$

Known F(x)s:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \operatorname{arccosh} + C$$

$\int \cosh = \sinh \quad \int \sinh = \cosh$