MathMeth.

ETH zürich

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ComA

$\sum_{n\in\mathbb{N}_0} az^n \overset{|z|<1}{=} \frac{a}{1-z}$

$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\exp(z) \coloneqq \sum_{n\in\mathbb{N}_0} \frac{z^n}{n!}$

$sin(z) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = -isinh(iz)$

$Z_{sin} = \{z\in\mathbb{C} : z = k\pi, k\in\mathbb{Z}\}$

$cos(z) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \frac{z^{2n}}{(2n)!} = cosh(iz)$

$Z_{cos} = \{z\in\mathbb{C} : z = \left(k+\frac{1}{2}\right)\pi, k\in\mathbb{Z}\}$

$w^n = z \rightsquigarrow w = r^{\frac{1}{n}} \exp(i\frac{t+2\pi(k\in\mathbb{N}_0)}{n}) = \exp(\frac{1}{n}\log(z))$

$$\int_a^b f(t)dt \coloneqq \int_a^b \Re\left(f(t)\right)dt + i\int_a^b \Im\left(f(t)\right)dt$$

Def.: Wegint.
$\int_\gamma f(z)dz \coloneqq \int_a^b \langle f(\gamma(t)), \dot\gamma(t)\rangle dt$

Def.: Abl.
$U\subseteq\mathbb{C}$, f diff.bar in z_0 : $(\frac{df}{dz}(z_0))$
$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$
$u_x(z_0)+iv_x(z_0) = v_y(z_0) - iu_y(z_0)$
CR-eq. $\iff f\in\mathcal{H}$:

$$u_x = v_y$$
$$u_y = -v_x$$

$$\rightsquigarrow \begin{cases}\partial_x^2 u + \partial_y^2 u = 0 \\ \partial_x^2 v + \partial_y^2 v = 0\end{cases} \hat{=} \Delta f(u,v) = 0$$

Def.: Potenzreihe/Taylor $f(z)\in\mathcal{H}(D_{R_{\geq 0}}(z_0))$

$$f(z) = \sum_{n\in\mathbb{N}_0} a_n(z-z_0)^n = \sum_{n\in\mathbb{N}_0} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n\in\mathbb{N}_0} \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz\cdot(z-z_0)^n$$

Laurent $f(z)\in\mathcal{H}(A_{r,R}(z_0))$
$A_{r,R}(z_0) = ann(z_0;r,R) \coloneqq \{z : r < |z-z_0| < R\}$

$$f(z) = \sum_{n\in\mathbb{Z}_+} a_n(z-z_0)^n + \sum_{n\in\mathbb{Z}_-^\times} b_n(z-z_0)^n$$
$$= \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n$$
$$= \ldots\frac{c_{-n}}{(z-z_0)^n} + \ldots + \frac{c_{-1} \coloneqq res_{z_0}f}{(z-z_0)} + G(z)$$
$$= P_{z_0}(f,z) + G(z) \in \mathcal{H}(D_r(z_0))$$
$$\forall z\in D_r^*(z_0)$$
$$\rightsquigarrow c_{-1} \coloneqq res_{z_0}f$$
$$(n=\infty \rightsquigarrow ess.sing.)$$

Def.: Konvergenzradius
$R_{\geq 0} \coloneqq \frac{1}{\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}} = \lim_{n\to\infty}|\frac{a_n}{a_{n+1}}|$

$r_{\geq 0} \coloneqq \limsup_{n\to\infty}|(c_{-n})|^{\frac{1}{n}} = \lim_{n\to\infty}|\frac{c_{-(n+1)}}{c_{-n}}|$

Cauchy Thm.:
$f\in\mathcal{H}(U_{sc}) \implies \exists F\in C^1 : F' = f, \gamma\in C^0_{pw.}(U_{sc})$

$$\oint_\gamma f(z)dz = 0$$
$$\iff \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$
$$\iff \int_\gamma f(z)dz = F(z)\Big|_{\gamma(a)}^{\gamma(b)}$$
$$\rightsquigarrow F' = f, G' = f \implies F - G = C\in\mathbb{C}$$
$$|\int_\gamma f| \leq \int_\gamma |f|$$
$$|\int_\gamma f| \overset{|f|\leq M}{\leq} ML = M\int_a^b |\dot\gamma|dt$$

CIF: $f\in\mathcal{H} \implies f^{(n)}\in\mathcal{H}$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Liouville:
$f \overset{!}{\in} \mathcal{H}(\mathbb{C})$ and bdd. ($|f|\leq M\in\mathbb{R}$) $\implies f = const.$
$\rightsquigarrow f' \overset{!}{=} 0$
Generalized Liouville:
$|f(z)| \overset{|z|\to\infty}{=} O(|z|^n) \implies f\in\mathbb{C}[z]_n$
$\iff |f(z)| \overset{|z|\to\infty}{\leq} C_{>0}|z|^n \implies f\in\mathbb{C}[z]_n$
Fund. Thm. of Alg.:
$\forall$ non-const. $p\in\mathbb{R}[z]_{n\geq 1}$ has n roots $\in\mathbb{C}$ counting multiplicities $\rightsquigarrow p(z) = a_n(z-w_1)\cdots(z-w_n)$
Def.: lim.pt. z_0 of $\Omega \iff$
$\exists(z_n)_n\subseteq\Omega\setminus\{z_0\} : \lim z_n = z_0$
hence $\Omega\cap\left(D^*_{r>0}(z_0)\right)\neq\emptyset$
Def.: ord. of zero z_0 of $f\in\mathcal{H}$

$$ord_{z_0}f \coloneqq \min\left\{n\geq 0 : f^{(n)}(z_0)\neq 0\right\}$$

$\downarrow$
1) $ord_{z_0}f = \infty \implies f(z) = 0\ \forall z\in D_r(z_0)$
2) $ord_{z_0}f = n \neq \infty \implies \exists! h\in\mathcal{H}(D_r(z_0))$:
$f(z) = (z-z_0)^n h(z), h(z_0)\neq 0$
Thm.: princ. of analy. contin.
$f\in\mathcal{H}(U_c), \mathcal{Z}$ inf. set w/lim. pt. $z_0\in U_c, z_0\notin\mathcal{Z}$
if $f(z) = 0\ \forall z\in\mathcal{Z} \implies f\equiv 0$
$\rightsquigarrow$**Thm.**: $f\in\mathcal{H}(U_c)$
TFAE:
1) $f(z)\equiv 0$
2) $\exists a\in U_c : f^{(n)}(a) = 0\ \forall n\geq 0$
3) $\{z\in U_c : f(z) = 0\}$ has lim. pt. in U_c
$\implies$ **Id. thm.**: $f,g\in\mathcal{H}(U_c\neq\emptyset)$
TFAE:
1) $f = g$
2) $\exists a\in U_c : f^{(n)}(a) = g^{(n)}(a)\ \forall n\geq 0$
3) $\{z\in U_c : f(z) = g(z)\}$ has lim. pt. in U_c
Thm.: $f,g\in\mathcal{H}(U_c)$
$fg = 0 \implies f\equiv 0 \vee g\equiv 0$

$\rightsquigarrow f,g\in\mathcal{H}(D_r(z_0))$:
$re(f) =$ const. $\implies f$ const.
$re(f) = re(g) \implies f = g + ic, c\in\mathbb{R}$
$\overline{f}\in\mathcal{H}(D_r(z_0)) \implies f$ const.
$|f|$ const $\implies f$ const.
Def.: iso. sing. z_0 if: $f\in\mathcal{H}(D_r^*(z_0))$
remov. if f holom. ext. to $D_r(z_0)$
Riemann contin. thm.: $f\in\mathcal{H}(U\setminus\{z_0\})$
TFAE: (remov. sing. if:)
1) f holom ext to U
2) f contin. " — "
3) f bdd in $D_r^*(z_0)$
4) $\lim_{z\to z_0}(z-z_0)f(z) = 0$
$\rightsquigarrow$ **Riemann thm. on remov. sing.**:
$f\in\mathcal{H}(U\setminus\{z_0\})$, f bdd in $D_r^*(z_0)\subset U$
$\implies z_0$ removable singularity of f
Def.: ord. of pole z_0 of $f\in\mathcal{H}(D_r^*(z_0))$

$$ord_{z_0}f \coloneqq \min\left\{n\in\mathbb{N} : (z-z_0)^n f(z) \text{ is bdd near } z_0\right\}$$

$(\lim_{z\to z_0}(z-z_0)^n f(z)\in\mathbb{C})$
$\downarrow$
TFAE:
1) f has pole of ord. m at z_0
(i.e. $(z-z_0)^m f(z)$ bdd near z_0)
2) $g\in\mathcal{H}(D_r(z_0)) : g(z_0)\neq 0$ and
$f(z) = (z-z_0)^{-m}g(z)\ \forall z\in D_r^*(z_0)$
3) $h\in\mathcal{H}(D_r(z_0)) : h(z_0)\neq 0\ \forall z\in D_r^*(z_0)$
h has zero of ord. m at z_0
$\rightsquigarrow f(z) = \frac{1}{h(z)}\ \forall z\in D_r^*(z_0)$
Thm.: res. pole ord. n at z_0

$$res_{z_0}f \coloneqq \lim_{z\to z_0}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z)$$

Lemma:
$f,g\in\mathcal{H}(\{z_0\}), g$ simple zero at z_0
$\implies \frac{f}{g}$ has simple pole at z_0 and

$$res_{z_0}\left(\frac{f}{g}\right) = \frac{f'(z_0)}{g'(z_0)}$$

Def.: Meromorphic, $f : U\to\hat{\mathbb{C}}$
1) $S_f \coloneqq \{z\in U : f(z) = \infty\} (= f^{-1}(\{\infty\}))$
has no lim. pt. in U
2) $z_0\in S_f$ are poles of f
3) $f\in\mathcal{H}(U\setminus S_f)$
$\rightsquigarrow \mathcal{M}(U) \coloneqq \left\{\frac{f}{g} : f,g\in\mathcal{H}(U), g\not\equiv 0\right\}$
$\mathcal{M}(\hat{\mathbb{C}})\cong\mathbb{C}[z]$, $(f = \frac{P}{Q})$
Def.: ord. of zero/pole z_0 of $f\not\equiv 0\in\mathcal{M}(U)$
$ord_{z_0}f \coloneqq n\in\mathbb{Z}$:
1) z_0 not a pole of f (i.e.: $f(z_0)\neq\infty$)
$\implies n\geq 0$ is **ord. of zero** of f at z_0
2) z_0 pole of f (i.e.: $f(z_0) = \infty$)
$\implies n < 0$ is **minus ord. of pole** of f at z_0

$$\rightsquigarrow \begin{cases} ord_{z_0}f > 0 \implies z_0 \text{ zero} \\ ord_{z_0}f < 0 \implies z_0 \text{ pole} \\ ord_{z_0}f = 0 \implies z_0 \text{ neither}\end{cases}$$

Prop.: $f\not\equiv 0\in\mathcal{M}(U), z_0\in U$
1) $\mathbf{n = ord_{z_0}f \iff h\in\mathcal{H}(D_r(z_0)) : h(z_0)\neq 0}$
and $\mathbf{f(z) = (z-z_0)^n h(z)}$
2) $ord_{z_0}(fg) = ord_{z_0}f + ord_{z_0}g$
3) $ord_{z_0}(f/g) = ord_{z_0}f - ord_{z_0}g$
4) $f+g\not\equiv 0 \implies ord_{z_0}(f+g)\geq\min\{ord_{z_0}f, ord_{z_0}g\}$
Residue Formula:
$f\in\mathcal{H}(U\setminus S_f)\hat{=}\mathcal{M}(U_{sc})$

$$\oint_{\gamma=\partial D} f(z)dz = 2\pi i\sum_{\substack{z_0\in S_f\cap D^\circ \\ \gamma\cap S_f=\emptyset}} w_\gamma(z_0)res_{z_0}f$$

$w_\gamma(z_0) \coloneqq \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}dz\in\mathbb{Z}$
Arg. Princ.:

$$\frac{1}{2\pi i}\oint_{\gamma=\partial D}\frac{f'(z)}{f(z)}dz = \sum_{\substack{z_0\in Z_f\cap D^\circ \\ \gamma\cap Z_f=\emptyset}} ord_{z_0}f + \sum_{\substack{z_0\in S_f\cap D^\circ \\ \gamma\cap S_f=\emptyset}} ord_{z_0}f = \#\{Z_f\cap D^\circ\} - \#\{S_f\cap D^\circ\}$$

Picard Thm.:
$f\in\mathcal{H}(D_r^*(z_0))$, ess. sing. at z_0
$\implies |\mathbb{C}\setminus f(D_r^*(z_0))|\leq 1$
$\Downarrow$
Casorati-Weierstrass:
$\implies \forall z\in\mathbb{C}\varepsilon > 0\exists w\in D_r^*(z_0) : |z - f(w)| < \varepsilon$
Rouché's Thm.: $f,g\in\mathcal{H}(U)$
$\overline{D}\subset U : |f(z)| > |g(z)|\ \forall z\in\partial D$
$\implies \#Z_f = \#Z_{f+g}$ on D°
Max. mod. princ.:
$f\in\mathcal{H}(U_c)$, non-const.
$\implies \nexists z_0\in U_c : |f(z)|\leq|f(z_0)|\ \forall z\in U_c$
i.e. f cannot attain max. in U_c in part. if $\overline{U_c}$ bdd.
and $f\in C^0(\overline{U_c}) \implies \max_{z\in\overline{U_c}}|f(z)| = \max_{z\in\partial U_c}|f(z)|$
$(|f(z)|\leq|f(z_0)|\ \forall z\in U_c \implies f(z) = f(z_0)$ konst.)
Def.: Homotop
$\gamma_0,\gamma_1 : [a,b]\to U$:
$\begin{cases}\gamma_0(a) = \gamma_1(a) = \alpha, \\ \gamma_0(b) = \gamma_1(b) = \beta\end{cases}$
$\gamma_0\sim_U\gamma_1 \iff \exists H\in C^0 : [0,1]\times[a,b]\to U$:
$\begin{cases}H(0,t) = \gamma_0(t), \\ H(1,t) = \gamma_1(t) \quad \forall t\in[a,b]\end{cases}$ and
$\begin{cases}H(s,a) = \alpha, \\ H(s,b) = \beta \quad \forall s\in[0,1]\end{cases}$
$\rightsquigarrow \gamma_s(t) \coloneqq H(s,t)$
$\downarrow$
Def.: simply conn./wegzsmhgd. ($\sim$ 1 blob, no holes)
U_{sc} simply conn. $\iff$
$\exists\gamma$ der z_1 & z_2 verbindet & $\gamma_i\sim_{U_{sc}}\gamma_j$
Homotopy Thm.: $f\in\mathcal{H}(U)$
either
1) γ_1,γ_2 closed and $\gamma_1\sim_U\gamma_2$
or
2) $\gamma_1\sim_U\gamma_2$ w/ fixed endpoints
$\implies \int_{\gamma_1} fdz = \int_{\gamma_2} fdz$
Def.: Log., $\exp(\log(z)) \overset{!}{=} z$
$\forall z\in\mathbb{C}^\times\exists Log(z) = w = s + it, t = arg(w)\in(-\pi,\pi]$
andere log: $w + 2\pi ik, k\in\mathbb{Z}$
$\rightsquigarrow log_U(z) = log(|z|) + iarg_U(z)$
Princ. branch:

$$Log(z) \coloneqq \log(|z|) + iArg(z), \quad Arg(z)\in(-\pi,\pi]$$
$$= \sum_{n\in\mathbb{N}}\frac{(-1)^{n+1}}{n}(z-1)^n \quad \text{Konv. für } |z-1| < 1$$

$\rightsquigarrow$ jeder weitere log: $Log(z) + 2\pi ik, k\in\mathbb{Z}$
$Log(z^2)\neq 2Log(z)$, $z^\alpha \coloneqq \exp(\alpha(\log_U(z) + 2\pi ik))$
Def.:

$$P.V.\cdot\int_{\mathbb{R}} f(z)dz = \lim_{R\to\infty}\int_{[-R,R]} f(z)dz$$

$P.V. = 1$ für f gerade i.e. $f(t) = f(-t)$
MWS:
$f\in\mathcal{H}(U), D_r(z_0)\subseteq U$:

$$f(z_0) = \int_0^1 f(z_0 + r\exp(2\pi it))dt$$

$\hat{=}$ MW von f auf $D_r(z_0)$

Fourier

$\chi_T(t) := \begin{cases} 1, & t \in T \\ 0, & t \notin T \end{cases}$

Def.: Fund.Period., $f : \mathbb{R} \to \mathbb{C}$
falls $\exists p \in \mathbb{R}$:

$$f(t+p) = f(t) \quad \forall t \in \mathbb{R}$$

p-period., p:periode von f, freq. von f: $\frac{1}{p}$
kleinste Periode := Fundamentalperiode

Def.: FR/trig.poly. ord. N

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right) \right)$$

von Fkt. $\in \{\sin \frac{n\pi}{L}t, \cos \frac{n\pi}{L}t : n \in \mathbb{N}\}$, $a_N, b_N \neq 0$
$\Updownarrow$

$$\sum_{n=-N}^{N} c_n e^{i\frac{n\pi}{L}t}$$

von Fkt. $\in \left\{ e^{i\frac{n\pi}{L}t} : n \in \mathbb{Z} \right\}$, $c_N \neq 0$

$f(t) \overset{!}{=} \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{i\frac{n\pi}{L}t}$

$\rightsquigarrow f(t) \sim \sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi}{L}t}$

Lemma: ON-Beziehung $e^{i\frac{n\pi}{L}t}$
$m, n \in \mathbb{Z}$

$$\frac{1}{2L} \int_{-L}^{L} e^{i\frac{m\pi}{L}t} e^{-i\frac{n\pi}{L}t} dt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Lemma: ON-Beziehung $sin, cos(\frac{n\pi}{L}t)$
$m, n \in \mathbb{N}_0$

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right) \cos\left(\frac{m\pi}{L}t\right) dt = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}t\right) \sin\left(\frac{m\pi}{L}t\right) dt = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right) \sin\left(\frac{m\pi}{L}t\right) dt = 0 \quad \forall n, m$$

$$\boxed{\begin{aligned} c_0 &= \frac{a_0}{2} \\ c_n &= \frac{1}{2}(a_n - ib_n) \\ c_{-n} &= \frac{1}{2}(a_n + ib_n) \\ a_n &= c_n + c_{-n} \\ b_n &= i(c_n - c_{-n}) \end{aligned}}$$

Thm.: Fourier-Koeff., f $2L$-period.
$f(t) \sim \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)\right)$
$= \sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi}{L}t}$
n=1: 1.Harm. od. Grundschwingung
n=m: m.Harm. od. (m-1).Oberschwingung
$\Downarrow$

$$\boxed{\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right) dt, \quad n \geq 0 \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right) dt, \quad n > 0 \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(t) e^{-i\frac{n\pi}{L}t} dt, \quad n \in \mathbb{Z} \end{aligned}}$$

Def.: $\mathbb{C}^\infty$-VR $\mathcal{V}$ der 2L-period. Fkt. $f : \mathbb{R} \to \mathbb{C}$
$f, g \in \mathcal{V}$:
$\langle f, g \rangle = \frac{1}{2L} \int_{-L}^{L} f(t)\overline{g(t)} dt$
$\langle f, g \rangle = 0 \implies f, g$ orthog.
$\|f\|_{\mathcal{V}} := \sqrt{\langle f, f \rangle} = \left(\frac{1}{2L}\int_{-L}^{L} |f(t)|^2 dt\right)^{\frac{1}{2}}$

Thm.: symm.
f 2L-period.

$$\boxed{\begin{aligned} &\text{f gerade} \iff f(t) = f(-t) \text{ (symm. bzgl. } \vec{e}_y) \\ &\implies b_n = 0 \quad \forall n \\ &a_n = \frac{2}{L}\int_0^L f(t) cos\left(\frac{n\pi}{L}t\right) dt, \quad n \geq 0 \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{f ungerade} \iff f(t) = -f(-t) \text{ (pkt.symm. bzgl. 0)} \\ &\implies a_n = 0 \quad \forall n \\ &b_n = \frac{2}{L}\int_0^L f(t) \sin\left(\frac{n\pi}{L}t\right) dt, \quad n > 0 \end{aligned}}$$

Basel Problem.:
FR: $f(t) = (t-\pi)^2, t \in [0, 2\pi]$ gerade fortgesetzt
$\rightsquigarrow \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$

Dirichlet:
$f \in C^0_{pw}(I := [-L, L])$ 2L-period.
linke & rechte Abl. ex. $\forall z \in I$
$\implies$ FR von f auf I ist konv.

$$f \in C^0(\{t : t \in I\}):$$
$$\frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right) \right)$$
$$f \notin C^0(\{t : t \in I\}) \text{ i.e. Sprungstelle } t_0 \in I:$$
$$FR = \frac{1}{2}\left(\lim_{t\to t_0^-} f(t) + \lim_{t\to t_0^+} f(t) \right)$$

Thm.:
$f \in C^0_{pw}([-L, L])$ 2L-period.
linke & rechte Abl. ex. $\forall z \in I$
$\implies \exists! FR$ F, beste & eind. Approx. von f,
FR hat kleinste Abst. $\forall N$ i.e.: $d(f, s_N) := \|f - s_N\|^2$

Gibbs-Phänomen.: Überschwinger an Sprungstellen
$s_N := \sum_{n=-N}^{N} c_n e^{i\frac{n\pi}{L}t}$

$\rightsquigarrow \sup_{t\in[-L,L]} |f(t) - s_N(t)| \sim 0.18\frac{1}{2}\left(\lim_{t\to t_0^-} f(t) + \lim_{t\to t_0^+} f(t) \right)$
pktw. konv. aber **nicht** glm./unif. konv.

Thm.:
trig. poly. ord. N das am besten 2π-period. Fkt. auf $[-\pi, \pi]$ approx. (kleinster quadr. error): s_N der FR
$E^*_N(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 - \sum_{n=-N}^{N} |c_n|^2 dt$
$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt - \frac{1}{2}\left(\frac{a_0^2}{2} + \sum_{n=-N}^{N} (a_n^2 + b_n^2) \right)$
E^*_N monoton abnehmend mit zunehmendem N

Bessel ineq.: $\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt$

Parseval id.: $\sum_{n\in\mathbb{Z}} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt$

Amp.- & Phasenspektrum:
$n \geq 0$:

$$a_n \cos n\omega t + b_n \sin n\omega t = A_n \cos(n\omega t + \phi_n)$$
$$= \sqrt{a_n^2 + b_n^2} \cos\left(n\omega t + \left(-atan\left(\frac{b_n}{a_n}\right) + \begin{cases} 0, & a_n > 0 \\ \pi, & a_n < 0 \end{cases} \right) \right)$$
$$\rightsquigarrow f(t) = \frac{A_0}{2} + \sum_{n\in\mathbb{N}} A_n \cos(n\omega t + \phi_n)$$

$\rightsquigarrow \{A_n\}_{n\in\mathbb{N}_0} \hat{=}$Amp.spektrum, $\{\phi_n\}_{n\in\mathbb{N}_0} \hat{=}$Phasenspektrum
$\left(\rightsquigarrow c_k = |c_k| e^{i\arg(c_k)} \rightsquigarrow \{|c_k|\}_{k\in\mathbb{Z}}, \{\arg(c_k)\}_{k\in\mathbb{Z}} \right)$

Fourierkoeff. von f / FT:
$\hat{f} : \mathbb{Z} \to \mathbb{C}, n \mapsto \hat{f}(n) := \langle f, \vec{e}_{n,L} \rangle = c_n = \frac{1}{2L}\int_{-L}^{L} f(\xi) e^{-i\frac{n\pi}{L}\xi} d\xi$

Def.: $f : \mathbb{R} \to \mathbb{C}$ abs intbar: $\int_{\mathbb{R}} |f(t)| dt < \infty$

Dirichlet FT:
$f \in C^0_{pw}(\mathbb{R})$ abs. intbar. (links & rechts Abl. ex.):

$$f \in C^0:$$
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{-i\omega\xi} d\xi \right) e^{i\omega t} d\omega$$
$$f \notin C^0:$$
$$f(t) = \frac{1}{2}\left(\lim_{t\to t_0^-} f(t) + \lim_{t\to t_0^+} f(t) \right)$$

FT: $\hat{\mathbf{f}} \iff \mathcal{F}(\mathbf{f})$
$f : \mathbb{R} \to \mathbb{C}$ abs. intbar.
$\hat{f} : \mathbb{R} \to \mathbb{R}$:

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{-i\omega\xi} d\xi$$

$\rightsquigarrow \lim_{\omega\to\pm\infty} \hat{f}(\omega) = 0$
$|f(\xi)| = |f(\xi) e^{-i\omega\xi}| \rightsquigarrow$ int. wohldef. da f abs. intbar.
$|\hat{f}(\omega)| \hat{=}$ Gesamtenergie der Freq. ω

FT 2π-Period. Fkt.:

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt, \quad n \in \mathbb{Z}$$

FT inv.: $\mathcal{F}^{-1}(\hat{f})$
$f : \mathbb{R} \to \mathbb{C}, \hat{f}$ abs. intbar.
inv. von $\hat{f}$:

$$\mathcal{F}^{-1}(\hat{f})(t) = \mathcal{F}^{-1}(\mathcal{F}(f))(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega t} d\omega \overset{\text{Dirichlet FT}}{=} f(t)$$

$$\left\{ f : \mathbb{R}_{(t)} \to \mathbb{C} : \int_{\mathbb{R}} |f(t)| dt < \infty \right\}$$
$$\mathcal{F}\Big\downarrow\Big\uparrow\mathcal{F}^{-1}$$
$$\left\{ \hat{f} : \mathbb{R}_{(\omega)} \to \mathbb{C} : \int_{\mathbb{R}} |\hat{f}(\omega)| d\omega < \infty \right\}$$

Eigenschaften FT:
$f, g : \mathbb{R} \to \mathbb{C}, \hat{f}, f^{(n)}$ abs. intbar:

- Linearität, $\forall \alpha, \beta \in \mathbb{C}$:
  $(\widehat{\alpha f + \beta g})(\omega) = \alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$
- Verschiebung in t, $a \in \mathbb{R}, T_a f := f(t-a)$:
  $\widehat{T_a f}(\omega) = e^{-i\omega a}\hat{f}(\omega)$
- Versch. in ω, $a \in \mathbb{R}$:
  $\widehat{e^{iat}f(t)}(\omega) = \hat{f}(\omega - a)$
- Streckung, $a \in \mathbb{R}, S_a f(t) := f(at)$:
  $\widehat{S_a f}(\omega) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$
- $\hat{\hat{f}} = f(-t) \left(= \mathcal{F}^{-1}(\hat{f})(-t) \right)$
- FT Abl., $n \in \mathbb{N}$:
  $\widehat{f^{(n)}}(\omega) = (i\omega)^n \hat{f}(\omega)$
- Potenz, $n \in \mathbb{N}$:
  $\widehat{t^n f(t)}(\omega) = i^n \frac{d^n}{d\omega^n}\hat{f}(\omega)$

Def.: Integraltrafo.
$Tf(y) := \int_X K(x, y) f(x) dx$
f def. auf X, K: Kern der Int.trafo., def. auf $X \times Y$

Plancherel: (E_{tot} unter FT erhalten)
$f : \mathbb{R} \to \mathbb{C}, \hat{f}$ abs. intbar.

$$\implies \int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega$$

Faltung

Def.: Faltung
$f, g : \mathbb{R} \to \mathbb{C}$ abs. intbar.

$$(f * g)(x) := \int_{\mathbb{R}} f(x-t)g(t)dt = \int_{\mathbb{R}} f(t)g(x-t)dt$$

f, g verschwinden auf $(-\infty, 0)$ od. $(-\infty, 0]$:

$$(f * g)(x) = \begin{cases} 0, & x < 0 \\ (f * g)(x \geq 0), & x \geq 0 \end{cases}$$

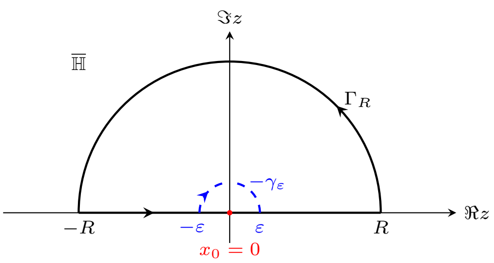
$$(f * g)(x \geq 0) = \int_{\mathbb{R}} f(x-t)g(t)dt$$
$$\overset{g(t)=0 \forall t \leq 0}{=} \int_{\mathbb{R}_+} f(x-t)g(t)dt$$
$$\overset{f(x-t)=0 \forall x \leq t}{=} \int_0^x f(x-t)g(t)dt$$

Eigenschaften Faltung:
$f, g, h : \mathbb{R} \to \mathbb{C}$

- Kommutativität:
  $(f * g) = (g * f)$
- Assoziativität:
  $(f * g) * h = f * (g * h)$
- Distributivität, $\alpha, \beta \in \mathbb{C}$:
  $(\alpha f + \beta g) * h = \alpha f * h + \beta g * h$
- $a \in \mathbb{R}, T_a(f)(x) := f(x-a)$:
  $(T_a f) * g = T_a(f * g)$
- FT der Faltung:
  $\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)$
- inv. FT der Faltung, $\hat{f}, \hat{g}$ abs. intbar.:
  $\mathcal{F}^{-1}(\hat{f} * \hat{g}) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(\hat{f})\mathcal{F}^{-1}(\hat{g})$
- FT des Prod., $\mathcal{F}(f), \mathcal{F}(f), f \cdot g$ abs. intbar.:
  $\mathcal{F}(f \cdot g) = \sqrt{2\pi}\mathcal{F}(f) * \mathcal{F}(g)$
- inv. FT des Prod., $f, g, f \cdot g$ abs. intbar.:
  $\mathcal{F}^{-1}(f \cdot g) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(g)$

Integrals:

$$\gamma_R = \Gamma_R + [-R, R]$$
$$= \Gamma_R + \lim_{\varepsilon \to 0}([-R, -\varepsilon] - \gamma_\varepsilon + [\varepsilon, R])$$
$$= \partial\{D_R \cap \mathbb{H}\}$$
$$\rightsquigarrow [-R, R] = \lim_{\varepsilon \to 0}([-R, -\varepsilon] + [\varepsilon, R])$$
$$= \gamma_R - \Gamma_R + \lim_{\varepsilon \to 0} \gamma_\varepsilon$$
$$\overline{\mathbb{H}} = \{z \in \mathbb{C} : \Im z \geq 0\}$$



Lemma:
$f \in \mathcal{H}(D_R^*(x_0)), x_0 \in \mathbb{R}$ simple pole of f
$\gamma_\varepsilon := x_0 + \varepsilon e^{it}, t \in [0, \pi], \varepsilon < R, im(\gamma_\varepsilon) > 0$

$$\implies \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)dz = \pi i \cdot res_{x_0} f$$

Type 1:

$$I := \int_{\mathbb{R}} \frac{P(x)}{Q(x)}dx$$

$P(x), Q(x) \in \mathbb{C}[x]$, Q has no zeros on $\mathbb{R}$:

$$degQ(x) \overset{!}{\geq} degP(x) + 2$$

$$\oint_{\gamma_R} \frac{P(z)}{Q(z)}dz = \int_{-R}^{R} \frac{P(x)}{Q(x)}dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)}dz$$
$$= 2\pi i \sum_{\substack{z_0 \in S_f \cap \{D_R \cap \mathbb{H}\}^\circ \\ \gamma_R \cap S_f = \emptyset}} res_{z_0} f$$

if
$$degQ = n, degP = m$$

on semi-circ Γ_R, for large R:

$$|Q(z)| > C\,|z|^n$$

for some C, and:

$$\left|\frac{P(z)}{Q(z)}\right| < C\frac{R^m}{R^n} = CR^{m-n}$$

$$\rightsquigarrow \left|\int_{\Gamma_R} \frac{P(z)}{Q(z)}dz\right| < CR^{m-n} \cdot \pi R = C\frac{1}{R^{n-m-1}}$$

$$\rightsquigarrow \int_{\Gamma_R} f \overset{R \to \infty}{\to} 0 \iff n - m - 1 \overset{!}{>} 0$$

(od.: $n > m + 1/n \geq m + 2$)
allg.:

$$\rightsquigarrow I = \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \int_{[-R, -\varepsilon] + [\varepsilon, R]} \frac{P(x)}{Q(x)}dx$$
$$= \lim_{R \to \infty} \oint_{\gamma_R} \frac{P(z)}{Q(z)}dz$$
$$- \lim_{R \to \infty} \int_{\Gamma_R} \frac{P(z)}{Q(z)}dz + \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{P(z)}{Q(z)}dz$$
$$= 2\pi i \sum_{\substack{z_0 \in S_f \cap \{D_R \cap \overline{\mathbb{H}}\}^\circ \\ \gamma_R \cap S_f = \emptyset}} res_{z_0} f$$
$$+ \pi i \sum_{\substack{x_0 \in S_f \cap \{D_R \cap \overline{\mathbb{H}}\}^\circ \\ \gamma_R \cap S_f = \emptyset \\ x_0 \in \mathbb{R}}} res_{x_0} f$$

Type 2:

$$I := \int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}dt$$

$P, Q \in \mathbb{C}[x], Q(x, y) \neq 0$ for $x^2 + y^2 = 1\ \forall x, y \in \mathbb{R}$
substitution on D_1:

$$\boxed{\begin{aligned} z &:= e^{it} \\ \rightsquigarrow \frac{1}{iz}dz &= dt \end{aligned}}$$

$$\rightsquigarrow \cos t = \frac{e^{it} + e^{-it}}{2}$$
$$= \frac{z + \frac{1}{z}}{2}$$
$$\rightsquigarrow \sin t = \frac{e^{it} - e^{-it}}{2i}$$
$$= \frac{z - \frac{1}{z}}{2i}$$

$$\rightsquigarrow \oint_{\partial D_1} f(z)\frac{1}{iz}dz = 2\pi i \sum_{\substack{z_0 \in S_f \cap D_1^\circ \\ \gamma_R \cap S_f = \emptyset}} res_{z_0} f$$

Type 3:
Same contour γ_R as Type 1
used for rational func. $\cdot \sin(ax), \cos(ax)$:

$$\int_{\mathbb{R}} \frac{P(x)}{Q(x)}\cos(ax)dx$$

$$\boxed{f(z) := \frac{P(z)}{Q(z)}e^{iaz}}$$

$$\oint_{\gamma_R} f(z)dz$$

$$\left|e^{iz}\right| = \left|e^{i(x+iy)}\right| = \left|e^{ix}\right| \cdot e^{-y} = e^{-y}$$

$$\implies \left|e^{iz}\right| \overset{z \in \mathbb{H} \iff im(z) > 0}{\leq} 1$$

$$\implies \int_{\Gamma_R} f(z)dz \overset{R \to \infty}{\to} 0$$

(je nachdem ob $\sin(ax)$ od. $\cos(ax)$ im ursprünglich zu berechnendem Integral vorkommt am Schluss Realteil oder Imaginärteil nehmen unter Berücksichtigung vom Vorfaktor i beim Imaginärteil)

Random

Trig.:
$\sin(\alpha + \frac{\pi}{2}) = \cos(\alpha)$
$\cos(\alpha - \frac{\pi}{2}) = \sin(\alpha)$
$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n \in \mathbb{N}_0} \frac{z^{2n}}{(2n)!}$
$\sinh(z) = \frac{e^z - e^{-z}}{2} = \sum_{n \in \mathbb{N}_0} \frac{z^{2n+1}}{(2n+1)!}$
$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$
$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$
$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$ / 0	0	1	0
$30°$ / $\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$45°$ / $\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$ / $\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$ / $\pi/2$	1	0	–

$f^{(n)}(e^{az}) = \frac{d^n}{dz^n}e^{az} = a^n e^{az}$
$\int_{\gamma \in \mathbb{C}} \Re(f(z) \in \mathcal{H})dz \not\in \mathbb{R}$
$f \in \mathcal{H} : |f'(z)| \leq$ auf $C := \{|z| = 1\}, f'(0) = 1$
$\implies \exists a \in \mathbb{C} : f(z) = z + a\ \forall z \in C$
PBZ: e.g.:
Ansatz zu $(x-q)^3$: $\frac{A}{(x-q)} + \frac{B}{(x-q)^2} + \frac{C}{(x-q)^3}$
Ansatz zu $(x^2+1)^2$: $\frac{Dx+E}{(x^2+1)} + \frac{Fx+G}{(x^2+1)^2}$
Quadratic Equation Formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Re\hat{f}(\xi) = 0, \forall\xi \iff f$ ungerade und reellwertig
$\Im\hat{f}(\xi) = 0, \forall\xi \iff f(-t) = \overline{f(t)}$
Integrale:
$\int \sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$
$\int \sin(ax)\sin(bx)dx \overset{a^2 \neq b^2}{=} \frac{\sin(ax-bx)}{2(a-b)} - \frac{\sin(ax+bx)}{2(a+b)}$
$\int \cos^2(ax)dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$
$\int \cos(ax)\cos(bx)dx \overset{a^2 \neq b^2}{=} \frac{\sin(ax-bx)}{2(a-b)} + \frac{\sin(ax+bx)}{2(a+b)}$
$\int \sin(ax)\cos(ax)dx = \frac{1}{2a}\sin^2(ax)$
$\int \sin(ax)\cos(bx)dx \overset{a^2 \neq b^2}{=} -\frac{\cos(ax-bx)}{2(a-b)} - \frac{\cos(ax+bx)}{2(a+b)}$
$\int x\sin(ax)dx = \frac{\sin(ax)}{a^2} - \frac{x}{a}\cos(ax)$
$\int x\cos(ax)dx = \frac{\cos(ax)}{a^2} + \frac{x}{a}\sin(ax)$
FT example:
$f(t) = e^{-|t|}, \hat{f}(\xi) = \frac{2}{1+\xi^2}$
$g(t) := f(-3t-3) = e^{-|-3t-3|}$
$\hat{g}(\xi) =?$

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(-3t-3)e^{-it\xi}dt$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{|-3|}\int_{\mathbb{R}} f(s)e^{-i\frac{(s+3)\xi}{-3}}ds$$
$$= \frac{1}{\sqrt{2\pi}}\frac{e^{i\xi}}{3}\int_{\mathbb{R}} f(s)e^{-is\frac{\xi}{-3}}ds$$
$$= \frac{e^{i\xi}}{3}\hat{f}\left(\frac{\xi}{-3}\right)$$
$$= \frac{1}{3}\frac{2}{1+\frac{\xi^2}{9}}e^{i\xi}$$
$$= \frac{6}{9+\xi^2}e^{i\xi}$$

Laurent example:
$f(z) = \frac{1}{(z-i)(z+3i)}$ auf $\{z \in \mathbb{C} : 1 < |z| < 3\} = A_{1,3}$
$\overset{PBZ.}{\rightsquigarrow} \frac{1}{4i}\left(\frac{1}{z-i} - \frac{1}{z+3i}\right) = \frac{1}{4i}\left(\frac{1}{z}\frac{1}{1-\frac{i}{z}} - \frac{1}{3i}\frac{1}{1-\frac{-z}{3i}}\right)$
auf $A_{1,3} : \left|\frac{i}{z}\right| < 1 \wedge \left|\frac{-z}{3i}\right| < 1$

$$\overset{geom.Reihe}{\rightsquigarrow} f(z) =$$
$$\ldots = \frac{1}{4i}\left(\sum_{n \in \mathbb{N}} \frac{i^{n-1}}{z^n} - \sum_{n \in \mathbb{N}_0} \frac{(-z)^n}{(3i)^{n+1}}\right)$$

Laplace

Laplace-Trafos.:

f(t)	$\mathcal{L}[f](s)$
1	$\frac{1}{s},\ \Re s > 0$
t^n	$\frac{n!}{s^{n+1}},\ \Re s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2},\ \Re s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2},\ \Re s > 0$
e^{at}	$\frac{1}{s-a},\ \Re s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ \Re s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ \Re s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}},\ \Re s > a$
$af(t) + bg(t)$	$a\mathcal{L}[f](s) + b\mathcal{L}[g](s)$
$tf(t)$	$-\frac{d}{ds}\,\mathcal{L}[f](s)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\,\mathcal{L}[f](s)$
$f'(t)$	$s\mathcal{L}[f](s) - f(0)$
$f''(t)$	$s^2\mathcal{L}[f](s) - sf(0) - f'(0)$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$\mathcal{L}[f](s)\cdot\mathcal{L}[g](s)$
$\frac{1}{a}e^{-\frac{t}{a}}$	$\frac{1}{as+1}$
$\frac{1}{a^2}te^{-\frac{t}{a}}$	$\frac{1}{(as+1)^2}$
$1-e^{-\frac{t}{a}}$	$\frac{1}{s(as+1)}$
$\frac{1}{a-b}\left(e^{-\frac{t}{a}} - e^{-\frac{t}{b}}\right)$	$\frac{1}{(as+1)(bs+1)}$
$\frac{1}{2a^3}t^2e^{-\frac{t}{a}}$	$\frac{1}{(as+1)^3}$
$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
$\sin(at+\varphi)$	$\frac{s\sin(\varphi)+a\cos(\varphi)}{(s^2+a^2)}$
$\cos(at+\varphi)$	$\frac{s\cos(\varphi)-a\sin(\varphi)}{(s^2+a^2)}$
$\sin^2(at)$	$\frac{2a^2}{s(s^2+4a^2)}$
$\cos^2(at)$	$\frac{s^2+2a^2}{s(s^2+4a^2)}$
$e^{-bt}\sin(at)$	$\frac{a}{(s+b)^2+a^2}$
$e^{-bt}\cos(at)$	$\frac{s+b}{(s+b)^2+a^2}$
$e^{-bt}\sin(at+\varphi)$	$\frac{(s+b)\sin\varphi+a\cos\varphi}{(s+b)^2+a^2}$
$e^{-bt}\cos(at+\varphi)$	$\frac{(s+b)\cos\varphi-a\sin\varphi}{(s+b)^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t^2\sin(at)$	$2a\frac{3s^2-a^2}{(s^2+a^2)^3}$
$t^2\cos(at)$	$2\frac{s^3-3a^2s}{(s^2+a^2)^3}$
$\sinh(at)$	$\frac{a}{s^2-a^2},\ \Re s > \lvert a\rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2},\ \Re s > \lvert a\rvert$
$\frac{t}{2a}\sinh(at)$	$\frac{s}{(s^2-a^2)^2}$

$\mathcal{L}\left[\frac{1}{2a^3}(at\cosh(at) - \sinh(at))\right] = \frac{1}{(s^2-a^2)^2}$
$\mathcal{L}\left[\frac{1}{2a}(at\cosh(at) + \sinh(at))\right] = \frac{s^2}{(s^2-a^2)^2}$

$f : \mathbb{R} \to \mathbb{C} : f(t) \overset{!}{=} 0\ \forall t \leq 0$
Def.: Heaviside-Fkt. $H(t)$

$$H(t) := \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

$\rightsquigarrow \chi_{(a,b)} := H(t-a) - H(t-b)$
$f \not\equiv 0\ \forall t < 0 \rightsquigarrow f \cdot H(t) \equiv 0\ \forall t < 0$
Laplace-Trafo.: $\mathcal{L}$
$s \in \mathbb{C}$:

$$\mathcal{L}[f](s) := \int_{\mathbb{R}_+} e^{-st} f(t)dt$$

Thm.:
$\mathcal{E}$ Raum von Fkt. $f : \mathbb{R} \to \mathbb{C}$ (Originalraum):

1. $f(t) = 0\ \forall t < 0$
2. $\exists \sigma, M \in \mathbb{R} : |f(t)| \leq Me^{\sigma t}\ \forall t > 0$
3. $f \in C^0_{pw.}$ & $\lim_{t \to t_0^\pm} f(t)$ ex. $\forall$ Sprungstellen $t_0 \in \mathbb{R}_+$ (insb. $t_0 = 0^+$)

$\implies \mathcal{L}[f]\forall f \in \mathcal{E}$ auf $\{s \in \mathbb{C} : re(s) > \sigma\}$ (Halbebene) wohldef. und kompl. analy. Fkt. der Var. s
$\rightsquigarrow \lim_{re(s)\to\infty} \mathcal{L}[f](s) = 0$
Def.: Wachstumskoeff.
$\sigma' > \sigma$: $e^{\sigma t} < e^{\sigma' t}$
$|f(t)| < Ce^{\sigma t} \implies |f(t)| < Ce^{\sigma' t}$
das kleinste σ_f :
$|f(t)| < Ce^{\sigma t}\ \forall \sigma_f < \sigma$ heisst Wachstumskoeff.
($\sigma_H = 0$)
Eigenschaften Laplace-Trafo.:
$f \in \mathcal{E}$:

- Linearität, $f, g \in \mathcal{E}, \alpha, \beta \in \mathbb{C}$:
  $\mathcal{L}[\alpha f + \beta g](s) = \alpha\mathcal{L}[f](s) + \beta\mathcal{L}[g](s)$
- Verschiebung in t:
  $\mathcal{L}[f(t-a)](s) = e^{-as}\mathcal{L}[f](s)$
- Verschiebung in s, $\alpha \in \mathbb{C}$:
  $\mathcal{L}[e^{\alpha t}f(t)](s) = \mathcal{L}[f](s-\alpha)$
- Ähnlichkeit, $a \in \mathbb{R}_+, S_a f(t) := f(at)$:
  $\mathcal{L}[S_a f](s) = \frac{1}{a}\mathcal{L}[f](\frac{s}{a})$
- Laplace-Trafo. Abl., $f' \in \mathcal{E}, f(t) \in C^0$ für $t > 0$:

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - \lim_{t\to 0^+} f(t)$$

$$f', \ldots, f^{(n)} \in \mathcal{E}, f, \ldots, f^{(n-1)} \in C^0 \quad \forall t > 0 :$$

$$\begin{aligned}
\mathcal{L}[f^{(n)}](s) &= s^n\mathcal{L}[f](s) - s^{n-1}\lim_{t\to 0^+} f(t) - s^{n-2}\lim_{t\to 0^+} f'(t) - \ldots - \lim_{t\to 0^+} f^{(n-1)}(t) \\
&= s^n\mathcal{L}[f](s) - \sum_{k\in\mathbb{N}_0 \leq n-1} s^{n-1-k}\lim_{t\to 0^+} f^{(k)}(t) \\
&\overset{f\in C^0(\{0^+\})}{=} s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \\
&= s^n\mathcal{L}[f](s) - \sum_{k\in\mathbb{N}_0 \leq n-1} s^{n-1-k} f^{(k)}(0)
\end{aligned}$$

- Abl. der Laplace-Trafo., $n \in \mathbb{N}_0$:
  $\frac{d^n}{ds^n}(\mathcal{L}[f])(s) = (-1)^n\mathcal{L}[t^n f(t)](s)$
- $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s) = \frac{1}{s}\mathcal{L}[f](s)$
- σ_f Wachstumskoeff. von f, $x > \sigma_f$:
  $\mathcal{L}\left[\frac{f(t)}{t}\right](x+iy) = \int_{x+iy}^{\infty+iy}\mathcal{L}[f](\tau)d\tau$
- $T > 0$, f T-period. Fkt. (f(t+T) = f(t)) $\forall t \geq 0$,
  $s \in \mathbb{C} : re(s) > 0$:
  $\mathcal{L}[f](s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$
- Faltungssatz, $f, g \in \mathcal{E}$:
  $\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]$
- $a \in \mathbb{R}$:
  $\mathcal{L}[\delta(t-a)](s) = e^{-as}$

Dirac-δ:

$$\delta_\varepsilon(t) := \frac{1}{2\varepsilon}\chi_{(-\varepsilon,\varepsilon)}(t)$$
$$\delta(t) := \lim_{\varepsilon\to 0}\delta_\varepsilon(t)$$

Eigenschaften-$\delta(t)$:

- $\int_{\mathbb{R}}\delta(t)dt = 1$
- $\forall f \in C^0$:
  $\int_{\mathbb{R}}\delta(t-t_0)f(t) = f(t_0)$
- $H(t) = \int_{-\infty}^t \delta(s)ds$

Lerch:
$f_1, f_2 \in \mathcal{E}, \sigma_1, \sigma_2$
$\mathcal{L}[f_1](s) = \mathcal{L}[f_2](s)\ \forall s : re(s) > max\{\sigma_1, \sigma_2\}$
$\implies f_1(t) = f_2(t)\ \forall t : f_1, f_2 \in C^0$
Inv. Laplace-Trafo.: $\mathcal{L}^{-1}$
$f \in \mathcal{E}$, σ_f, Laplace-Trafo $F(s)$, $\beta_c(y) := c + iy, y \in (-\infty, \infty)$ Pfad : $c > \sigma_f$ bel., $\forall$ Stetigkeitsstellen $t \in (0, \infty)$ von f:

$$f(t) = \frac{1}{2\pi i}\int_{\beta_c} e^{st}F(s)ds$$

für Unstetigkeitsstellen $t_0 \in (0, \infty)$:

$$\frac{1}{2}\left(\lim_{t\to t_0^-} F(t) + \lim_{t\to t_0^+} F(t)\right) = \frac{1}{2\pi i}\int_{\beta_c} e^{st}F(s)ds$$

Cheatsheet based on 401-0302-10 Mathematische Methoden, FS2025
This document is intended as a personal summary. It may contain mistakes, omissions, or additional information/notation borrowed from the Complex Analysis course (401-2303-00S HS23) for BSc Mathematics / BSc Physics. For questions or suggestions for improvement, feel free to write me an email: `atuzlak@ethz.ch`. A cheatsheet version for the BSc Mathematics / BSc Physics course can be found here: `https://n.ethz.ch/~atuzlak/ComA`