First passage percolation and large deviations

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Chapter 1

Introduction of the model and main questions

The model of first-passage percolation (FPP) was introduced by Hammersley and Welsh [14] in 1965 to study the spread of a fluid in a porous medium. Since then, it has been studied extensively in the probability and the statistical physics literature. We refer to [17] for a general background and to [3] for a review of more recent results.

1.1 Introduction of the model of FPP

Consider the lattice $(\mathbb{Z}^d, \mathbb{E}^d)$ for $d \geq 1$ where $\mathbb{E}^d := \{\{x, y\} : \|x - y\|_2 = 1, x, y \in \mathbb{Z}^d\}$ is the set of edges. Consider an independent and identically distributed (IID) family $(t_e)_{e \in \mathbb{E}^d}$ distributed according to some *weight distribution* G, a probability measure on $\mathbb{R}_+$. In our interpretation of the model, the random variable t_e represents the time needed to cross the edge e.

A path $p = (x_0, e_1, x_1, \ldots, e_n, x_n)$ connecting x to y is an alternative sequence of vertices and edges such that $x_i \in \mathbb{Z}^d$, $e_i \in \mathbb{E}^d$,for all $i \in \{1, \ldots, n\}$, $e_i = \{x_{i-1}, x_i\}$, $x_0 = x$ and $x_n = y$. To each finite path $p = (x_0, e_1, x_1, \ldots, e_n, x_n)$, we associate a passage time

$$T(p) := \sum_{i=1}^{n} t_{e_i}. \tag{1.1.1}$$

We define a random pseudo-metric T on $\mathbb{Z}^d$ by setting the *passage time between* $x, y \in \mathbb{Z}^d$ to be

$$T(x, y) := \inf\{T(p) : \ p \text{ is a path connecting } x \text{ and } y\}. \tag{1.1.2}$$

Any path achieving the infimum is called a *geodesic* between x and y. One can check that when G is atomless, there exists a unique geodesic.

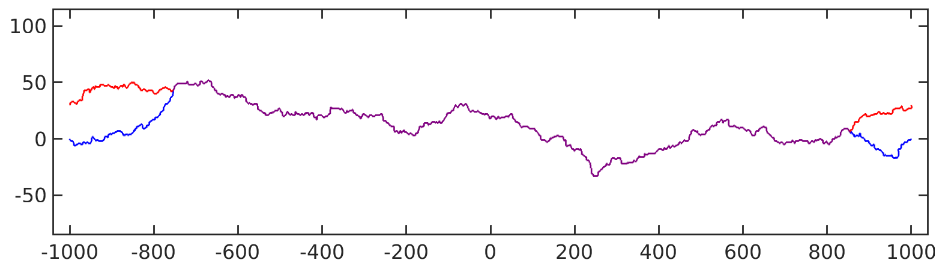


Figure 1.1: Simulation of two geodesics with different endpoints that have most of their edges in common.

Exercise

1. Prove that T is a pseudo-metric, i.e., it satisfies the triangular inequality and the symmetry. Argue why the separation property does not hold in general.

2. Prove that there exists almost surely a unique geodesic when G is atomless (i.e. has a density with respect to Lebesgue measure).

3. Prove that the infimum in the definition of T is almost surely attained when $G(\{0\}) = 0$.

For $t \geq 0$, denote $\mathcal{B}(t)$ the set of points that can be reached starting from 0 before time t, *i.e.*,

$$\mathcal{B}(t) := \left\{ x \in \mathbb{Z}^d : T(0,x) \leq t \right\}.$$

This set may represent the spread of an infection until time t; the infection starts at the origin 0 and for $e = \{x, y\} \in \mathbb{E}^d$ the time needed for x to transmit the infection to y is given by t_e. The set $\mathcal{B}(t)$ represents the set of vertices that have been infected before time t.

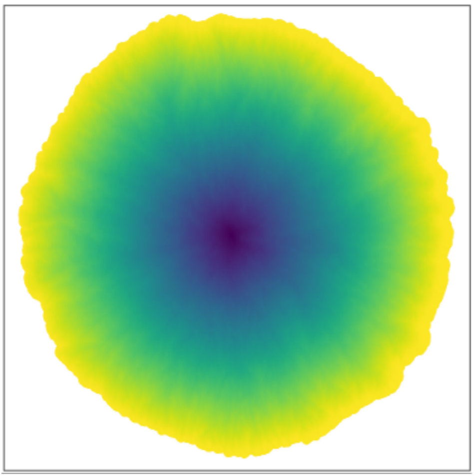


Figure 1.2: Simulation of $\mathcal{B}(150)$ for G uniform random variable on $[0, 1]$. The color scale represents the distance to 0.

In this lecture, we will be interested in the following questions.

1. The study of the large-scale properties of the random metric T and its geodesics. In particular, how does the random variable $T(0, nx)$ behaves when n is large ? What does $\mathcal{B}(t)$ look like for large t ? What is the impact of perturbing the edge weights on the asymptotic properties ?
2. The fluctuations of the random variable $T(0, nx)$ as well as the spatial fluctuations of the geodesics. How much do the geodesics deviate from the straight lines ?
3. The large deviations of the random variable $T(0, nx)$. We will see that the random variable $T(0, nx)/n$ converges, we will study at what rate this convergence occurs and what is the probability that this random variable has atypical deviations from its mean.

1.2 FPP in dimension 1

As a warm-up, let us start by studying the dimension $d = 1$. Denote for $n \geq 1$, $T_n := T(0, n)$. It is easy to check that

$$T_n = T(0, n) = \sum_{i=1}^{n} t_{\{i-1,i\}}.$$

In dimension 1, first passage percolation boils down to studying a sum of IID random variables. It follows from basic probability theory that if G is integrable, that is $\mathbb{E}[t_e] = \int_{\mathbb{R}_+} x dG(x) < \infty$ (where e is a fixed edge in $\mathbb{E}^1$), then we have the *strong law of large numbers*

$$\lim_{n\to\infty} \frac{T_n}{n} = \mathbb{E}[t_e] \qquad \text{almost surely.}$$

The law of large numbers characterizes the asymptotic properties of T_n. One can also wonder about the fluctuations of T_n, how much T_n typically deviates from its expected value $n\mathbb{E}[t_e]$. If G has a finite second moment, that is $\int_{\mathbb{R}_+} x^2 dG(x) < \infty$, then the *central limit theorem* holds, we have

$$\lim_{n\to\infty} \sqrt{n} \left(\frac{T_n}{n} - \mathbb{E}[t_e] \right) \overset{d}{=} \mathcal{N}(0, \mathrm{Var}(t_e))$$

where $\mathcal{N}(0,\sigma^2)$ is a normal distribution with mean 0 and variance σ^2. The fluctuations are the typical deviations from the mean value. One can also be interested in atypical deviations also called *large deviations*, that is the probability that the random variable T_n has non-negligible deviations from its mean. More precisely, we are interested in the decay rate of the following probability for $\varepsilon > 0$

$$\mathbb{P}\left(\frac{T_n}{n} - \mathbb{E}[t_e] \geq \varepsilon\right).$$

If the distribution G has light tails in the sense that the distribution G has an exponential moment, i.e. there exists $\theta > 0$ such that $\mathbb{E}[e^{\theta t_e}] < \infty$ then the decay rate is exponential. We will prove at the end of the lecture, that there exists a function $I : \mathbb{R}_+ \to [0, +\infty]$ such that

$$\mathbb{P}\left(\frac{T_n}{n} - \mathbb{E}[t_e] \geq \varepsilon\right) = e^{-nI(\varepsilon)+o(n)}$$

as n goes to infinity. We will prove that when G has light tails, the most likely scenario causing the upper large deviations event is when each input makes a tiny contribution to the overall deviation by tilting its distribution slightly.

In what follows, we will also be interested in asymptotic properties (law of large numbers), fluctuations, and large deviations. We will see that our understanding of the model of FPP is far from complete in $d \geq 2$.

1.3 Asymptotic properties of FPP in $d \geq 2$: definition of the time constant

A natural question is to understand the large scale properties of this random pseudo-metric. In particular, what is the asymptotic behavior of the quantity $T(0, nx)$ when n goes to infinity ? If G has a finite moment, one can prove that we have a *law of large numbers*: asymptotically when n is large, the random variable $T(0, nx)$ behaves like $n\,\mu_G(x)$ where $\mu_G(x)$ is a deterministic constant depending only on the distribution G and the point $x \in \mathbb{Z}^d$. More precisely, we have the following theorem.

Theorem 1.3.1 (Law of large numbers for passage times). *Let G be a distribution on $\mathbb{R}_+$ with a finite moment. For every $x \in \mathbb{Z}^d$, there exists a deterministic constant $\mu_G(x)$ such that*

$$\lim_{n\to\infty} \frac{T(0,nx)}{n} = \mu_G(x) \qquad \text{almost surely and in } L^1.$$

This constant $\mu_G(x)$ is the so-called time constant *in the direction x.*

This constant may be interpreted as an inverse speed in the direction x. The convergence of this quantity comes from the fact that the family $(T(nx, mx))_{1\leq n\leq m}$ is *sub-additive*. Sub-additivity is a central notion in statistical mechanics. To get a better understanding of this notion, we can start with the deterministic setting with Fekete's lemma (whose proof is left as an exercise):

Lemma 1.3.2 (Fekete's lemma). *Let $(u_n)_{n\geq 1}$ be a sub-additive sequence of real numbers, that is, a sequence such that*

$$\forall n \geq 1 \quad \forall m \geq 1 \qquad u_{n+m} \leq u_n + u_m\,.$$

Then,

$$\lim_{n\to\infty} \frac{u_n}{n} = \inf_{n\geq 1} \frac{u_n}{n} \in \mathbb{R}\cup\{-\infty\}\,.$$

Proof. It is sufficient to prove that

$$\inf_{n\geq 1} \frac{u_n}{n} \leq \liminf_{n\to\infty} \frac{u_n}{n} \leq \limsup_{n\to\infty} \frac{u_n}{n} \leq \inf_{n\geq 1} \frac{u_n}{n}.$$

The left hand-side follows easily from the definition of lim inf. Let us prove the right-hand side. Let $\varepsilon > 0$ and $q \geq 1$ such that

$$\frac{u_q}{q} \leq \inf_{n\geq 1} \frac{u_n}{n} + \varepsilon.$$

Let $n \geq 1$ write $n = qp + r$ with $0 \leq r < q$. Using the subadditivity, it yields

$$\frac{u_n}{n} \leq \frac{pu_q + u_r}{n} \leq \frac{qp}{qp+r}\frac{u_q}{q} + \frac{1}{n}\inf_{0\leq l<q} u_r.$$

By taking the limsup when n goes to infinity

$$\limsup_{n\to\infty} \frac{u_n}{n} \le \frac{u_q}{q} \le \inf_{n\ge 1} \frac{u_n}{n} + \varepsilon .$$

The result follows by letting ε goes to 0. □

We claim that the sequence of real numbers $(\mathbb{E}[T(0,nx)])_{n\ge 1}$ is sub-additive. Indeed, using the triangular inequality and taking the expectation, we get

$$\forall n \ge 1 \quad \forall m \ge 1 \qquad \begin{aligned} \mathbb{E}[T(0,(n+m)x)] &\le \mathbb{E}[T(0,nx)] + \mathbb{E}[T(nx,(n+m)x)] \\ &= \mathbb{E}[T(0,nx)] + \mathbb{E}[T(0,mx)] \end{aligned}$$

where we use in the last equality that the model is invariant in law under translations by a vector in $\mathbb{Z}^d$. It follows using Fekete's lemma that the limit of $\mathbb{E}[T(0,nx)]/n$ when n goes to infinity exists and is finite when $\mathbb{E}[T(0,x)] < \infty$. The proof of the convergence of $T(0,nx)/n$ requires more powerful tools. We need the following theorem that comes proved by Kingman [18] using ergodic theory (the proof was later simplified by Steele [19]:

Theorem 1.3.3 (Kingman's subadditive ergodic theorem)**.** *Let $(X_{n,m},\, 0 \le n < m)$ be a collection of random variables such that*

1. $X_{l,n} \le X_{l,m} + X_{m,n}$ *whenever* $0 \le l < m < n$,
2. *For each $m \ge 0$, the joint distributions of $(X_{m,n},\, 0 \le m \le n)$ are the same as those of $(X_{m+1,n+1},\, 0 \le m \le n)$*
3. *For each $n \ge 1$, $\mathbb{E}[|X_{0,n}|] < \infty$ and $\mathbb{E}[X_{0,n}] \ge -cn$ for some constant c.*

Then

$$\lim_{n\to\infty} \frac{X_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}[X_{0,n}]}{n} = \inf_{n\ge 1} \frac{\mathbb{E}[X_{0,n}]}{n} \quad \text{a.s. and in } L^1.$$

Let us prove how we can deduce Theorem 1.3.1 from Theorem 1.3.3. Let $x \in \mathbb{Z}^d$, consider the sequence $(T(nx,mx),\, 0 \le n < m)$. Thanks to the triangular inequality, this sequence satisfies the condition 1. Because the environment $(t_e)_{e\in\mathbb{E}^d}$ is i.i.d., the sequence also satisfies the condition 2. When the distribution G has a first moment, *i.e.*, $\mathbb{E}[t_e] < \infty$, then

$$\mathbb{E}[T(0,nx)] \le \|nx\|_1 \mathbb{E}[t_e] < +\infty$$

and condition 3 is satisfied. It follows that

$$\lim_{n\to\infty} \frac{T(0,nx)}{n} = \lim_{n\to\infty} \frac{\mathbb{E}[T(0,nx)]}{n} = \mu_G(x) \quad \text{a.s. and in } L^1 .$$

The subaditive ergodic theorem is a powerful tool but it has two main limitations: it cannot be applied to irrational directions $x \in \mathbb{R}^d$ (because of the property 2) and it cannot be applied to a distribution G without a first moment.

1.4 Fluctuations in $2D$: Last passage percolation

Even though deriving the precise behaviour of fluctuations in FPP is expected to be a very hard problem, there is an analogous model, called last passage percolation, where we have a precise picture of the fluctuations. This model has the advantage of being *solvable*, meaning that the problem of fluctuations can be solved by making explicit computations.

Define the model of last passage percolation (LPP) on $\mathbb{Z}_+^2$, we consider an IID family $(\omega_x)_{x\in\mathbb{Z}_+^2}$ distributed according to geometric distribution of parameter $q \in (0,1)$. Let $M,N \ge 1$. Let $\Pi_{M,N}$ be the set of all up/right paths from $(1,1)$ to (M,N), that is the set of $p = (x_0,\dots,x_n)$ such that $x_0 = (1,1)$, $x_n = (M,N)$ and for all $i \in \{1,\dots,n\}$ $x_i - x_{i-1} \in \{(1,0),(0,1)\}$.

Set

$$G(M,N) := \max_{p\in\Pi_{M,N}} \omega_x .$$

Let $\gamma \ge 1$ be a parameter characterising the slope of the paths. Using the subadditive ergodic theorem 1.3.3 (here the process is super-additive), we can prove that a law of large numbers holds; there exists $\omega(\gamma,q) > 0$ such that

$$\lim_{n\to\infty} \frac{G(\lfloor \gamma n\rfloor, n)}{n} = \sup_{n\ge 1} \frac{\mathbb{E}[G(\lfloor \gamma n\rfloor, n)]}{n} := \omega(\gamma,q) \quad \text{a.s.}. \tag{1.4.1}$$

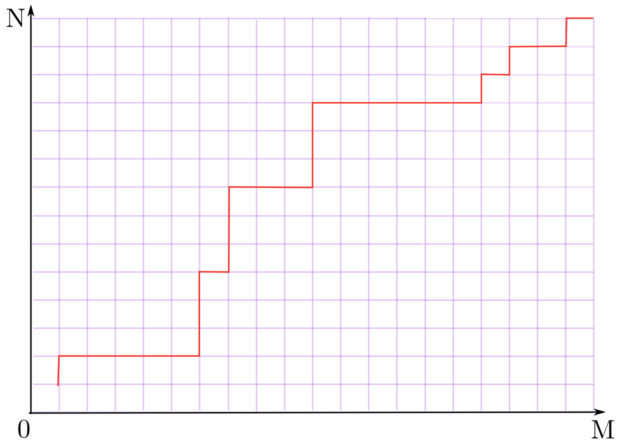


Figure 1.3: A path $p \in \Pi_{M,N}$

Exercise Prove the convergence in (1.4.1). Is the function $q \mapsto \omega(\gamma, q)$ monotone ?

The Tracy–Widom cumulative distribution function F_{TW} was introduced by Tracy and Widom [23] in the context of the study of the largest eigenvalue of hermitian random matrices. The function F_{TW} has a semi-explicit expression. In particular, this distribution is *not symmetric* (see figure 1.4). We refer to Johansson [23] for the precise expression of F_{TW}.

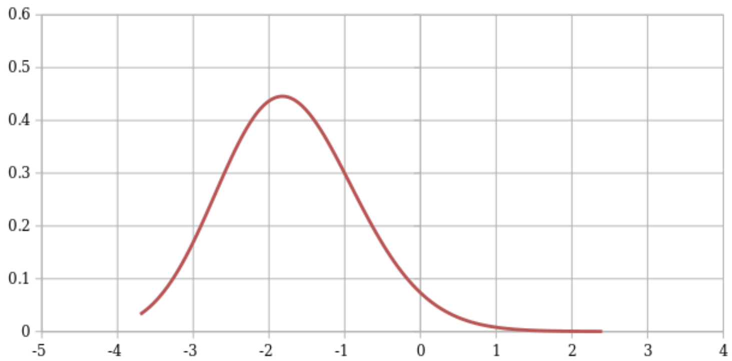


Figure 1.4: Graph of the density function f_{TW} corresponding to F_{TW}

There exists $\sigma(\gamma, q) > 0$ that has an explicit expression in terms of γ and q.

Theorem 1.4.1 (Johansson [15]). *For each $q \in (0,1)$, $\gamma \geq 1$ and $s \in \mathbb{R}$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{G(\lfloor \gamma n \rfloor, n) - n\omega(\gamma, q)}{\sigma(\gamma, q) n^{1/3}} \leq s\right) = F_{TW}(s).$$

This theorem is of a similar nature to the central limit theorem. Similar to how the normal distribution describes the fluctuations of various quantities, Tracy-Widom fluctuations belong to another universality class of fluctuations that encompass several models, such as the distribution of the largest eigenvalues of certain types of random matrices and the size of the longest increasing subsequence in a random permutation. The concept of *universality* is very important in statistical mechanics. A *universality class* is a collection of models that share the same macroscopic behavior even though they can drastically differ at the microscopic level.

For instance, the model of planar FPP is conjectured to be in the same universality class than the model of LPP. It enables to have insights on the fluctuations of FPP. In particular, let us informally define the scaling exponents χ, ξ:

$$\mathrm{Var}(T((0,0),(0,n))) \approx n^{2\chi} \quad \text{and} \quad h_n \approx n^{\xi}$$

where h_n corresponds to the maximal height attained by the geodesic γ between $(0,0)$ and $(n,0)$, i.e., $h_n := \sup\{|y| : (x,y) \in \gamma\}$. It is believed that the FPP model is part of the KPZ universality class, which includes various stochastic growth models first introduced by Kardar–Parisi–Zhang [16]. This universality class is characterized by shared scaling exponents ($\chi = 1/3$ and $\xi = 2/3$) that govern the large-scale properties of their interfaces or surfaces, such as their roughness and growth rates. While it has been proven that the LPP model belongs to the KPZ universality class, the FPP model is still conjectured to belong to this class.

At first sight, it may be surprising that the density function f_{TW} does not have null mean value (see figure 1.4). It is due to the so-called *non-random fluctuations*

$$\mathbb{E}[G(\lfloor \gamma n \rfloor, n)] - n\omega(\gamma, q) < 0.$$

An important point to note is that, unlike the central limit theorem, the fluctuations in this case are not symmetric once centered. Intuitively, it is easier for $G(M,N)$ to be above the mean because only one path needs to have a larger weight than the mean, while to be below the mean, all paths must have weights smaller than the mean. We will observe that this asymmetry is further amplified when analyzing upper and lower large deviations.

Chapter 2

Asymptotic properties

2.1 Limit shape theorem

In this section, we are interested in the behavior of $\mathcal{B}(t)$ when t is large. In the previous section, we have seen the existence of a time constant. The emergence of a limit shape boils down to prove that the convergence towards the time constant is uniform in the direction.

2.1.1 Extending the time constant to $\mathbb{R}^d$ and properties of the time constant

In this section, we aim at extending μ_G to all $\mathbb{R}^d$ and proving basic properties of this function. Denote by $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ the canonical basis of $\mathbb{R}^d$.

Proposition 2.1.1. *Let G be a distribution on $\mathbb{R}_+$ with a finite moment. The time constant μ_G can be extended to $\mathbb{R}^d$. Moreover, the function μ_G is homogeneous and satisfies the following inequality*

$$\forall x, y \in \mathbb{R}^d \qquad |\mu_G(y) - \mu_G(x)| \leq \mu_G(y - x) \leq \mu_G(\mathbf{e}_1)\|y - x\|_1. \tag{2.1.1}$$

Proof. Let us start with rational points in $\mathbb{R}^d$. Let $x \in \mathbb{Q}^d$ and $N \geq 1$ such that $Nx \in \mathbb{Z}^d$. It follows that

$$\mu_G(Nx) = \lim_{n\to\infty} \frac{T(0, nNx)}{n} = N \lim_{n\to\infty} \frac{T(0, nNx)}{Nn}.$$

Hence, we set

$$\mu_G(x) := \frac{\mu_G(Nx)}{N}.$$

By the same arguments, one can prove μ_G is homogeneous, that is

$$\forall r \in \mathbb{Q} \quad \forall x \in \mathbb{Q}^d \qquad \mu_G(rx) = |r|\mu_G(x). \tag{2.1.2}$$

Now that μ_G is defined on $\mathbb{Q}^d$, we need to prove that the map $x \mapsto \mu_G(x)$ is continuous in order to define μ_G on the whole $\mathbb{R}^d$. Let $x, y \in \mathbb{Z}^d$, we have by triangular inequality

$$\mu(x + y) = \lim_{n\to\infty} \frac{\mathbb{E}T(0, n(x + y))}{n} \leq \lim_{n\to\infty} \frac{\mathbb{E}T(0, nx))}{n} + \lim_{n\to\infty} \frac{\mathbb{E}T(nx, n(x + y))}{n} = \mu(x) + \mu(y).$$

Let $x \in \mathbb{Q}^d$, we can write

$$x = \sum_{i=1}^{d} \lambda_i \mathbf{e}_i, \quad \lambda_i \in \mathbb{Q}, 1 \leq i \leq d.$$

By triangular inequality and inequality (2.1.2), we have

$$\mu_G(x) \leq \sum_{i=1}^{d} |\lambda_i| \mu_G(\mathbf{e}_i).$$

Since μ_G is invariant under permutation of the coordinate axes, it yields that

$$\mu_G(x) \leq \sum_{i=1}^{d} |\lambda_i| \mu_G(\mathbf{e}_1) \leq \|x\|_1 \mu_G(\mathbf{e}_1).$$

Moreover, by triangular inequality, we have

$$\forall x, y \in \mathbb{Q}^d \qquad \mu_G(y) \le \mu_G(x) + \mu_G(y - x)\,.$$

It follows that

$$\forall x, y \in \mathbb{Q}^d \qquad |\mu_G(y) - \mu_G(x)| \le \mu_G(y - x) \le \mu_G(\mathbf{e}_1)\|y - x\|_1$$

The latter inequality enables to extend μ_G to $\mathbb{R}^d$ by continuity and density of $\mathbb{Q}^d$ in $\mathbb{R}^d$. We can prove that for any $r \ge 0$ and $x \in \mathbb{R}^d$, we have $\mu_G(rx) = r\mu_G(x)$. □

Exercises

1. Prove that there exists $p_0 > 0$ such that if $G(\{0\}) < p_0$ then $\mu_G(x) > 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$.
2. Prove that μ_G is convex.
3. Prove that $\mu_G = 0$ if and only if $\mu_G(\mathbf{e}_1) = 0$.

2.1.2 Limit shape theorem

We have seen in Theorem 1.3.1, that there exists a law of large numbers, when considering some direction $x \in \mathbb{Z}^d$, we have the almost sure convergence of $T(0, nx)$ towards $\mu_G(x)$. If the latter convergence is uniform in all directions, then an asymptotic shape emerges. This was first proved by Cox and Durrett in [11]. Set

$$\forall t \ge 0 \qquad \widetilde{\mathcal{B}}(t) = \mathcal{B}(t) + \left[-\frac{1}{2}, \frac{1}{2}\right)^d.$$

Theorem 2.1.2 (Cox-Durrett [11]). *Let G be a distribution on $\mathbb{R}^+$ such that $\mu_G > 0$ and*

$$\mathbb{E}[\min(t_1^d, \dots, t_{2d}^d)] < \infty \tag{2.1.3}$$

where t_i, $i = 1, \dots 2d$, are independent copies of t_e. Then, there exists a deterministic convex compact shape $\mathcal{B}_{\mu_G}$ in $\mathbb{R}^d$ such that

$$\forall \varepsilon > 0 \quad a.s. \quad \exists t_0 > 0 \qquad \forall t \ge t_0 \quad (1 - \varepsilon)\mathcal{B}_{\mu_G} \subset \frac{\widetilde{\mathcal{B}}(t)}{t} \subset (1 + \varepsilon)\mathcal{B}_{\mu_G}\,.$$

Furthermore, $\mathcal{B}_{\mu_G} := \{x \in \mathbb{R}^d : \mu_G(x) \le 1\}$ is the unit ball for the norm μ_G, it has non-empty interior and is symmetric about the axes of $\mathbb{R}^d$.

Little is known about the limit shape $\mathcal{B}_{\mu_G}$. There is no distributions G for which we can prove that the limit shape is not a polygon. One of the main open problem is to prove that the limit shape is strictly convex for continuous distributions. We will see that the limit shape encodes information about the geometry of geodesics.

To prove Theorem 2.1.2, we need to interpolate between a finite number of rational directions to obtain the linear growth of $\mathcal{B}(t)$ for any direction. We can apply the subadditive ergodic theorem to a fixed rational direction to obtain the linear growth of $\mathcal{B}(t)$ in that direction. It follows that with probability one, the linear growth holds simultaneously for a finite set of rational directions. To be able to interpolate, we also need to prove that if two vertices are close then their passage time is not too large. To simplify the proof, we will work under the stronger assumption that G has a finite 3-moment.

The following lemma ensures that the passage time between close vertices is not too large.

Lemma 2.1.3. *Assume $\mathbb{E}[t_e^4] < \infty$ There exists $\kappa > 0$ such that for any $\varepsilon > 0$, there exists almost surely $M > 0$ random such that*

$$\forall x, y \in \mathbb{Z}^d \qquad \|x\|_1 \ge M \ \text{and}\ \|x - y\|_1 \le \varepsilon\|x\|_1 \implies T(x, y) \le \kappa\varepsilon\|x\|_1. \tag{2.1.4}$$

Proof of Theorem 2.1.2. Let G be a distribution such that μ_G is a norm. Let $c_0, C_0 > 0$ be such that

$$\forall x \in \mathbb{R}^d \qquad c_0\|x\|_1 \le \mu_G(x) \le C_0\|x\|_1.$$

Let $\kappa > 0$ be as in the statement of Lemma 2.1.3. Let $\varepsilon > 0$ and M be large enough such that (2.1.4) holds. There exists a finite set $(y_1, \dots, y_m)$ of rational points in $\{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}$ such that

$$\{x \in \mathbb{R}^d : \|x\|_1 = 1\} \subset \bigcup_{i=1}^{m} \left\{ x \in \mathbb{R}^d : \|y_i - x\|_1 \leq \frac{\varepsilon}{2} \right\}.$$

Let $\lambda_i > 0$ be such that $\lambda_i y_i \in \mathbb{Z}^d$. Thanks to the subadditive ergodic theorem, there exists a random $n_i \geq 1$ such that

$$\forall n \geq n_i \quad \left| \frac{T(0, n\lambda_i y_i)}{n} - \mu_G(\lambda_i y_i) \right| \leq \varepsilon.$$

Let $t_0 = \max(M, \lambda_i n_i, \lambda_i/2\varepsilon)$. Let $t \geq t_0$. Let $x \in \mathbb{R}^d$ such that $\|x\|_1 \geq t_0$, let y_i the closest point from $x/\|x\|_1$. We have

$$\| \|x\|_1 y_i - x\|_1 \leq \frac{\varepsilon}{2}\|x\|_1.$$

There exists $n \geq n_i$ such that

$$\|\lambda_i n y_i - x\|_1 \leq \frac{\varepsilon}{2}\|x\|_1 + \lambda_i \leq \varepsilon\|x\|_1.$$

Let $\varepsilon' = (\mu_G(\mathbf{e}_1) + 1 + \kappa)\varepsilon/c_0$. Hence for $x \in \mathbb{Z}^d$ such that $\mu_G(x) \leq (1 - \varepsilon')t$, using (2.1.4) and (2.1.1), we have

$$T(0,x) \leq T(0, \lambda_i n y_i) + T(\lambda_i n y_i, x) \leq n\mu_G(\lambda_i y_i) + \varepsilon n + \kappa\varepsilon\|x\|_1 \leq \mu_G(x) + (\mu_G(\mathbf{e}_1) + 1 + \kappa)\varepsilon\|x\|_1 \leq t.$$

It yields the leftmost inclusion.

Conversely, let $x \in \mathbb{Z}^d$ such that $T(0,x) \leq t$. We have

$$n(\mu_G(\lambda_i y_i) - \varepsilon) \leq T(0, \lambda_i n y_i) \leq T(0,x) + T(x, \lambda_i n y_i) \leq t + \kappa\varepsilon\|x\|_1.$$

By (2.1.1), it yields that

$$\mu_G(x) \leq n\mu_G(\lambda_i y_i) + \mu_G(\mathbf{e}_1)\varepsilon\|x\|_1 \leq t + (\kappa + \mu_G(\mathbf{e}_1) + 1)\varepsilon\|x\|_1 \leq t + (\kappa + \mu_G(\mathbf{e}_1) + 1)\varepsilon C_0 \mu_G(x)$$

and

$$\mu_G(x) \leq \frac{1}{1 - (\kappa + \mu_G(\mathbf{e}_1) + 1)\varepsilon C_0} t.$$

This yields the second inclusion with $\varepsilon'' = \frac{(\kappa + \mu_G(\mathbf{e}_1) + 1)\varepsilon C_0}{1 - (\kappa + \mu_G(\mathbf{e}_1) + 1)\varepsilon C_0}$. To conclude the proof, for all $\varepsilon_0 > 0$, we choose ε small enough such that $\varepsilon', \varepsilon'' \leq \varepsilon_0$. □

Proof of Lemma 2.1.3. For simplicity, we will assume that $\mathbb{E}[|t_e|^3] < \infty$. We claim that there exists $\kappa_0 \geq 1$ such that for any $x, y \in \mathbb{Z}^d$, there exist $2d$ edge disjoint paths between x and y of length less than $\kappa_0\|x - y\|_1$. We leave this claim as an exercise. Let p be a path of length less than $\kappa_0\varepsilon\|x\|_1$ between x and y. Let $\kappa \geq 2\kappa_0\mathbb{E}[t_e]$ to be choose later. We have

$$\begin{aligned}
\mathbb{P}\left(\sum_{e \in p} t_e \geq \kappa\varepsilon\|x\|_1\right) &\leq \mathbb{P}\left(\sum_{e \in p}(t_e - \mathbb{E}[t_e]) \geq \kappa_0\varepsilon\|x\|_1\mathbb{E}[t_e]\right) \\
&\leq \frac{1}{(\kappa_0\varepsilon\|x\|_1\mathbb{E}[t_e])^4}\left(\sum_{e \in p}\mathbb{E}[(t_e - \mathbb{E}[t_e])^4] + 12\sum_{e \neq f \in p}\mathbb{E}[(t_e - \mathbb{E}[t_e])^2(t_f - \mathbb{E}[t_f])^2]\right) \\
&\leq \frac{24|p|^2}{(\kappa_0\varepsilon\|x\|_1\mathbb{E}[t_e])^4}\mathbb{E}[(t_e - \mathbb{E}[t_e])^4].
\end{aligned}$$

It follows using the $2d$ edge disjoint paths $(p_1, \dots, p_{2d})$ joining x and y

$$\begin{aligned}
\mathbb{P}(T(x,y) \geq \kappa\varepsilon\|x\|_1) &\leq \mathbb{P}(\forall i \in \{1, \dots, 2d\} \quad T(p_i) \geq \kappa\varepsilon\|x\|_1) \\
&= \prod_{i \in \{1,\dots,2d\}} \mathbb{P}(T(p_i) \geq \kappa\varepsilon\|x\|_1) \\
&\leq \left(\frac{24(\kappa_0\|x - y\|_1)^2}{(\kappa_0\varepsilon\|x\|_1\mathbb{E}[t_e])^4}\mathbb{E}[(t_e - \mathbb{E}[t_e])^4]\right)^{2d}.
\end{aligned}$$

Denote

$$\mathcal{E}_n := \{\exists x, y \in \mathbb{Z}^d \quad \|x\|_1 = n,\ \|x-y\|_1 \le \varepsilon n,\ T(x,y) \ge \kappa\varepsilon n\}.$$

Note that

$$\mathcal{E}_n \subset \bigcup_{x:\|x\|_1=n} \bigcup_{y:\|y-x\|_1\le\varepsilon n} \{T(x,y) \ge \kappa\varepsilon n\}.$$

We have by union bounding

$$\mathbb{P}(\mathcal{E}_n) \le \sum_{x:\|x\|_1=n} \sum_{y:\|y-x\|_1\le\varepsilon n} \mathbb{P}(T(x,y) \ge \kappa\varepsilon n) \le C \sum_{x:\|x\|_1=n} \sum_{y:\|y-x\|_1\le\varepsilon n} \frac{1}{n^{4d}} \le \frac{C}{n^{2d}} \tag{2.1.5}$$

It follows that

$$\sum_{n\ge1} \mathbb{P}(\mathcal{E}_n) < \infty. \tag{2.1.6}$$

By Borel-Cantelli Lemma, we deduce that there is almost surely only a finite number of occurrence of $\mathcal{E}_n$. Hence, there exists almost surely $M \ge 1$ such that (2.1.4) holds. □

2.2 Positivity of the time constant

In this section, we are interested in characterizing the distributions G such that $\mu_G = 0$. This will require some background on percolation theory.

2.2.1 Percolation

Bernoulli bond percolation was introduced by Broadbent and Hammersley in 1957 to model the circulation of water in a porous medium [14]. The model is defined as follows. Let $d \ge 2$. We consider an i.i.d. family of Bernoulli random variables $(B_e)_{e\in\mathbb{E}^d}$ of parameter $p \in [0,1]$. We will denote by $\mathbb{P}_p$ the distribution of this family. If $B_e = 1$, then we say that the edge e is *open*; otherwise, we say the edge is *closed*. This model was introduced to model the circulation of water through microscopic pores of a rock, with the open edges representing the holes through which water can flow.

Let $\mathcal{G}_p$ be the graph of the open edges:

$$\mathcal{G}_p := \left(\mathbb{Z}^d, \{e \in \mathbb{E}^d :\ e \text{ is open}\}\right).$$

We write $x \leftrightarrow y$ if x and y are connected in $\mathcal{G}_p$. For $n \ge 1$, define the box $\Lambda_n := \{-n, \dots, n\}^d$ and its inner boundary $\partial\Lambda_n := \{y \in \mathbb{Z}^d : \|y\|_\infty = n\}$.

Define for $p \in [0,1]$

$$\theta(p) := \lim_{n\to\infty} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n).$$

We can prove that $p \to \theta(p)$ is non-decreasing. Define

$$p_c(d) := \sup\{p \ge 0 : \theta(p) = 0\}.$$

Theorem 2.2.1. *For $d \ge 2$, we have $p_c(d) \in (0,1)$.*

We say that this model exhibits a phase transition at $p_c(d)$: for $p > p_c(d)$ (*supercritical regime*), there almost surely exists a unique infinite open cluster $\mathcal{C}_\infty$ in $\mathcal{G}_p$. In contrast, for $p < p_c(d)$ (*subcritical regime*), there is no infinite open cluster [13].

We will also need the following estimates. The following theorem controls the probability of having a long open path in the subcritical regime.

Theorem 2.2.2. *Let $p < p_c(d)$, there exist $c, C > 0$ such that*

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \le Ce^{-cn}.$$

The following theorem controls the size of the holes of the infinite cluster in the supercritical regime.

Theorem 2.2.3. *Let $p > p_c(d)$, there exist $c, C > 0$ such that*

$$\mathbb{P}_p(\mathcal{C}_\infty \cap \Lambda_n = \emptyset) \le Ce^{-cn^{d-1}}.$$

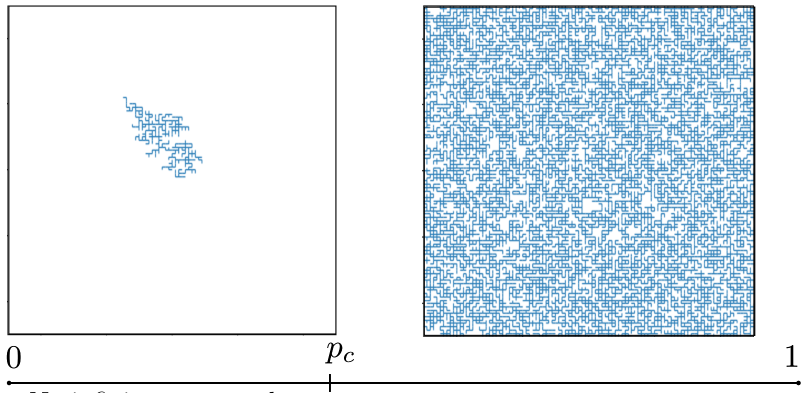


Figure 2.1: Simulation of the connected component of 0 in $\mathcal{G}_p$ in the subcritical regime $p < p_c$ and in the supercritical regime $p > p_c$ (in this simulation $0 \in \mathcal{C}_\infty$).

Exercises

1. Prove that $p \mapsto \theta(p)$ is non-decreasing.
2. Prove that $p_c(d) > 0$.
3. Prove that the function $d \mapsto p_c(d)$ is non-increasing.

2.2.2 Control of the length of the geodesics

Theorem 2.2.4. *Let G such that $G(\{0\}) < p_c(d)$, there exist $\rho, c, C > 0$ such that for all $n \geq 1$*

$$\mathbb{P}(\textit{There exists a self-avoiding path } \gamma \textit{ starting at } 0 \textit{ such that } T(\gamma) \leq n \textit{ and } |\gamma| \geq \rho n) \leq C \exp(-cn).$$

In order to prove this theorem, we will need a combinatorial result on the number of connected component of size n containing 0. We say that $\Gamma \subset \mathbb{Z}^d$ is connected if for any $x, y \in \Gamma$, there exists a sequence of vertices $x_0, \dots, x_k \in \Gamma$ such that $\|x_i - x_{i-1}\|_\infty = 1$. Let Animal_x^k be the set of connected macroscopic sites of $\mathbb{Z}^d$ of size k containing the site x. We have the following control

$$|\mathrm{Animal}_x^k| \leq 7^{dk} \tag{2.2.1}$$

(see for instance Grimmett [13, (4.24)]).

Exercise Let $\mathcal{C}(0)$ be the connected component of 0 in $\mathcal{G}_p$. By considering $\mathbb{P}_p(|\mathcal{C}(0)| = n)$ prove inequality (2.2.1).

Proof. Let $\rho > 1$ be a constant we will choose later. Let $\varepsilon > 0$ such that $G([0, \varepsilon]) < p_c(d)$. Let $N \geq 1$. We do a renormalization of size N; that is we divide the space into mesoscopic components of size N. We define the N-box for $\mathbf{k} \in \mathbb{Z}^d$

$$\underline{\Lambda}_N(\mathbf{k}) := [-N, N)^d \cap \mathbb{Z}^d + 2\mathbf{k}N,$$

and the enlarged box

$$\underline{\Lambda}'_N(\mathbf{k}) = \bigcup_{\|\mathbf{i}-\mathbf{k}\|_\infty \leq 1} \underline{\Lambda}_N(\mathbf{i}).$$

Note that

$$\mathbb{Z}^d = \bigsqcup_{\mathbf{i} \in \mathbb{Z}^d} \underline{\Lambda}_N(\mathbf{i})$$

where $\sqcup$ denotes the disjoint union. The sites corresponding to the boxes are the so-called *macroscopic lattice*; whereas the standard vertices in $\mathbb{Z}^d$ correspond to the *microcopic lattice*. We say that a box $\mathbf{i} \in \mathbb{Z}^d$ is good if there is no path of diameter at least N made of edges with passage time less than ε inside $\underline{\Lambda}'_N(\mathbf{i})$. Otherwise, we say that the box is bad. Let us compute the probability that a box is bad. Set $p = G([0,\varepsilon])$. We can couple the model of FPP with a subcritical percolation in such a way that an edge e is open if and only if $t_e \le \varepsilon$.

$$\mathbb{P}(\mathbf{i} \text{ is bad}) \le \mathbb{P}_p\left(\bigcup_{x\in\underline{\Lambda}'_N(\mathbf{i})}\{x \longleftrightarrow (\partial\Lambda_N + x)\}\right) \le |\underline{\Lambda}'_N(\mathbf{i})|\,\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_N) \le (6N)^d C\exp(-cN) \tag{2.2.2}$$

where we used Theorem 2.2.2 in the last inequality.

We will need the following result that controls the probability that there exists a set of $\mathbb{Z}^d$-connected boxes containing a positive fraction of bad boxes. Let $\Gamma \subset \mathbb{Z}^d$ be a fixed set. The state of the boxes in Γ are not necessarily independent. Though it is easy to check that if $\|\mathbf{i}-\mathbf{j}\|_\infty \ge 3$ then the state of the boxes $\mathbf{i}$ and $\mathbf{j}$ are independent. Note that

$$\mathbb{Z}^d = \bigsqcup_{x\in\{0,1,2\}^d}(3\mathbb{Z}^d + x).$$

Then, if Γ contains at least a $|\Gamma|/2$ bad boxes, by pigeon-hole principle, there exists $x \in \{0,1,2\}^d$ such that $\Gamma\cap(3\mathbb{Z}^d+x)$ contains at least $\frac{1}{2\cdot 3^d}|\Gamma|$ bad sites. In particular, these sites are independent. It yields

$$\begin{aligned}\mathbb{P}\left(|\{\mathbf{i}\in\Gamma : \mathbf{i} \text{ is bad}\}| \ge \frac{1}{2}|\Gamma|\right) &\le \sum_{A\subset\Gamma:|A|\ge|\Gamma|/2}\mathbb{P}\left(\forall \mathbf{i}\in A \quad \mathbf{i} \text{ is bad}\right)\\ &\le \sum_{A\subset\Gamma:|A|\ge|\Gamma|/2}\mathbb{P}(\mathbf{i} \text{ is bad})^{|A|/3^d}\\ &\le 2^{|\Gamma|}C\exp\left(-cN\frac{1}{3^d}|\Gamma|\right).\end{aligned} \tag{2.2.3}$$

Denote

$$\mathcal{F}_n := \left\{\exists k\ge n \quad \exists\Gamma\in \text{Animal}_0^k \quad |\{\mathbf{i}\in\Gamma : \mathbf{i} \text{ is bad}\}| \ge \frac{1}{2}|\Gamma|\right\}.$$

By union bound, using (2.2.1) and (2.2.3), we get

$$\begin{aligned}\mathbb{P}(\mathcal{F}_n) &\le \sum_{k\ge n}\sum_{\Gamma\in\text{Animal}_0^k}\mathbb{P}\left(|\{\mathbf{i}\in\Gamma : \mathbf{i} \text{ is bad}\}| \ge \frac{1}{2}|\Gamma|\right)\\ &\le \sum_{k\ge n}7^{dk}2^k C\exp\left(-cN\frac{k}{3^d}\right).\end{aligned}$$

Finally, we choose N large enough depending on d such that

$$\mathbb{P}(\mathcal{F}_n) \le \exp\left(-cN\frac{n}{6^d}\right). \tag{2.2.4}$$

Let γ be a self-avoiding path such that $T(\gamma) \le n$ and $|\gamma| \ge \rho n$. Denote by $\Gamma(\gamma)$ the set of boxes intersecting γ, that is

$$\Gamma(\gamma) := \{\mathbf{i}\in\mathbb{Z}^d : \underline{\Lambda}_N(\mathbf{i})\cap\gamma \ne \emptyset\}.$$

It is easy to check that when γ is self avoiding then $|\Gamma(\gamma)| \ge |\gamma|/(2N)^d$ and that the set $\Gamma(\gamma)$ is $\mathbb{Z}^d$ connected. In particular, on the event $\mathcal{F}^c_{\rho n/(2N)^d}$, we have that γ crosses at least $|\Gamma(\gamma)|/2$ boxes where γ has at least one edge of passage time larger than ε. It yields that

$$T(\gamma) \ge \frac{|\gamma|}{2(2N)^d}\varepsilon \ge \frac{\rho}{2(2N)^d}\varepsilon n.$$

Finally we set $\rho := \frac{4(2N)^d}{\varepsilon}$. Thanks to this choice, on the event $\mathcal{F}^c_{\rho n/(2N)^d}$, there exists no path γ starting at 0 such that $|\gamma| \ge \rho n$ and $T(\gamma) \le n$. The proof follows from inequality (2.2.4).

□

2.2.3 Positivity of the time constant

Theorem 2.2.5. *We have that $\mu_G > 0$ if and only if $G(\{0\}) < p_c$.*

Proof. Let us assume $G(\{0\}) > p_c$ and G is integrable. Let $\mathcal{C}_\infty$ denote the infinite cluster of edges with null passage time. Let $n \geq 1$. Set

$$\mathcal{F}_n := \{\mathcal{C}_\infty \cap \partial\Lambda_{\log^2 n} \neq \emptyset\} \cap \{\mathcal{C}_\infty \cap (\partial\Lambda_{\log^2 n} + n\mathbf{e}_1) \neq \emptyset\}.$$

Note that for $x, y \in \mathcal{C}_\infty$, we have $T(x, y) = 0$. On the event $\mathcal{F}_n$, let $x \in \mathcal{C}_\infty \cap \partial\Lambda_{\log^2 n}$ and $y \in \mathcal{C}_\infty \cap (\partial\Lambda_{\log^2 n} + n\mathbf{e}_1)$. It follows that

$$T(0, n\mathbf{e}_1) \leq T(0, x) + T(x, y) + T(y, x) = T(0, x) + T(y, n\mathbf{e}_1).$$

Denote π_n the straight path joining 0 and $n\mathbf{e}_1$. It follows that

$$T(0, n\mathbf{e}_1) \leq \sum_{e\in\pi_n} t_e \mathbf{1}_{\mathcal{F}_n^c} + \sum_{e\in\Lambda_{\log^2 n}\cup(n\mathbf{e}_1+\Lambda_{\log^2 n})} t_e. \tag{2.2.5}$$

By Theorem 2.2.3, we have

$$\mathbb{P}(\mathcal{F}_n^c) \leq 2Ce^{-c(\log n)^{2(d-1)}}$$

and

$$\sum_{n\geq 1} \mathbb{P}(\mathcal{F}_n^c) < \infty.$$

By Borel-Cantelli Lemma, it yields that

$$\lim_{n\to\infty} \mathbf{1}_{\mathcal{F}_n^c} = 0 \qquad \text{a.s.}.$$

By strong law of large numbers, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{e\in\pi_n} t_e = \mathbb{E}[t_e] \quad \text{a.s.}.$$

Besides,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n} \sum_{e\in\Lambda_{\log^2 n}\cup(n\mathbf{e}_1+\Lambda_{\log^2 n})} t_e\right] = 0.$$

In particular, it implies that there exists a subsequence $(n_k)_{k\geq 1}$ such that the integrand converges almost surely to 0. Finally, we get

$$\mu_G(\mathbf{e}_1) = \lim_{n\to\infty} \frac{T(0, n\mathbf{e}_1)}{n} = \lim_{k\to\infty} \frac{T(0, n_k\mathbf{e}_1)}{n_k} = 0.$$

The case where $G(\{0\}) = p_c(d)$ will follow by continuity of $G \mapsto \mu_G$ (we refer to the next section).

Conversely, let us now assume that $G(\{0\}) < p_c(d)$ and let us prove that $\mu_G > 0$. The geodesic between 0 and $n\mathbf{e_1}$ contains at least n edges. Thanks to Theorem 2.2.4, we have

$$\mathbb{P}(T(0, n\mathbf{e}_1) \geq n/\rho) \geq 1 - Ce^{-cn}.$$

By Borel–Cantelli Lemma, it follows that $\mu_G(\mathbf{e}_1) \geq 1/\rho$. This concludes the proof. □

2.3 Continuity of the time constant

In this section, we aim to estimate the impact of perturbating the distribution G on the time constant μ_G. Recall the notion of weak convergence of measures. Let $(G_n)_{n\in\mathbb{N}}$, and G be probability measures on $[0, +\infty)$, we say that G_n weakly converges toward G if for all continuous bounded functions $f : [0, +\infty) \to [0, +\infty)$, we have

$$\lim_{n\to+\infty} \int_{[0,+\infty)} f dG_n = \int_{[0,+\infty)} f dG\,.$$

Theorem 2.3.1. *Let $(G_n)_{n\in\mathbb{N}}$, and G be integrable probability measures on $[0, +\infty)$ such that G_n weakly converges toward G. We have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| = 0$$

where $\mathbb{S}^{d-1}$ is the unit sphere for the Euclidean norm.

A key part of the proof of this theorem is a continuity result for truncated distributions.

Proposition 2.3.2. *Let G be a distribution on $\mathbb{R}_+$ such that $G(\{0\}) < p_c$. For $M > 0$, set $G^M := G\mathbf{1}_{[0,M]} + G((M,+\infty))\delta_M$, then*

$$\forall x \in \mathbb{Z}^d \qquad \lim_{M\to\infty} \mu_{G^M}(x) = \mu_G(x)\,.$$

Lemma 2.3.3. *Let $M > 0$. Let $(G_n)_{n\in\mathbb{N}}$, and G be probability measures on $[0,M]$ such that G_n weakly converges toward G and $G_n(\{0\}), G(\{0\}) < p_c$. We have*

$$\forall x \in \mathbb{Z}^d \quad \lim_{n\to\infty} \mu_{G_n}(x) = \mu_G(x).$$

Proof. Let $\delta > 0$. Let $\varepsilon > 0$ we will choose later depending on δ. Since G_n weakly converges towards G, there exists a coupling of $(t_e^G)_e$ and $(t_e^{G_n})_e$ such that $(t_e^G)_e$ (respectively $(t_e^{G_n})_e$) is an IID family distributed according to G (respectively G_n) and there exists $n_0 \geq 1$ such that for all $n \geq n_0$

$$\forall e \in \mathbb{E}^d \qquad \mathbb{P}(|t_e^G - t_e^{G_n}| \geq \varepsilon) \leq \varepsilon.$$

We denote by T_G (respectively T_{G_n}) the pseudo-metric associated to the family of edge weights $(t_e^G)_e$ (respectively $(t_e^{G_n})_e$).

Let $x \in \mathbb{Z}^d$ and $k \geq 1$. Denote by γ the geodesic between 0 and kx for T_{G_n}. We have

$$T_G(0,kx) \leq \sum_{e\in\gamma} t_e^G \leq \sum_{e\in\gamma} t_e^{G_n} + \varepsilon + M\mathbf{1}_{|t_e^G - t_e^{G_n}|\geq\varepsilon} \leq T_{G_n}(0,kx) + \varepsilon|\gamma| + M\sum_{e\in\gamma}\mathbf{1}_{|t_e^G - t_e^{G_n}|\geq\varepsilon}. \tag{2.3.1}$$

Let p be a fixed path, let us compute the following using Chebyshev inequality for $\alpha > 0$

$$\mathbb{P}\left(\sum_{e\in p}\mathbf{1}_{|t_e^G - t_e^{G_n}|\geq\varepsilon} \geq \delta|p|\right) \leq \frac{\mathbb{E}[\exp(\alpha\mathbf{1}_{|t_e^G - t_e^{G_n}|\geq\varepsilon})]^{|p|}}{e^{\alpha\delta|p|}} \leq \left(\frac{1+\varepsilon e^{\alpha}}{e^{\delta\alpha}}\right)^{|p|}. \tag{2.3.2}$$

It follows that

$$\mathbb{P}\left(\exists p \text{ a path starting at 0 s.t. } |p| \geq k \text{ and } \sum_{e\in p}\mathbf{1}_{|t_e^G - t_e^{G_n}|\geq\varepsilon} \geq \delta|p|\right) \leq \sum_{j\geq k}(2d)^k\left(\frac{1+\varepsilon e^{\alpha}}{e^{\delta\alpha}}\right)^k.$$

We first choose α such that $2de^{-\delta\alpha} \leq \frac{1}{2}$ depending on δ and then ε such that $1 + \varepsilon e^{\alpha} \leq \frac{3}{2}$. It yields that

$$\mathbb{P}\left(\exists p \text{ a path starting at 0 s.t. } |p| \geq k \text{ and } \sum_{e\in p}\mathbf{1}_{|t_e^G - t_e^{G_n}|\geq\varepsilon} \geq \delta|p|\right) \leq 4\left(\frac{3}{4}\right)^k. \tag{2.3.3}$$

Besides, by Theorem 2.2.4, since $T_{G_n}(0,kx) \leq M\|kx\|_1$, it yields

$$\mathbb{P}(|\gamma| \geq \rho M k\|x\|_1) \leq Ce^{-ck}.$$

It follows that for large enough k

$$\mathbb{P}(T_G(0,kx) \leq T_{G_n}(0,kx) + \varepsilon\rho M k + M^2\rho\delta k) \geq \frac{3}{4}.$$

Furthermore, by subadditive ergodic theorem, we have that for k large enough

$$\mathbb{P}(T_G(0,kx) \geq k(\mu_G(x) - \delta)) \geq \frac{3}{4}$$

and

$$\mathbb{P}(T_{G_n}(0,kx) \leq k(\mu_{G_n}(x) + \delta)) \geq \frac{3}{4}.$$

Hence, the intersection of these three events occurs with positive probability and

$$\mu_G(x) - \delta \leq \mu_{G_n}(x) + \delta + \varepsilon\rho M + M^2\rho\delta.$$

By taking first the liminf in n and then the limit when δ goes to 0, it yields

$$\mu_G(x) \leq \liminf_{n\to\infty} \mu_{G_n}(x).$$

To conclude, we prove similarly by considering the geodesic for T_G that

$$\limsup_{n\to\infty} \mu_{G_n}(x) \leq \mu_G(x).$$

□

Lemma 2.3.4. *Let $(G_n)_{n\in\mathbb{N}}$, and G be probability measures on $\mathbb{R}_+$ such that G_n weakly converges toward G and $G_n \succ G$ and G is integrable. We have*

$$\forall x \in \mathbb{Z}^d \qquad \lim_{n\to\infty} \mu_{G_n}(x) = \mu_G(x).$$

Proof. Due to the stochastic domination, we can couple the weight distributions $(t_e^{G_n})_e$ and (t_e^G) in such a way that for all $e \in \mathbb{E}^d$ $t_e^{G_n} \geq t_e^G$ and

$$\forall e \in \mathbb{E}^d \qquad \lim_{n\to\infty} t_e^{G_n} = t_e^G \quad \text{a.s..}$$

To do so we use the standard coupling using uniform random variable on $[0,1]$, that is $t^G = F_G^{-1}(U)$ and $t^{G_n} = F_{G_n}^{-1}(U)$ where F_G^{-1} is the inverse of the partition function of G. We recall that

$$\forall t \in [0,1] \qquad F_G^{-1}(t) := \sup\{x \geq 0 : G([0,x]) \leq t\}.$$

It yields that

$$\forall x, y \in \mathbb{Z}^d \qquad T_{G_n}(x,y) \geq T_G(x,y)$$

and

$$\liminf_{n\to\infty} \mu_{G_n}(x) \geq \mu_G(x). \tag{2.3.4}$$

Besides for $x, y \in \mathbb{Z}^d$ and γ be a path between x and y, we have

$$T_G(\gamma) = \lim_{n\to\infty} T_{G_n}(\gamma) \geq \limsup_{n\to\infty} T_{G_n}(x,y).$$

By taking the infimum over all path γ joining x and y in the previous inequality, it follows that

$$T_G(x,y) \geq \limsup_{n\to\infty} T_{G_n}(x,y)$$

and

$$T_G(x,y) = \lim_{n\to\infty} T_{G_n}(x,y) \quad \text{a.s..}$$

Let H be the distribution corresponding to $\sup_n t_e^{G_n}$. In particular, we have that $H \succ G$. There exists $M > 0$ such that $H([0,M)) > p_c$. Denote by $\mathcal{C}_\infty$ the infinite cluster of edges such that $t_e^H \leq M$ and denote $\widetilde{x}$ the closest point to x in $\mathcal{C}_\infty$ with a deterministic rule to break ties. We have

$$\forall x \in \mathbb{Z}^d \qquad T_{G_n}(\widetilde{0}, \widetilde{x}) \leq T_H(\widetilde{0}, \widetilde{x}).$$

We admit the following result that is related to the study of graph distance in the infinite cluster of percolation. There exists $\beta > 0$ such that

$$\forall x \in \mathbb{Z}^d \qquad \mathbb{E}[T_H(\widetilde{0}, \widetilde{x})] \leq \beta M \|x\|_1.$$

By Fekete Lemma, we have

$$\lim_{k\to\infty} \frac{\mathbb{E}[T_{G_n}(\widetilde{0}, \widetilde{kx})]}{k} = \inf_{k\geq 1} \frac{\mathbb{E}[T_{G_n}(\widetilde{0}, \widetilde{kx})]}{k}.$$

Besides, we have

$$|T_{G_n}(\widetilde{0}, \widetilde{kx}) - T_{G_n}(0, kx)| \leq T_{G_n}(\widetilde{0}, 0) + T_{G_n}(kx, \widetilde{kx})$$

For every $\varepsilon > 0$

$$\mathbb{P}(|T_{G_n}(\widetilde{0}, \widetilde{kx}) - T_{G_n}(0, kx)| \geq 2\varepsilon k) \leq 2\mathbb{P}(T_{G_n}(\widetilde{0}, 0) \geq \varepsilon k).$$

Since by Theorem 2.2.3, the random variable $T_{G_n}(\widetilde{0}, 0)$ is almost surely finite, it yields that

$$\lim_{k\to\infty} \frac{1}{k} |T_{G_n}(\widetilde{0}, \widetilde{kx}) - T_{G_n}(0, kx)| = 0 \quad \text{in probability,}$$

and

$$\inf_{k\geq 1} \frac{\mathbb{E}[T_{G_n}(\widetilde{0}, \widetilde{kx})]}{k} = \mu_{G_n}(x).$$

Since $T_H(\widetilde{x}, \widetilde{y})$ is integrable, we have using dominated convergence theorem

$$\mathbb{E}[T_G(\widetilde{x}, \widetilde{y})] = \mathbb{E}[\limsup_{n\to\infty} T_{G_n}(\widetilde{x}, \widetilde{y})] = \limsup_{n\to\infty} \mathbb{E}[T_{G_n}(\widetilde{x}, \widetilde{y})].$$

Let $x \in \mathbb{Z}^d$ and $\varepsilon > 0$, by definition of the time constant, there exists $k \geq 1$ such that

$$\mu_G(x) \geq \frac{\mathbb{E}[T_G(\widetilde{0}, \widetilde{kx})]}{k} - \varepsilon.$$

Combining the previous inequalities it yields that

$$\mu_G(x) \geq \limsup_{n\to\infty} \frac{\mathbb{E}[T_{G_n}(\widetilde{0}, \widetilde{kx})]}{k} - \varepsilon \geq \limsup_{n\to\infty} \mu_{G_n}(x) - \varepsilon.$$

By letting ε go to 0, it follows that

$$\mu_G(x) \geq \limsup_{n\to\infty} \mu_{G_n}(x). \tag{2.3.5}$$

Combining (2.3.4) and (2.3.5) yields the result.

□

Proof of Theorem 2.3.1. We will only prove that for $x \in \mathbb{Z}^d$, $\mu_{G_n}(x)$ converges towards $\mu_G(x)$ and we leave the remaining of the proof as an exercise. Let $x \in \mathbb{Z}^d$. Let U be a uniform random variable on $[0,1]$. We couple t^G and t^{G_n} using the standard coupling using uniform random variable on $[0,1]$. Denote $\underline{G}_n$ and $\overline{G}_n$ the probability distribution of respectively $\min(t^G, t^{G_n})$ and $\max(t^G, t^{G_n})$. In particular, we have

$$\mu_{\underline{G}_n}(x) \leq \mu_G(x) \leq \mu_{\overline{G}_n}(x).$$

The proof will follow from the following inequality

$$\limsup_{n\to\infty} \mu_{\overline{G}_n}(x) \leq \mu_G(x) \leq \liminf_{n\to\infty} \mu_{\underline{G}_n}(x).$$

The left hand side of the inequality follows easily from Lemma 2.3.4. To prove the right hand side, consider $G^K = G\mathbf{1}_{[0,K]} + G([K, +\infty))\delta_K$. Note that $\underline{G}_n$ weakly converges towards G. By Lemma 2.3.3, we have

$$\lim_{n\to\infty} \mu_{\underline{G}_n^K}(x) = \mu_{G^K}(x).$$

It yields that

$$\liminf_{n\to\infty} \mu_{\underline{G}_n}(x) \geq \liminf_{n\to\infty} \mu_{\underline{G}_n^K}(x) = \mu_{G^K}(x).$$

Finally, by Proposition 2.3.2 when taking the limit when K goes to infinity

$$\liminf_{n\to\infty} \mu_{\underline{G}_n}(x) \geq \mu_G(x).$$

This concludes the proof.

□

Exercise Let G be a distribution with a finite moment. Let $(G_n)_{n\geq 1}$ such that G_n weakly converges towards G.

1. We proved that for every $x \in \mathbb{Z}^d$, $\mu_{G_n}(x)$ converges towards $\mu_G(x)$. Deduce that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{S}^{d-1}} |\mu_G(x) - \mu_{G_n}(x)| = 0.$$

2. Set

$$\mathrm{d}_2(x, A) := \inf_{y\in A} \|x - y\|_2.$$

Define the Hausdorff distance between two sets A and B

$$\mathrm{d}_{\mathcal{H}}(A, B) := \max\left(\sup_{x\in A} \mathrm{d}_2(x, B), \sup_{x\in B} \mathrm{d}_2(x, A)\right).$$

Prove that

$$\lim_{n\to\infty} \mathrm{d}_{\mathcal{H}}(\mathcal{B}_{\mu_{G_n}}, \mathcal{B}_{\mu_G}) = 0.$$

Proof of Proposition 2.3.2. Let $(t_e)_{e\in\mathbb{E}^d}$ be an IID family distributed according to G. Write T_G and T_{G_M} the pseudo-metric associated to the family of edge-weight respectively $(t_e)_{e\in\mathbb{E}^d}$ and $(\min(t_e, M))_{e\in\mathbb{E}^d}$.

Let $\varepsilon > 0$ we will choose later. Let $M_0 > 0$ such that $G([M_0, +\infty)) < \varepsilon/2d$. We say that $w \in \mathbb{Z}^d$ is good if all the edges incident to x have a time less than M_0. Otherwise, we say that w is bad. In particular, we have

$$\mathbb{P}(w \text{ is bad}) \leq 2dG([M_0, +\infty)) \leq \varepsilon.$$

We say that $u, v \in \mathbb{Z}^d$ are $*$-connected by a bad path if there exists a sequence $(x_0, \dots, x_m)$ such that $x_0 = u$, $x_m = v$, for all $i \in \{1, \dots, m\}$ $\|x_i - x_{i-1}\|_\infty = 1$ and all x_i are bad. We denote by $\mathcal{C}(x)$ the bad connected component of x, that is all the bad sites that are connected to x by a bad path. We use the convention that $\mathcal{C}(x) = \emptyset$ if x is good. For $\mathcal{C}$ a $*$-connected cluster, we define the exterior boundary as follows

$$\partial^{ext}\mathcal{C} := \{x \in \mathbb{Z}^d : \exists y \in \mathcal{C} \quad \|x - y\|_\infty = 1 \quad \text{and } y \text{ is connected to infinity in } \mathbb{Z}^d \setminus \mathcal{C}\}.$$

By Lemma 2.23 in [17], the set $\partial^{ext}\mathcal{C}$ is $\mathbb{Z}^d$-connected. Moreover, it is easy to check that the sites in $\partial^{ext}\mathcal{C}(x)$ are good.

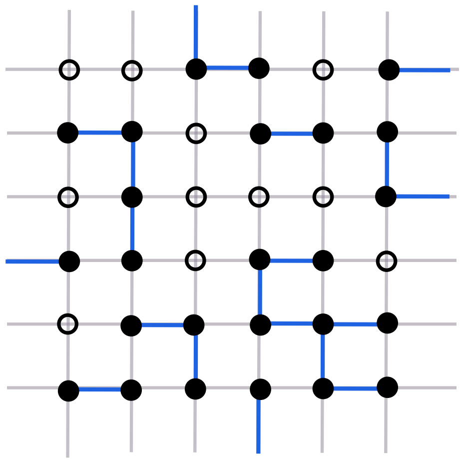


Figure 2.2: Bad and good vertices. The edges in blue have a passage time larger than M_0. The sites in white are good vertices

Let $x \in \mathbb{Z}^d$. Let $n \geq 1$. Let $M \geq M_0$. Let γ be the geodesic between 0 and nx for T_{G^M}. Denote by $\mathcal{E}$ the event that both 0 and nx are in the infinite cluster of good site. In particular, using percolation estimates, we can prove that for ε small enough we have

$$\mathbb{P}(\mathcal{E}) \geq \frac{1}{2}.$$

Write $\gamma = (\gamma_i)_{i=1}^r$. Denote by $V_M(\gamma)$ the set of vertices whose bad cluster intersects an edge of γ with passage time larger than M, that is

$$V_M(\gamma) := \{w \in \mathbb{Z}^d : \exists e \in \gamma : t_e \geq M,\, e \cap \mathcal{C}(w) \neq \emptyset\}.$$

On the event $\mathcal{E}$, we define inductively the following sequence

$$\Psi_{in}(1) = \min\left\{\, j \geq 1 : \gamma_j \in V_M(\gamma) \,\right\} - 1$$

and

$$\Psi_{out}(1) = \max\left\{\, j \geq \Psi_{in}(1) :\, \gamma_j \in \mathcal{C}(\gamma_{\Psi_{in}(1)+1}) \,\right\} + 1\,.$$

Assume $\Psi_{in}(1), \dots, \Psi_{in}(k)$ and $\Psi_{out}(1), \dots, \Psi_{out}(k)$ are constructed then

$$\Psi_{in}(k+1) = \min\left\{\, j \geq \Psi_{out}(k) :\, \gamma_j \in V_M(\gamma) \,\right\} - 1$$

and

$$\Psi_{out}(k+1) = \max\left\{\, j \geq \Psi_{in}(k+1) :\, \gamma_j \in \mathcal{C}(\gamma_{\Psi_{in}(k+1)+1}) \,\right\} + 1\,.$$

This process eventually stops for an integer that we call m. By definition, all the edges between $\gamma_{\Psi_{out}(k)}$ and $\gamma_{\Psi_{in}(k+1)}$ have a time less than M as the path does not intersect $V_M(\gamma)$.

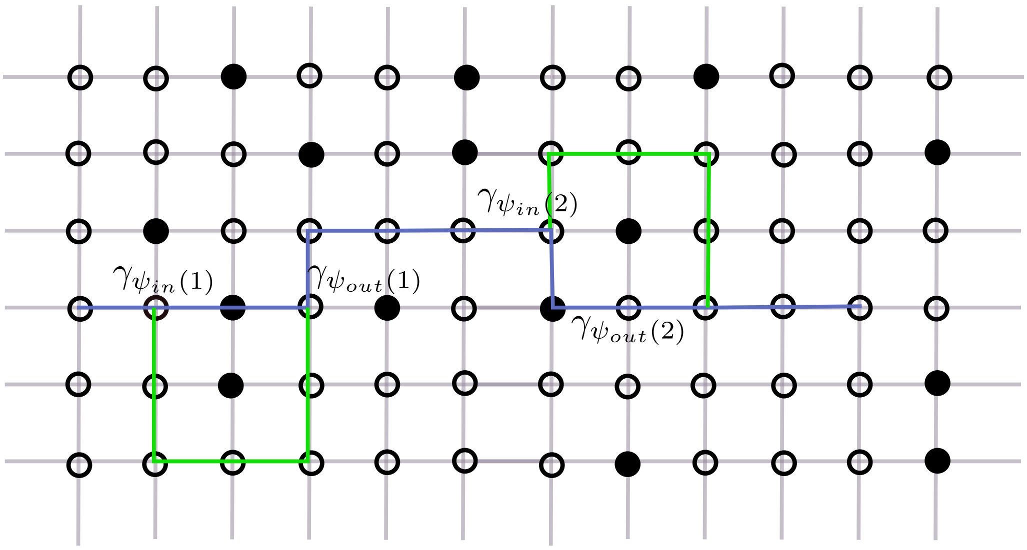


Figure 2.3: The path γ is in blue, in green are the $\overline{\gamma}^{(k)}$

By construction, we have $\gamma_{\psi_{in}(k)}, \gamma_{\psi_{out}(k)} \in \partial^{ext}\mathcal{C}(\gamma_{\psi_{in}(k)+1})$. In particular, these two vertices are connected by $\overline{\gamma}^{(k)}$ a $\mathbb{Z}^d$-path of good sites in $\partial^{ext}\mathcal{C}(\gamma_{\psi_{in}(k)+1})$. Since all the sites in $\partial^{ext}\mathcal{C}(\gamma_{\psi_{in}(k)+1})$ are good, then the edges between these sites have passage time less than M_0 and

$$T_G(\overline{\gamma}^{(k)}) \leq M_0|\partial^{ext}\mathcal{C}(\gamma_{\psi_{in}(k)})| \leq 3^d M_0 |\mathcal{C}(\gamma_{\psi_{in}(k)})|.$$

We write γ' the path where we concatenate the paths $(\gamma_i)_{i=1}^{\Psi_{in}(1)}$, $\overline{\gamma}^{(1)}$, $\dots$, $(\gamma_i)_{i=\Psi_{out}(k)}^{\Psi_{in}(k)}$, $\overline{\gamma}^{(k)}$,$\dots$, $(\gamma_i)_{\Psi_{out}(m)}^{r}$ in this order. In particular, the path γ' is a path from 0 to nx made of edges with passage time less than M. It follows that

$$\begin{aligned} T_G(0,nx) \leq \sum_{e\in\gamma'} t_e = \sum_{e\in\gamma'} \min(t_e, M) &\leq \sum_{e\in\gamma} \min(t_e,M) + \sum_{k=1}^{m} T_G(\overline{\gamma}^{(k)}) \\ &\leq \sum_{e\in\gamma} \min(t_e,M) + 3^d M_0 \sum_{C\in \mathrm{Bad}: C\cap V_M(\gamma)\neq\emptyset} |C| \\ &\leq T_{G^M}(0,nx) + 3^d M_0 \sum_{C\in \mathrm{Bad}: C\cap V_M(\gamma)\neq\emptyset} |C| \end{aligned}$$

where Bad is the set of bad clusters. We claim that for any $\delta > 0$ there exists $M \geq M_0$ such that for all n large enough

$$\mathbb{P}\left(\exists p \text{ starting at } 0 \text{ such that } |p|\geq n, \sum_{C\in \mathrm{Bad}: C\cap V_M(p)\neq\emptyset} |C| \geq \delta|p|\right) \leq \frac{1}{8}. \tag{2.3.6}$$

Let us postpone the proof of this claim and first conclude. By Theorem 2.2.4, there exists $\rho > 1$ such that

$$\begin{aligned} &\mathbb{P}(|\gamma| \geq 2\rho\mathbb{E}[t_e]\|x\|_1 n) \\ &\quad\leq \mathbb{P}(|\gamma| \geq \rho\mathbb{E}[t_e]\|x\|_1 n, T_{G_n}(0,nx) \leq 2\|x\|_1\mathbb{E}[t_e]n) + \mathbb{P}(T_{G_n}(0,nx) \geq 2\mathbb{E}[t_e]\|x\|_1 n) \leq \frac{1}{8}. \end{aligned}$$

Furthermore, by subadditive ergodic theorem, we have that for n large enough

$$\mathbb{P}(T_G(0,nx) \geq n(\mu_G(x)-\delta)) \geq \frac{7}{8}$$

and

$$\mathbb{P}(T_{G^M}(0,nx) \leq n(\mu_{G^M}(x)+\delta)) \geq \frac{7}{8}.$$

On the intersection of the four latter events together with the event $\mathcal{E}$ that occur with positive probability, we have

$$-\delta + \mu_G(x) \leq \mu_{G^M}(x) + \delta + 2 \cdot 3^d M_0 \mathbb{E}[t_e]\rho\delta$$

By letting first M go to infinity and then δ go to 0, we get

$$\mu_G(x) \leq \lim_{M\to\infty} \mu_{G^M}(x).$$

The other inequality being trivial, it completes the proof.

We are now left to prove inequality (2.3.6). The combinatorial term arising when summing over all possible path p is too large to be counterbalanced. To solve this issue, we introduce the skeleton of the path (instead of summing on all possible paths, we only reveal the positions of the path at regularly space points). Let us first control the size of $V_M(\gamma)$. Let $\xi > 0$ depending on δ we will choose later. By similar argument as in the proof of inequality (2.3.3), we have for M large enough depending on ξ

$$\mathbb{P}\left(\exists p \text{ a path starting at 0 s.t. } |p| = n \text{ and } \sum_{e\in p} \mathbf{1}_{t_e \geq M} \geq \xi n\right) \leq \left(\frac{3}{4}\right)^n . \tag{2.3.7}$$

This yields

$$\mathbb{P}\left(\exists p \text{ a path starting at 0 s.t. } |p| = n \text{ and } \#\{C \in \text{Bad} : C \cap V_M(p) \neq \emptyset\} \geq \xi n\right) \leq \left(\frac{3}{4}\right)^n . \tag{2.3.8}$$

Let K be an integer that we will chose later. For $v \in \mathbb{Z}^d$, we denote by $\Lambda_K(v)$ the hypercube of side-length $2K$ centered at v and by $\partial\Lambda_K(v)$ its inner boundary:

$$\Lambda_K(v) := \{w \in \mathbb{Z}^d : \|w - v\|_\infty \leq K\} \quad \text{and} \quad \partial\Lambda_K(v) := \{w \in \mathbb{Z}^d : \|w - v\|_\infty = K\} .$$

Let $p = (p_i)_{0\leq i\leq n}$ be a self-avoiding path. We define $v_0 = p_0$, $\ell_0 = 0$. If $\ell_0, \dots, \ell_k$ and $v_0, \dots, v_k$ are constructed, we define if any

$$\ell_{k+1} := \min\{i \in \{\ell_k + 1, \dots, n\} : p_i \in \partial\Lambda_K(v_k)\} \quad \text{and} \quad v_{k+1} := p_{\ell_{k+1}} .$$

If there is no such index we stop the process. Since $\ell_{k+1} - \ell_k \geq K$, there are at most $1 + n/K$ such ℓ_k. We define $\tau = \lfloor 1 + n/K \rfloor$. The path p is contained in

$$D(v_0, \dots, v_\tau) := \bigcup_{i=0}^{\tau} \Lambda_K(v_i) .$$

If we stop the process for a $k < \tau$, we artificially complete the sequence until attaining τ by setting for $k < j \leq \tau$, $v_j = v_k$.

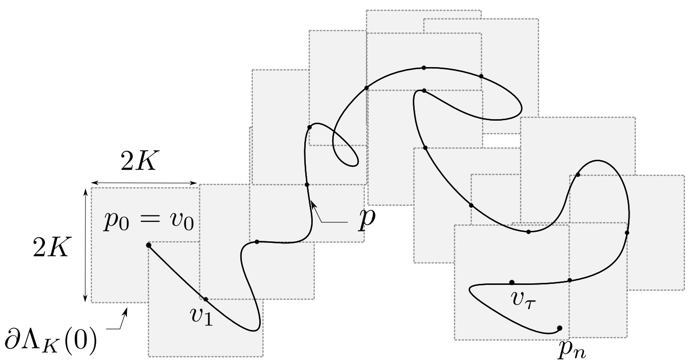


Figure 2.4: The corridor $D(v_0, \dots, v_\tau)$ associated to the path p is represented by the grey section

Define $\mathbf{A}_x^k$ be the set of $*$-connected macroscopic sites of size k containing the site x. We have $|\mathbf{A}_x^k| \leq 7^{dk}$ (see for instance Grimmett [13], p85).

Thus, we obtain

$$
\begin{aligned}
&\mathbb{P}\left(\exists p \text{ starting at 0 such that } |p| = n, \quad \sum_{\substack{C\in\text{Bad}:\\ C\cap V_M(p)\neq\emptyset}} |C| \geq \delta n\right) \\
&\leq \mathbb{P}\left(\bigcup_{v_0,\ldots,v_\tau}\left\{\begin{array}{c}\exists p \text{ starting at 0 such that } |p| = n \\ \sum_{\substack{C\in\text{Bad}:\\ C\cap V_M(p)\neq\emptyset}} |C| \geq \delta n,\, p\subset D(v_0,\ldots,v_\tau)\end{array}\right\}\right) \\
&\leq \sum_{v_0,\ldots,v_\tau} \mathbb{P}\left(\begin{array}{c}\exists p \text{ starting at 0 such that } \#\{C\in \text{Bad} : C\cap V_M(p)\neq\emptyset\}\leq \xi n \text{ and} \\ \sum_{\substack{C\in\text{Bad}:\\ C\cap V_M(p)\neq\emptyset}} |C| \geq \delta n,\, p\subset D(v_0,\ldots,v_\tau)\end{array}\right) + \left(\frac{3}{4}\right)^n \\
&\leq \sum_{v_0,\ldots,v_\tau} \mathbb{P}\left(\exists x_1,\ldots,x_k\in D(v_0,\ldots,v_\tau) : \begin{array}{c} k\leq \xi n,\, \sum_{i=1}^k |\mathcal{C}(x_i)|\geq \delta n, \\ \forall i\neq j \quad \mathcal{C}(x_i)\neq \mathcal{C}(x_j)\end{array}\right) + \left(\frac{3}{4}\right)^n \\
&\leq \sum_{v_0,\ldots,v_\tau}\sum_{k=1}^{\xi n}\sum_{j\geq\delta n}\sum_{\substack{j_1+\cdots+j_k=j\\ j_1\geq 1,\ldots,j_k\geq 1}}\sum_{x_1,\ldots,x_k\in D(v_0,\ldots,v_\tau)}\sum_{\substack{C_i\in\mathbf{A}_{x_i}^{j_i},i\in\{1,\ldots,k\}\\ \forall i\neq j \quad C_i\cap C_j=\emptyset}} \mathbb{P}\left(\forall 1\leq l\leq k \quad \mathcal{C}(x_l) = C_l\right) + \left(\frac{3}{4}\right)^n
\end{aligned}
$$

where the first sum is over the sites $v_0,\ldots,v_\tau$ satisfying $v_0 = p_1$ and for all $0\leq k<\tau$, $v_{k+1}\in \partial\Lambda_K(v_k)\cup\{v_k\}$. Since $\partial\Lambda_K(v)\cup\{v\}$ contains at most $(c_dK)^{d-1}$ sites where $c_d\geq 1$ is a constant depending only on the dimension, the sum over the sites $v_0,\ldots,v_\tau$ contains at most

$$(c_dK)^{(d-1)\tau} \leq (c_dK)^{\frac{2n(d-1)}{K}} := \kappa_2^n$$

terms for n large enough. For any fixed $v_0,\ldots,v_\tau$, $D(v_0,\ldots,v_\tau)$ contains at most

$$(\tau+1)(2K+1)^d \leq (n/K+2)(2K+1)^d \leq 2n(3K)^d := \kappa_3 n$$

sites. The constants κ_2 and κ_3 only depend on K and d. There are at most $\binom{\kappa_3 n}{k}$ ways of choosing the sites $x_1,\ldots,x_k$. Thus, if we fix the sites $x_1,\ldots,x_k$ the number of possible choices of the connected components $\bar{C}_1,\ldots,\bar{C}_k$ such that for all $1\leq l\leq k$, $C(x_l)=\bar{C}_l$ and $\sum_{l=1}^k|\bar{C}_l| = j$ is at most:

$$\sum_{\substack{j_1,\ldots,j_k\geq 1\\ j_1+\cdots+j_k=j}} (7^d)^{j_1}\cdots(7^d)^{j_k} = (7^d)^j \sum_{\substack{j_1,\ldots,j_k\geq 1\\ j_1+\cdots+j_k=j}} 1\,.$$

Consider fixed disjoint $C_1,\ldots,C_k$. Note that there is a short range dependence between the state of the vertices. But if $\|x-y\|_\infty\geq 3$ then the fact that x is bad is independent from the fact that y is bad. Note that the sets $(3\mathbb{Z}^d + x, x\in\{0,1,2\}^d)$ are disjoint. By pigeon-hole principle, there exists $x\in\{0,1,2\}^d$ such that

$$|(3\mathbb{Z}^d+x)\cap(\cup_{1\leq l\leq k}C_l)| \geq \frac{1}{3^d}|\cup_{1\leq l\leq k} C_l|$$

It follows that

$$\mathbb{P}\left(\forall 1\leq l\leq k \quad \mathcal{C}(x_l) = C_l\right) \leq \mathbb{P}(0 \text{ is bad})^{j/3^d} \leq \varepsilon^{j/3d}$$

Notice that

$$\sum_{k=1}^{\xi n}\sum_{x_1,\ldots,x_k\in D(v_0,\ldots,v_\tau)}\sum_{\substack{j_1,\ldots,j_k\geq 1\\ j_1+\cdots+j_k=j}} 1 = \binom{\kappa_3 n}{\xi n}\sum_{\substack{j_1,\ldots,j_{\xi n}\geq 0\\ j_1+\cdots+j_{\xi n}=j}} 1 = \binom{\kappa_3 n}{\xi n}\binom{\xi n+j-1}{j}.$$

Finally combining all the previous inequalities, we have

$$\mathbb{P}\left(\exists p \text{ starting at 0 such that } |p| = n, \quad \sum_{\substack{C\in\text{Bad}:\\ C\cap V_M(p)\neq\emptyset}} |C| \geq \delta n\right) \leq \kappa_2^n\binom{\kappa_3 n}{\xi n}\sum_{j\geq\delta n}\binom{\xi n+j-1}{j}7^{dj}\varepsilon^{j/3d}$$

To bound these terms we will need the following inequality, that we leave as an exercise.

Exercise Let $r \geq 3$, $N \geq 1$ and a real z such that $0 < ez(1 + \frac{r}{N}) < 1$:

$$\sum_{j=N}^{\infty} z^j \binom{r+j-1}{j} \leq \nu \frac{(ez(1+\frac{r}{N}))^N}{1 - ez(1+\frac{r}{N})} \tag{2.3.9}$$

where ν is a universal constant.

Let us choose $\xi = \delta^2$ such that for ε small enough we have

$$e7\varepsilon^{1/3d}\left(1 + \frac{\xi}{\delta}\right) < \varepsilon^{1/6d}$$

Besides, using Stirling approximation, we have

$$\binom{\kappa_3 n}{\xi n} \leq e^{\xi \log(\frac{\kappa_3}{\xi})n + o(n)}$$

Thanks to these estimates we have choosing $K \geq \delta^{-2}$

$$\begin{aligned}\mathbb{P}\left(\exists p \text{ starting at } 0 \text{ such that } |p| = n, \quad \sum_{\substack{C \in \mathrm{Bad}: \\ C \cap V_M(p) \neq \emptyset}} |C| \geq \delta n\right) &\leq (c_d K)^{\frac{2n(d-1)}{K}} e^{\xi(\log \frac{(6K)^d}{\xi})n + o(n)} \nu e^{\frac{\delta}{6d}(\log \varepsilon)} \\ &\leq e^{\frac{\delta}{10d}(\log \varepsilon)n}.\end{aligned}$$

Inequality (2.3.6) follows by summing over all $k \geq n$. $\square$

Chapter 3

Fluctuations

3.1 Application of noise sensitivity to FPP

3.1.1 A useful formula for the variance

The proof of this section is an adaptation of the ones in [22]. We will consider here distributions G that can take two values $0 < a < b$ such that $G(\{a\}) < p_c$. Let $x \in \mathbb{Z}^d$ and $n \geq 1$. In particular, the geodesics between 0 and nx stay almost surely in a bounded region such that $T(0, nx)$ only depends on a finite number of edges $(t_{e_1}, \dots, t_{e_N})$. Denote

$$f(t_{e_1}, \dots, t_{e_N}) := T(0, nx).$$

We are interested in how f and the geodesics are affected when noising the environment. Set $X := (t_{e_1}, \dots, t_{e_N})$. Let X' be an independent vector distributed as X. Consider $(U_1, \dots, U_N)$ an i.i.d. family of uniform random variables on $[0, 1]$. For any $s_1, \dots, s_N \in [0, 1]$, we define

$$\forall 1 \leq i \leq N \quad Y_i(s_1, \dots, s_N) := \begin{cases} X_i & \text{if } U_i \geq s_i \\ X'_i & \text{otherwise.} \end{cases}$$

For short, we write Y_s for $Y(s, \dots, s)$.

Definition 3.1.1 (Pivotal edge)**.** *We say that an edge e is pivotal for the configuration $(t_{e_1}, \dots, t_{e_N})$ if the value of $f(t_{e_1}, \dots, t_{e_N})$ depends on the value of t_e. More precisely, define for $i \in \{1, \dots, N\}$ $\sigma_i^a : \{a, b\}^N \to \{a, b\}^N$ the function that changes the value of the i-th coordinate to be a. Define similarly σ_i^b. We say that the edge e_i is pivotal for $f(X)$ if $f \circ \sigma_i^a(X) \neq f \circ \sigma_i^b(X)$.*

Let $\mathcal{P}_s$ be the set of edges that are pivotal for the function $f(Y(s, \dots, s))$. The following proposition establishes a relationship between the variance of $f(X)$ and the extent to which the pivotal edges in the original environment overlap with those in the perturbed (noised) environment.

Proposition 3.1.2. *For any $n \geq 1$ we have*

$$\mathrm{Var}(T(0, nx)) = \mathrm{Var}(t_e) \int_0^1 \mathbb{E}[|\mathcal{P}_0 \cap \mathcal{P}_s|] ds \,.$$

Moreover, the function $s \mapsto \mathbb{E}[|\mathcal{P}_0 \cap \mathcal{P}_s|]$ is non-increasing.

Remark 3.1.3. *This formula implies that if the variance is negligible compared to n, then a small noise modifies totally the geodesic in the sense that $\mathbb{E}[|\mathcal{P}_0 \cap \mathcal{P}_s|]$ is also negligible with respect to n.*

Proof. We have

$$\mathrm{Var}(f(X)) = \mathbb{E}[f(X)f(Y(0))] - \mathbb{E}[f(X)f(Y(1))] = \int_0^1 -\frac{d}{ds}\mathbb{E}[f(X)f(Y(s))] ds.$$

By the chain rule, we have

$$\frac{d}{ds}\mathbb{E}[f(X)f(Y(s))] = \sum_{i=1}^N \frac{d}{ds_i}\mathbb{E}[f(X)f(Y(s))].$$

Denote $\nabla_i f(X) := \mathbf{1}_{f\circ\sigma_i^b(X)-f\circ\sigma_i^a(X)>0} = \mathbf{1}_{e_i \text{ is pivotal for } f(X)}$. In particular, the value $\nabla_i f(X)$ does not depend on X_i. Note that

$$f(X) = (X_i - a)\nabla_i f(X) + f \circ \sigma_i^a(X).$$

Define

$$\forall s \in [0,1] \quad \varphi(s) = \mathbb{E}[f(X)f(Y(s, s_2, \dots, s_N))].$$

Let $\delta > 0$ such that $t_1 + \delta \in [0,1]$. Write $Y = Y(t_1, t_2, \dots, t_N)$ and $Y' = Y(t_1 + \delta, t_2, \dots, t_N)$. Then,

$$\begin{aligned}\varphi(t_1+\delta) - \varphi(t) &= \mathbb{E}[f(X)(f(Y') - f(Y))]\\ &= \mathbb{E}[((X_1 - a)\nabla_1 f(X) + f\circ\sigma_1^a(X))(Y_1' - Y_1)\nabla_1 f(Y)]\\ &= \mathbb{E}[(X_1-a)(Y_1'-Y_1)]\mathbb{E}[\nabla_1 f(X)\nabla_1 f(Y)] + \mathbb{E}[Y_1'-Y_1]\mathbb{E}[f\circ\sigma_1^a(X)\nabla_1 f(X)]\end{aligned}$$

Note that $\mathbb{E}[Y_1' - Y_1] = 0$ and $\mathbb{E}[X_1(Y_1' - Y_1)] = \mathbb{E}[X_1(Y_1'-Y_1)\mathbf{1}_{U_1\in[t_1,t_1+\delta]}] = \mathbb{E}[X_1X_1' - X_1^2]\delta$. It follows that

$$\varphi(t_1+\delta) - \varphi(t) = -\delta \operatorname{Var}(X_1)\mathbb{E}[\nabla_1 f(X)\nabla_1 f(Y)]$$

and $\varphi'(t_1) = -\operatorname{Var}(X_1)\mathbb{E}[\nabla_1 f(X)\nabla_1 f(Y)]$. Finally, we get

$$\begin{aligned}\frac{d}{ds}\mathbb{E}[f(X)f(Y(s))] &= -\sum_{i=1}^N \operatorname{Var}(X_1)\mathbb{E}[\nabla_1 f(X)\nabla_1 f(Y(s))]\\ &= -\operatorname{Var}(X_1)\sum_{i=1}^N \mathbb{E}[\mathbf{1}_{e_i\in\mathcal{P}_0\cap\mathcal{P}_s}] = -\operatorname{Var}(X_1)\mathbb{E}[|\mathcal{P}_0\cap\mathcal{P}_s|].\end{aligned} \tag{3.1.1}$$

We are now left to prove that the quantity $\mathbb{E}[\nabla_1 f(X)\nabla_1 f(Y(s))]$ is non-increasing in s. To do so, we consider the more general setting of a function $g : \{a,b\}^N \to \{0,1\}$ and we will prove that the following function is non decreasing

$$\forall s \in [0,1] \quad \psi(s) = \mathbb{E}[g(X)g(Y(s, s_2, \dots, s_N))].$$

Note that $g(X) = (2\mathbf{1}_{X_1=b} - 1)\nabla_1 g(X) + \mathbb{E}_1 g(X)$ where $\nabla_1 g = \frac{1}{2}(g\circ\sigma_1^b - g\circ\sigma_1^a)$ and $\mathbb{E}_1 g = \frac{1}{2}(g\circ\sigma_1^b + g\circ\sigma_1^a)$. By a same reasoning as in the case of f, we can prove that

$$\psi(t_1+\delta) - \psi(t) = -4\delta G(\{b\})G(\{a\})\mathbb{E}[\nabla_1 g(X)\nabla_1 g(Y)].$$

It is now left to prove that $\mathbb{E}[\nabla_1 g(X)\nabla_1 g(Y)] \geq 0$. This follows from the fact that the dynamic is reversible. Let $(s_1, \dots, s_N) \in [0,1]^N$ and $(u_1, \dots, u_N)$ such that $(1-u_i)^2 = 1 - s_i$. Now let $\overline{X}$ be distributed as X and $\overline{W}$ be the vector where the i-th coordinate of $\overline{X}$ is resampled with probability u_i, and $\overline{Y}$ is the vector where the i-th coordinate of $\overline{W}$ is resampled with probability u_i. It follows using that $\overline{X}$ and $\overline{Y}$ are independent conditionally on $\overline{W}$ and $(\overline{X}, \overline{Y})$ has the same law as (X,Y)

$$\mathbb{E}[\nabla_1 g(X)\nabla_1 g(Y)] = \mathbb{E}[\nabla_1 g(\overline{X})\nabla_1 g(\overline{Y})] = \mathbb{E}[\mathbb{E}[\nabla_1 g(\overline{X})|\overline{W}]^2] \geq 0.$$

This concludes the proof.

□

3.1.2 Lower bound on the vertical fluctuations

Denote $\operatorname{cyl}(0, n\mathbf{e}_1, h)$ the following cylinder:

$$\operatorname{cyl}(0, n\mathbf{e}_1, h) := \{x \in \mathbb{R}^d : \exists t \in [0,n] \quad \|x - t\mathbf{e}_1\|_\infty \leq h\}.$$

Let $\xi > 0$ be such that

$$\exists n_0 \geq 1 \quad \forall n \geq n_0 \quad \mathbb{P}(\gamma(0, n\mathbf{e}_1) \subset \operatorname{cyl}(0, n\mathbf{e}_1, n^\xi)) = 1 \tag{3.1.2}$$

where $\gamma(0, n\mathbf{e}_1)$ denotes a geodesic between 0 and $n\mathbf{e}_1$ chosen according to a deterministic rule in case of ties. The following proposition may be understood as a lower bound for vertical fluctuations of the geodesic.

Proposition 3.1.4. *Let $\xi > 0$ such that (3.1.2) holds. Then, we have*

$$\xi \geq \frac{1}{d+1}.$$

Proof. Let n be large enough, by definition of ξ the random variables $(T(0,n\mathbf{e_1})$ and $T(2n^\xi\mathbf{e}_2, n\mathbf{e_1}+2n^\xi\mathbf{e}_2)$ are independent. Besides, it is easy to check using triangular inequality that

$$|T(0,n\mathbf{e_1})-T(2n^\xi\mathbf{e}_2, n\mathbf{e_1}+2n^\xi\mathbf{e}_2)|\leq 4n^\xi b.$$

Hence,

$$\mathrm{Var}(T(0,n\mathbf{e_1}))=\mathbb{E}[(T(0,n\mathbf{e_1})-T(2n^\xi\mathbf{e}_2, n\mathbf{e_1}+2n^\xi\mathbf{e}_2))^2]\leq \mathbb{E}[(4n^\xi b)^2]\leq 16b^2n^{2\xi}.$$

Besides, using Proposition 3.1.2, we have

$$\begin{aligned}\mathrm{Var}(T(0,n\mathbf{e}_1))\geq \mathrm{Var}(t_e)\mathbb{E}[|\mathcal{P}_0\cap\mathcal{P}_1|]&=\mathrm{Var}(t_e)\sum_{e\in \mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)}\mathbb{P}(e\in\mathcal{P}_0,e\in\mathcal{P}_1)\\&=\mathrm{Var}(t_e)\sum_{e\in \mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)}\mathbb{P}(e\in\mathcal{P}_0)^2.\end{aligned}$$

Using Cauchy-Schwarz, it yields

$$\left(\sum_{e\in \mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)}\mathbb{P}(e\in\mathcal{P}_0)\right)^2\leq 2d|\,\mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)|\sum_{e\in \mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)}\mathbb{P}(e\in\mathcal{P}_0)^2.$$

Note that if $e\in\gamma(0,n\mathbf{e}_1)$ and $t_e=b$, then $e\in\mathcal{P}_0$.

Exercise Let G such that $G(\{a\})<p_c$. Prove that there exists $\rho>0$ such that for n large enough

$$\mathbb{E}[\#\{e\in\gamma(0,n\mathbf{e}_1):t_e=b\}]\geq\rho n.$$

Hint: Fist prove that

$$\mathbb{P}(\#\{e\in\gamma(0,n\mathbf{e}_1):t_e=b\}\leq\rho n)\leq\frac{1}{2}$$

by using a renormalization similar to the proof of Theorem 2.2.4.

It follows that

$$\sum_{e\in \mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)}\mathbb{P}(e\in\mathcal{P}_0)=\mathbb{E}[\#\{e\in \mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)\cap\mathcal{P}_0\}]\geq\mathbb{E}[\#\{e\in\gamma(0,n\mathbf{e}_1):t_e=b\}]\geq\rho n.$$

Combining all the previous inequalities, we get

$$16b^2n^{2\xi}\geq \mathrm{Var}(T(0,n\mathbf{e}_1))\geq\mathrm{Var}(t_e)\rho^2\frac{n^2}{2d|\,\mathrm{cyl}(0,n\mathbf{e}_1,n^\xi)|}\geq\frac{\mathrm{Var}(t_e)\rho^2}{4d}n^{1-\xi(d-1)}.$$

Finally, since the previous inequality holds for all large enough n, it follows that

$$\xi\geq\frac{1}{d+1}.$$

□

3.2 Sublinear upper bound on the variance

Let $b>a>0$ and $G=\frac{1}{2}\delta_a+\frac{1}{2}\delta_b$. The following Theorem due to Benjamini–Kalai–Schramm is currently the best upperbound on the variance. This result was extended to more general distributions by Benaim and Rossignol [5] and Damron, Hanson and Sosoe [12].

Theorem 3.2.1 (BKS[6])**.** *There exists a constant $C>0$ depending on G and d such that for every $v\in\mathbb{Z}^d$ such that $\|v\|_1\geq2$*

$$\mathrm{Var}(T(0,v))\leq C\frac{\|v\|_1}{\log\|v\|_1}.$$

The key input to prove this theorem is the following concentration inequality. Let J be a finite set of indices. For $\omega \in \{a,b\}^J$ and $j \in J$ denote $\sigma_j\omega$ the function that switches the value in the j-th coordinate. For $f : \{a,b\}^J \to \mathbb{R}$, denote

$$\partial_j f := \frac{f - f\circ\sigma_j}{2}.$$

Consider μ the product measure on $\{a,b\}^J$ which gives a with probability $1/2$ and b with probability $1/2$. We denote $\|f\|_2^2 = \int f^2 d\mu$ and $\|f\|_1 = \int |f| d\mu$

Theorem 3.2.2 (Talagrand's inequality [20] Theorem 1.5)**.** *Let $f : \{a,b\}^J \to \mathbb{R}$. We have*

$$\operatorname{Var}(f) \le C \sum_{j\in J} \frac{\|\partial_j f\|_2^2}{1 + \log(\|\partial_j f\|_2 / \|\partial_j f\|_1)} \tag{3.2.1}$$

where C is a universal constant.

We will need the following control on the expected intersection of a geodesic with a box. We leave this lemma as an exercise as an application of Theorem 2.2.4.

Lemma 3.2.3. *Let G be a distribution such that $G(\{0\}) < p_c$. For any geodesic γ between x and y, there exists $\kappa > 0$ such that for any $n \ge 1$*

$$\mathbb{E}[|\gamma \cap \Lambda_m|] \le \kappa m.$$

Proof of Theorem 3.2.1. Let $v \in \mathbb{Z}^d$ and $n = \|v\|_1$. Let $m = \lfloor n^{1/3} \rfloor$. Let $(X_i^1)_{1\le i\le m}, \dots, (X_i^d)_{1\le i\le m}$ be IID family of Rademacher random variable (that is X_1^1 takes the value 1 with probability $1/2$ and -1 with probability $1/2$). Denote by $z(X)$ the following random point of $\mathbb{Z}^d$

$$z = z(X) := \left(\sum_{i=1}^m X_i^1, \dots, \sum_{i=1}^m X_i^d\right).$$

Let E be the deterministic set of edges such that almost surely for any X the geodesic between $z(X)$ and $v + z(X)$ only uses edges in E. Let $(t_e)_{e\in E}$ be an IID family distributed according to G. We will need to define $\widetilde{f}$ the averaged version of $f = T(0,v)$:

$$\widetilde{f} := T(z(X), z(X) + v).$$

In particular, we have almost surely as $\|z(X)\|_1 \le dm$

$$|\widetilde{f} - f| \le 2dmb.$$

Moreover, we have $\mathbb{E}[\widetilde{f}] = \mathbb{E}[f]$ In particular, it follows that

$$\operatorname{Var}(f) = \mathbb{E}[(f - \mathbb{E}[f])^2] = \mathbb{E}[(f - \widetilde{f} + \widetilde{f}\mathbb{E}[f])^2] \le 2\mathbb{E}[(f - \widetilde{f})^2] + 2\operatorname{Var}(\widetilde{f}) \le 2\operatorname{Var}(\widetilde{f}) + 8d^2b^2n^{2/3}.$$

To conclude, it is therefore sufficient to prove that

$$\operatorname{Var}(\widetilde{f}) \le C \frac{\|v\|_1}{\log \|v\|_1}.$$

The reason for introducing this averaged version is that it will be easier to control the influence of edges for $\widetilde{f}$ than for f. The reason is that we don't have estimates of the probability of an edge being on the geodesic.

Let $I \subset \mathbb{N}$ be an indexing of the coordinates of X. For $i \in I$, it is easy to check that $\|\sigma_i X - X\|_1 \le 2$ and $|\sigma_i \widetilde{f} - \widetilde{f}| \le 4b$ It follows that

$$\|\partial_i \widetilde{f}\|_2 \le 2b.$$

Let $e \in E$. Using that $\partial_e \widetilde{f} \neq 0$ does not depend on the value of e, we have

$$\mathbb{P}(\partial_e \widetilde{f} \neq 0) = 2\mathbb{P}(\partial_e \widetilde{f} \neq 0, t_e = a) \le 2\mathbb{P}(e \in \gamma(z, z+v)).$$

Note that if $\partial_e \widetilde{f} \neq 0$ and $t_e = a$, then necessarily e has to belong to the geodesic. For $e \in E$, thanks to the previous inequality, we have

$$\|\partial_e \widetilde{f}\|_2^2 \le \frac{(b-a)^2}{4} \mathbb{P}(\partial_e \widetilde{f} \neq 0) \le \frac{(b-a)^2}{2} \mathbb{P}(e \in \gamma(z, z+v)).$$

Besides, we have by Cauchy–Schwarz inequality

$$\|\partial_e \widetilde{f}\|_1 = \mathbb{E}\left[\left|\partial_e \widetilde{f}\right|\right] \leq \sqrt{\mathbb{P}(\partial_e \widetilde{f} \neq 0)}\, \|\partial_e \widetilde{f}\|_2 \leq \sqrt{2\mathbb{P}(e \in \gamma(z, z+v))}\, \|\partial_e \widetilde{f}\|_2.$$

Let us assume that for all $e \in E$, $\mathbb{P}(e \in \gamma(z(X), z(X)+v)) \leq m^{-1/8}$. By applying Theorem 3.2.2, we have

$$\begin{aligned}\mathrm{Var}(\widetilde{f}) &\leq C\left(\sum_{i \in I} 2b + \sum_{e \in E} \frac{12(b-a)^2}{\log n}\mathbb{P}(e \in \gamma(z, z+v))\right)\\ &\leq 2Cdn^{1/3}b + 12C\frac{(b-a)^2}{\log n}\mathbb{E}[|\gamma(z, z+v)|] \leq C'\frac{n}{\log n}\end{aligned} \tag{3.2.2}$$

Let us now control $\mathbb{P}(e \in \gamma(z(X), z(X)+v))$. Let us assume that

$$\exists e \in E \quad \mathbb{P}(e \in \gamma(z(X), z(X)+v)) \geq m^{-1/8}. \tag{3.2.3}$$

Let $e \in E$ be such an edge. We aim at deriving a contradiction. Let $i \in \{1, \dots, d\}$, let us prove using a coupling that there exists a universal constant $C > 0$ such that

$$|\mathbb{P}(e \in \gamma(z(X), z(X)+v)) - \mathbb{P}(e + 2\mathbf{e}_i \in \gamma(z(X), z(X)+v))| \leq \frac{C}{\sqrt{m}}.$$

Consider the case where $i = 1$. The other cases are proved similarly. Let $(t_e)_{e \in E}$. We define t'_e as follows

$$t'_e := \begin{cases} t_{e+2\mathbf{e}_1} & \text{if } e + 2\mathbf{e}_1 \in E \\ t''_e & \text{otherwise}\end{cases}$$

where $(t''_e)_{e \in E}$ is independent from (t_e). Let $(Z_i)_{1 \leq i \leq m}$, $(Z'_i)_{1 \leq i \leq m}$, $(X^2_i)_{1 \leq i \leq m}$, $(X^d_i)_{1 \leq i \leq m}$ be independent families of Rademacher random variables. Set

$$S_k := \sum_{i=1}^k Z'_i \quad \text{and} \quad S'_k := \sum_{i=1}^k Z'_i.$$

Let

$$\tau := \inf\{k \in \{1, \dots, m\} : S_k \geq S'_k + 2\}$$

where we use the convention $\inf \emptyset = +\infty$. Finally, we set

$$z_1 := \sum_{k=1}^M Z_k \qquad \text{and} \qquad z'_1 := \sum_{k=1}^{\min(\tau,m)} Z'_k + \sum_{k=\min(\tau,m)+1}^m Z_k.$$

Denote by γ' the geodesic between z' and $z' + v$ corresponding to the family $(t'_e)_{e \in E}$. It is easy to check that it has the same law as γ the geodesic from z to $z + v$ with passage time $(t_e)_{e \in E}$. Moreover, there exists a universal $C > 0$ s.t.

$$\mathbb{P}(z_1 - z'_1 \neq 2) = \mathbb{P}(\tau = \infty) = \mathbb{P}(\forall k \in \{1, \dots, m\} \quad S'_k - S_k \geq 0) \leq \frac{C}{\sqrt{m}}.$$

On the event $\{z_1 = z'_1 + 2\}$, we have

$$\gamma = \gamma' + 2\mathbf{e}_1.$$

It yields

$$|\mathbb{P}(e \in \gamma(z', z'+v)) - \mathbb{P}(e + 2\mathbf{e}_1 \in \gamma(z, z+v))| \leq \mathbb{P}(z_1 - z'_1 \neq 2) \leq \frac{C}{\sqrt{m}}.$$

It follows that for all $w \in \mathbb{Z}^d$ such that $\|w\|_\infty \leq m^{1/4}$

$$|\mathbb{P}(e \in \gamma(z, z+v)) - \mathbb{P}(e + 2w \in \gamma(z, z+v))| \leq C\frac{\|w\|_1}{\sqrt{m}} \leq Cd\frac{1}{m^{1/4}}$$

and

$$\mathbb{P}(e + 2w \in \gamma(z, z+v)) \geq \mathbb{P}(e \in \gamma(z, z+v)) - Cdm^{-1/4} \geq m^{-1/8}/2.$$

This yields that

$$\mathbb{E}[|\gamma(z, z+v) \cap (e + \Lambda_{2m^{1/4}})|] \geq \sum_{w : \|w\|_\infty \leq m^{1/4}} \mathbb{P}(e + 2w \in \gamma(z, z+v)) \geq \frac{1}{2}m^{d/4 - 1/8} \geq \frac{1}{2}m^{3/8}.$$

We conclude that it contradicts (3.2.3) for n large enough thanks to Lemma 3.2.3. □

3.3 Scaling relation

The aim of this section is to establish the link between spatial and time fluctuations. The limit shape encodes information related to the fluctuations. In particular, in order to understand the fluctuations, one needs to understand what is the energetic cost for the geodesics to deviate from the straight line. This cost crucially relies on the curvature exponent.

Definition 3.3.1 (Curvature exponent)**.** *Let us assume that $\partial\mathcal{B}_\mu$ is differentiable. The curvature exponent $\kappa(u)$ in the direction u is a real number such that there exist positive constants c, C, ϵ such that for all z such that $\|z\| \leq \epsilon$ and $u+z$ belongs to the tangent plane to the surface at u (see Figure 3.1),*

$$c\|z\|^{\kappa(u)} \leq \mu(u+z) - \mu(u) \leq C\|z\|^{\kappa(u)}.$$

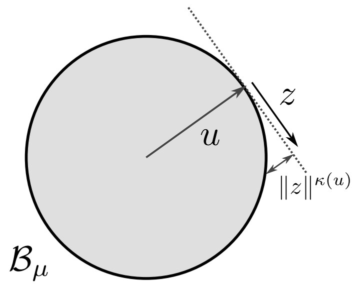


Figure 3.1: Definition of the exponent $\kappa(u)$

For a direction u, we define the transversal fluctuation exponent

$$\xi_u := \inf\left\{\xi \geq 0 : \lim_{n\to\infty} e^{\log^2 n}\mathbb{P}(\gamma(0,nu) \not\subset \mathrm{cyl}(0,nu,n^\xi)) = 0\right\}$$

where

$$\mathrm{cyl}(0,nu,h) := \{x \in \mathbb{R}^d : \exists t \in [0,n] \quad \|x - tu\|_\infty \leq h\}.$$

We will prove in the next section using concentration estimates that $\xi_u \leq 1$. Define the exponent related to time fluctuations

$$\chi := \inf\left\{\chi' \geq 0 : \quad \exists \alpha > 0 \quad \sup_{v\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}\left[\exp\left(\alpha\frac{|T(0,v)-\mu(v)|}{\|v\|^{\chi'}}\right)\right] < \infty\right\}. \tag{3.3.1}$$

Remark 3.3.2. *This exponent is not directly related to time fluctuations $T(0,v) - \mathbb{E}[T(0,v)]$. Though, thanks to a result of Alexander [1], one can prove that the definition is equivalent if we replace $\mu(v)$ by $\mathbb{E}[T(0,v)]$.*

Theorem 3.3.3. *Let us assume the direction u has curvature exponent $\kappa(u) = 2$. Then, we have*

$$\xi_u \leq \frac{\chi+1}{2}. \tag{3.3.2}$$

Proof. For simplicity, we will only consider $u = \mathbf{e}_1$. For short, write $\xi_{\mathbf{e}_1} = \xi$. Let $\varepsilon > 0$. Let $\xi_0 = \frac{\chi+1}{2} + \varepsilon$. Let us prove that $\xi_0 \geq \xi$. Denote by $V_n := \mathrm{cyl}(0, n\mathbf{e}_1, n^{\xi_0})$. Note that if the geodesic exits V_n it has to pass through the vertex boundary ∂V_n of V_n. It follows that

$$\mathbb{P}(\gamma(0,n\mathbf{e}_1) \not\subset V_n) \leq \mathbb{E}[|\gamma(0,n\mathbf{e}_1) \cap \partial V_n|] = \sum_{w\in\partial V_n} \mathbb{P}(w \in \gamma(0,n\mathbf{e}_1)).$$

Let $w \in \partial V_n$. Let us lower bound the following quantity which corresponds to the additional energetic cost of passing through w compared with energetic cost of the straight path

$$\mu(w) + \mu(w - n\mathbf{e}_1) - \mu(n\mathbf{e}_1).$$

Let w_0 be the projection of w on the line passing through 0 of direction $\mathbf{e}_1$. Write $w_0 = t_0\mathbf{e}_1$. Thanks to the symmetry of the model, we can assume without loss of generality that $t_0 \geq n/2$. By definition of the curvature exponent, we have for $t_0 \leq n + n^{\xi_0}$

$$\frac{c}{t_0^2}\|w - w_0\|^2 \leq \frac{1}{t_0}(\mu(w) - \mu(w_0)) \leq \frac{C}{t_0^2}\|w - w_0\|^2.$$

For such t_0, we have $\|w - w_0\| = \Omega(n^{\xi_0})$. This yields

$$n^{2\xi_0 - 1} \leq \mu(w) - \mu(w_0). \tag{3.3.3}$$

If $t_0 \geq n + n^{\xi_0}$, then we have $\mu(w) \geq \mu(n\mathbf{e}_1) + n^{\xi_0}$. Besides, using the convexity of μ, we have

$$\mu(w - n\mathbf{e}_1) \geq \mu(w_0 - n\mathbf{e}_1). \tag{3.3.4}$$

Note that for $\xi \leq 1$, we have $n^{\xi_0} \geq n^{2\xi_0 - 1}$. We get

$$\mu(w) + \mu(w - n\mathbf{e}_1) - \mu(n\mathbf{e}_1) = \mu(w) + \mu(w - n\mathbf{e}_1) - \mu(w_0) - \mu(w_0 - n\mathbf{e}_1) \geq cn^{2\xi_0 - 1}.$$

Note that $2\xi_0 - 1 = \chi + 2\varepsilon$. In particular, by definition of χ, there exists $\alpha > 0$ such that

$$M := \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}\left[\exp\left(\alpha \frac{|T(0,v) - \mu(v)|}{\|v\|^{\chi+\varepsilon}}\right)\right] < \infty.$$

It yields for $t \geq 0$

$$\mathbb{P}(|T(0,v) - \mu(v)| \geq t\|v\|^{\chi+2\varepsilon}) \leq \mathbb{E}\left[\exp\left(\alpha \frac{|T(0,v) - \mu(v)|}{\|v\|^{\chi+\varepsilon}}\right)\right] e^{-\alpha t\|v\|^{\varepsilon}} \leq M e^{-\alpha t \|v\|^{\varepsilon}}. \tag{3.3.5}$$

Finally, we have

$$\begin{aligned}
\mathbb{P}(w \in \gamma(0, n\mathbf{e}_1)) &= \mathbb{P}(T(0,w) + T(w, n\mathbf{e}_1) \leq T(0, n\mathbf{e}_1)) \\
&\leq \mathbb{P}(|T(0,w) - \mu(w)| \geq \frac{c}{3} n^{2\xi_0 - 1}) + \mathbb{P}(|T(0, n\mathbf{e}_1) - \mu(n\mathbf{e}_1)| \geq \frac{c}{3} n^{2\xi_0 - 1}) \\
&\quad + \mathbb{P}(|T(w, n\mathbf{e}_1) - \mu(w - n\mathbf{e}_1)| \geq \frac{c}{3} n^{2\xi_0 - 1}) \leq 3Me^{-\alpha c n^{\varepsilon}}.
\end{aligned}$$

Finally, it yields that

$$\lim_{n \to \infty} e^{\log^2 n} \mathbb{P}(\gamma(0, n\mathbf{e}_1) \not\subset V_n) = 0.$$

Hence, by definition of ξ, we have $\frac{\chi+1}{2} + \varepsilon = \xi_0 \geq \xi$. Since, the previous inequality holds for every ε, this yields the desired inequality. □

Let us define a new exponent for time fluctuations.

$$\underline{\chi} := \sup\left\{\chi' \geq 0 : \quad \inf_{v \in \mathbb{Z}^d \setminus \{0\}} \frac{\mathrm{Var}(T(0,v))}{\|v\|^{2\chi'}} > 0\right\}. \tag{3.3.6}$$

The following result was first proved by Chatterjee in [9] and later simplified by Auffinger–Damron in [2].

Theorem 3.3.4 (Hard-direction of the scaling relation)**.** *Let us assume the direction* $\mathbf{e}_1$ *has curvature exponent* $\kappa(\mathbf{e}_1) = 2$ *and* $\underline{\chi} = \chi$. *Then, we have*

$$\chi \leq 2\xi - 1. \tag{3.3.7}$$

Proof. Let $\varepsilon > 0$. Let $v_n = 4n^{\xi+\varepsilon}\mathbf{e}_2$. Denote by $\mathcal{F}_n$ the following good event

$$\mathcal{F}_n := \{\gamma(0, n\mathbf{e}_1) \subset \mathrm{cyl}(0, n\mathbf{e}_1, n^{\xi+\varepsilon})\} \cap \{\gamma(v_n, n\mathbf{e}_1 + v_n) \subset v_n + \mathrm{cyl}(0, n\mathbf{e}_1, n^{\xi+\varepsilon})\}.$$

In particular, we have by definition of ξ, that

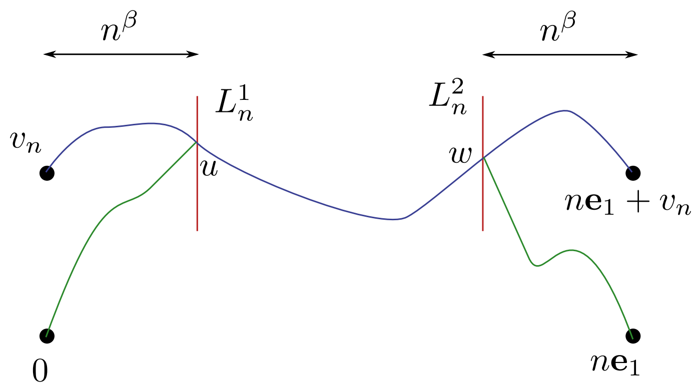
$$\lim_{n \to \infty} e^{\log^2 n} \mathbb{P}(\mathcal{F}_n^c) = 0.$$

Recall that $\mathrm{Var}(X) = \mathbb{E}[(X - X')^2]$ for X, X' independent and identically distributed. Let T, T' be independent random variables distributed as $T(0, n\mathbf{e}_1)$. It follows by Cauchy–Schwarz

$$\begin{aligned}
\mathrm{Var}(T(0, n\mathbf{e}_1)) &= \mathbb{E}[(T - T')^2 \mathbf{1}_{\mathcal{F}_n}] + \mathbb{E}[(T - T')^2 \mathbf{1}_{\mathcal{F}_n^c}] \\
&\leq \mathbb{E}[(T(0, n\mathbf{e}_1) - T(v_n, n\mathbf{e}_1 + v_n))^2 \mathbf{1}_{\mathcal{F}_n}] + Cn^2 \sqrt{\mathbb{P}(\mathcal{F}_n^c)}.
\end{aligned}$$

Using the definition of $\underline{\chi}$, it follows for n large enough

$$cn^{2\underline{\chi} - \varepsilon} \leq \frac{1}{2} \mathrm{Var}(T(0, n\mathbf{e}_1)) \leq \mathbb{E}[(T(0, n\mathbf{e}_1) - T(v_n, n\mathbf{e}_1 + v_n))^2 \mathbf{1}_{\mathcal{F}_n}].$$



Let $\beta < 1$. Let $L_n^1 := \{n^\beta\} \times [3n^\xi, 5n^\xi]$ and $L_n^2 := \{n - n^\beta\} \times [3n^\xi, 5n^\xi]$. On the event $\mathcal{F}_n$, let $u \in L_n^1$ and $w \in L_n^2$ such that the geodesic $\gamma(v_n, n\mathbf{e}_1 + v_n)$ goes through u and w. It follows that

$$\begin{aligned} T(0, n\mathbf{e}_1) - T(v_n, n\mathbf{e}_1 + v_n) &\le T(0,u) + T(u,w) + T(w, n\mathbf{e}_1) - T(v_n, n\mathbf{e}_1 + v_n) \\ &\le T(0,u) - T(v_n, u) + T(w, n\mathbf{e}_1) - T(w, n\mathbf{e}_1 + v_n) \\ &\le \max_{x \in L_n^1} |T(0,x) - T(v_n, x)| + \max_{x \in L_n^2} |T(x, n\mathbf{e}_1) - T(x, n\mathbf{e}_1 + v_n)|. \end{aligned}$$

Let $x \in L_n^1$. We have

$$|T(0,x) - T(v_n,x)| \le |T(0,x) - \mu(x)| + |\mu(x) - \mu(v_n - x)| + |T(v_n, x) - \mu(x_n - x)|. \tag{3.3.8}$$

Let us now upperbound

$$|\mu(x) - \mu(v_n - x)| \le |\mu(x) - \mu(x - n^\beta \mathbf{e}_1)| + |\mu(x - n^\beta \mathbf{e}_1) - \mu(x - n^\beta \mathbf{e}_1)| \le C n^{2\xi - \beta}. \tag{3.3.9}$$

Besides, we have by similar computations as in (3.3.5)

$$\mathbb{P}(|T(0,x) - \mu(x)| \ge n^{\beta(\chi + 2\varepsilon)}) \le C e^{-\alpha n^{\beta\varepsilon}}. \tag{3.3.10}$$

In particular, we have

$$\mathbb{E}\left[\max_{x \in L_n^1} |T(0,x) - \mu(x)|^2\right] \le C n^{2\beta(\chi + 2\varepsilon)}$$

and

$$\begin{aligned} \mathbb{E}\left[\max_{x \in L_n^1} |T(0,x) - T(v_n, x)|^2\right] &\le 3\mathbb{E}\left[\max_{x \in L_n^1} |T(0,x) - \mu(x)|^2\right] + 3\mathbb{E}\left[\max_{x \in L_n^1} |T(v_n,x) - \mu(x_n - x)|^2\right] \\ &\quad + 3 \max_{x \in L_n^1} |\mu(x) - \mu(v_n - x)|^2 \\ &\le C(n^{2(2\xi - \beta)} + n^{2\beta(\chi + 2\varepsilon)}). \end{aligned} \tag{3.3.11}$$

We can get a similar upper-bound for $T(v_n, n\mathbf{e}_1 + v_n) - T(0, n\mathbf{e}_1)$. Finally, we get

$$c n^{2\underline{\chi} - \varepsilon} \le \mathbb{E}[(T(0, n\mathbf{e}_1) - T(v_n, n\mathbf{e}_1 + v_n))^2] \le C(n^{2(2\xi - \beta)} + n^{2\beta(\chi + 2\varepsilon)}). \tag{3.3.12}$$

For small enough ε, since $\beta < 1$ and $\chi = \underline{\chi}$, it follows that $2\beta(\chi + 2\varepsilon) < 2\underline{\chi} - \varepsilon$. Since inequality (3.3.12) holds for any n, it yields that

$$2\underline{\chi} - \varepsilon \le 2(2\xi - \beta).$$

By letting first ε go to 0 and then β go to 1, we get

$$\chi \le 2\xi - 1. \tag{3.3.13}$$

This concludes the proof. □

3.4 Talagrand concentration estimate

In this section, we aim to prove the following strong concentration estimate originally proved by Talagrand [21]. The proofs of this section are inspired by the proofs in [3, Section 3.4] and [7, Section 6].

Theorem 3.4.1 (Talagrand concentration theorem). *Assume $G(\{0\}) < p_c$ and that G has an exponential moment. There exist $c, C > 0$ such that*

$$\forall x \in \mathbb{Z}^d \qquad \mathbb{P}(T(0,x) - \mathbb{E}[T(0,x)] \ge t\sqrt{\|x\|_1}) \le e^{-ct^2} \quad \textit{for } t \in (0, C\sqrt{\|x\|_1}).$$

Exercise Thanks to this theorem prove that $\chi \leq \frac{1}{2}$ and $\xi \leq 1$ using the definitions of the exponent from the previous section.

3.4.1 Logarithmic Sobolev inequalities

Let $X_1, \dots, X_n$ be independent random variables taking values in $\mathcal{X}$ and g be a positive-valued function on $\mathcal{X}^n$. We investigate concentration properties of $Z = g(X_1, \dots, X_n)$. Denote by $\mathbb{E}_i$ the expectation conditionally on $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$. Let $X'_1, \dots, X'_n$ be independent random variables distributed as $X_1, \dots, X_n$. Write $Z'_i = g(X_1, \dots, X_{i-1}, X'_i, X_{i+1}, \dots, X_n)$. As a warm-up let us prove subadditivity for the variance.

Theorem 3.4.2 (Efron-Stein's inequality)**.** *We have*

$$\mathrm{Var}(Z) \leq \sum_{i=1}^{n} \mathbb{E}[\mathrm{Var}_i(Z)]$$

where $\mathrm{Var}_i(Z) = \mathbb{E}_i[(Z - \mathbb{E}_i[Z])^2]$.

Proof. Write $\mathbb{E}^{(i)}$ the expectation conditionally on $X_1, \dots, X_i$. Denote $\Delta_i = \mathbb{E}^{(i)}[Z] - \mathbb{E}^{(i-1)}[Z]$. Note that

$$Z - \mathbb{E}Z = \sum_{i=1}^{n} \Delta_i.$$

It follows that

$$\mathrm{Var}(Z) = \mathbb{E}\left[\left(\sum_{i=1}^{n} \Delta_i\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}[\Delta_i^2] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}[\Delta_i \Delta_j].$$

Besides, we have for $i < j$

$$\mathbb{E}[\Delta_i \Delta_j] = \mathbb{E}[\Delta_i \mathbb{E}^{(i)}[\Delta_j]] = \mathbb{E}[\Delta_i (\mathbb{E}^{(i)}[Z] - \mathbb{E}^{(i)}[Z])] = 0.$$

It yields that

$$\mathrm{Var}(Z) = \sum_{i=1}^{n} \mathbb{E}[\Delta_i^2].$$

By Jensen's inequality, it follows that

$$\mathbb{E}[\Delta_i^2] = \mathbb{E}[\mathbb{E}^{(i)}[Z - Z'_i]^2] \leq \mathbb{E}[\mathbb{E}^{(i)}[(Z - Z'_i)^2]] = \mathbb{E}[(Z - Z'_i)^2] = \mathbb{E}[\mathbb{E}_i[(Z - Z'_i)^2]] = \mathbb{E}[\mathrm{Var}_i(Z)]$$

where we used in the last inequality that conditionally on $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$, the random variables Z and Z'_i are independent and identically distributed. This concludes the proof. □

Exercise Deduce from Theorem 3.4.2 that

$$\mathrm{Var}(Z) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}[(Z - Z'_i)^2 \mathbf{1}_{Z'_i > Z}].$$

Apply this inequality to get a linear upper-bound for the first passage time between two points.

Denote $\psi(x) = e^x - x - 1$ and $\tau(x) = x(e^x - 1)$ for $x > 0$. Define the entropy of X and the conditional entropy

$$\mathrm{Ent}\, X := \mathbb{E}X \log X - \mathbb{E}X \log \mathbb{E}X \quad \text{and} \quad \mathrm{Ent}_i\, X := \mathbb{E}_i X \log X - \mathbb{E}_i X \log \mathbb{E}_i X.$$

Roughly speaking, the entropy measures the amount of uncertainty of the measure X.

The following theorem will be the key inequality to prove Talagrand Theorem.

Theorem 3.4.3 (A logarithmic Sobolev inequality)**.** *For all* $s \in \mathbb{R}$

$$\mathrm{Ent}[e^{sZ}] \leq \sum_{i=1}^{n} \mathbb{E}[e^{sZ} \tau(s(Z'_i - Z)) \mathbf{1}_{Z < Z'_i}].$$

We will admit the following theorem (we refer to [7, Theorem 4.22]). This theorems says that the entropy is a sub-additive quantity. It may be seen as the analogous of Theorem 3.4.2.

Theorem 3.4.4 (Tensorization inequality of the entropy)**.** *We have*

$$\operatorname{Ent} Z \leq \sum_{i=1}^{n} \mathbb{E}[\operatorname{Ent}_i Z].$$

We will also need the following easy lemma.

Lemma 3.4.5. *Let Y denote a positive random variable and $u > 0$*

$$\mathbb{E}Y \log Y - \mathbb{E}Y \log \mathbb{E}Y \leq \mathbb{E}[Y \log Y - Y \log u - (Y - u)].$$

Proof. We need to check that

$$\mathbb{E}Y \log \frac{u}{\mathbb{E}Y} \leq u - \mathbb{E}Y.$$

It follows from $\log x \leq x - 1$ for any $x > 0$.

□

Proof of Theorem 3.4.3. Thanks to Lemma 3.4.5, we have

$$\operatorname{Ent}_i Y \leq \mathbb{E}_i[Y(\log Y - \log Y_i) - (Y - Y_i)].$$

By applying it for $Y = e^{sZ}$, we get

$$\operatorname{Ent}_i Y \leq \mathbb{E}_i[Y(s(Z - Z_i') - 1 + e^{s(Z_i' - Z)}]] = \mathbb{E}_i[e^{sZ}\psi(-s(Z - Z_i'))].$$

Write

$$e^{sZ}\psi(-s(Z - Z_i')) = e^{sZ}\psi(-s(Z - Z_i'))\mathbf{1}_{Z > Z_i'} + e^{sZ}\psi(s(Z_i' - Z))\mathbf{1}_{Z < Z_i'}.$$

By symmetry, we have

$$\mathbb{E}_i[e^{sZ}\psi(-s(Z - Z_i'))\mathbf{1}_{Z > Z_i'}] = \mathbb{E}_i[e^{sZ_i'}\psi(s(Z - Z_i'))\mathbf{1}_{Z < Z_i'}] = \mathbb{E}_i[e^{sZ}e^{s(Z_i' - Z)}\psi(s(Z - Z_i'))\mathbf{1}_{Z < Z_i'}].$$

Note that $\psi(x) + e^x\psi(-x) = \tau(x)$. It yields that

$$\begin{aligned}\mathbb{E}_i[e^{sZ}\psi(-s(Z - Z_i'))] &= \mathbb{E}_i[e^{sZ}\psi(s(Z_i' - Z))\mathbf{1}_{Z < Z_i'}] + \mathbb{E}_i[e^{sZ}e^{s(Z_i' - Z)}\psi(s(Z - Z_i'))\mathbf{1}_{Z < Z_i'}] \\ &= \mathbb{E}_i[e^{sZ}\tau(s(Z_i' - Z))\mathbf{1}_{Z < Z_i}].\end{aligned}$$

The result follows by applying Theorem 3.4.4. □

3.4.2 Proof of Talagrand concentration Theorem 3.4.1

Assume G has an exponential moment and $G(\{0\}) < p_c$. Let $x \in \mathbb{Z}^d$. We set $Z = T(0, x)$. By dominated convergence theorem, one can check that Theorem 3.4.3 extends to the case of $Z = T(0, x)$ where the random variables depend on a countable number of random variables as long s is small enough. We enumerate the edges $(e_1, e_2, \dots)$. We will denote by $(X_i)_{i \geq 1}$ and $(X_i')_{i \geq 1}$ two IID families of random variables distributed according to G. We set $t_{e_i} = X_i$ and $t'_{e_i} = X_i'$.

Denote by γ the geodesic between 0 and $n\mathbf{e}_1$ in the environment $(X_i)_i$ chosen with a deterministic rule to break ties. By Theorem 3.4.3, we have

$$\operatorname{Ent}[e^{sZ}] \leq \sum_{i=1}^{n} \mathbb{E}[e^{sZ}\tau(s(Z_i' - Z))\mathbf{1}_{Z < Z_i'}]$$

Note that if $Z < Z_i'$ it implies that the edge e_i is on the geodesic γ and that $Z_i' - Z \leq X_i' = t'_{e_i}$. It yields that

$$\operatorname{Ent}[e^{sZ}] \leq \sum_{i=1}^{n} \mathbb{E}[e^{sZ}\tau(st'_{e_i})\mathbf{1}_{e_i \in \gamma}] = \mathbb{E}[\tau(sX_1)]\mathbb{E}[e^{sZ}|\gamma|]$$

Since it is difficult to decouple $|\gamma|$ and e^{sZ}, we will use the following variational characterization of the entropy (see [7, Theorem 4.13])

$$\operatorname{Ent} X = \sup\{\mathbb{E}XY : \mathbb{E}e^Y \leq 1\}$$

which implies for $X \geq 0$ and any Y

$$\mathbb{E}XY \leq \operatorname{Ent} X + \mathbb{E}X \log \mathbb{E}e^Y.$$

It follows for any $a > 0$

$$\mathbb{E}[e^{sZ}|\gamma|] \leq a \operatorname{Ent} e^{sZ} + a\mathbb{E}e^{sZ} \log \mathbb{E} \exp\left(\frac{|\gamma|}{a}\right).$$

Combining with the previous inequalities, for a small enough such that $a\mathbb{E}[\tau(sX_1)] < 1$, we get

$$\operatorname{Ent} e^{sZ} \leq \frac{a\mathbb{E}[\tau(sX_1)]}{1 - a\mathbb{E}[\tau(sX_1)]}\mathbb{E}e^{sZ} \log \mathbb{E} \exp\left(\frac{|\gamma|}{a}\right).$$

We will need the following lemma that we leave as an exercise.

Lemma 3.4.6. *Assume that G has an exponential moment and $G(\{0\}) < p_c$. Then there exists $c_1 > 0$ depending on a such that*

$$\forall x \in \mathbb{Z}^d \quad \log \mathbb{E} \exp\left(\frac{|\gamma(0,x)|}{a}\right) \leq c_1 \|x\|_1.$$

Now set $F(s) := \mathbb{E}[e^{sZ}]$ and $H(s) := s^{-1} \log F(s)$. It follows that

$$H'(s) = \frac{sF'(s) - F(s) \log F(s)}{s^2 F(s)} \leq c_1 \|x\|_1 \frac{a\mathbb{E}[\tau(sX_1)]}{s^2(1 - a\mathbb{E}[\tau(sX_1)])}.$$

By dominated convergence, we have

$$\lim_{s \to 0} \frac{a\mathbb{E}[\tau(sX_1)]}{s^2(1 - a\mathbb{E}[\tau(sX_1)])} = a\mathbb{E}[X_1^2].$$

It follows that there exists $s_0 > 0$ such that for all $s \in (0, s_0)$

$$H'(s) \leq 2c_1 \|x\|_1 a\mathbb{E}[X_1^2] := C\|x\|_1.$$

Besides, we have by l'Hospital rule

$$\lim_{s \to 0} H(s) = \frac{F'(0)}{F(0)} = \mathbb{E}[Z].$$

By integrating the above inequality for some $s < s_0$, we get

$$H(s) \leq C\|x\|_1 s + \mathbb{E}Z.$$

Finally, it gives

$$F(s) \leq e^{s\mathbb{E}Z + s^2 C\|x\|_1}.$$

By Markov's inequality, we get

$$\mathbb{P}(Z > \mathbb{E}Z + t) \leq F(s)e^{-s\mathbb{E}Z - st} \leq e^{s^2 C\|x\|_1 - st}.$$

Finally, we choose $s = t/2C\|x\|_1$ for $t \leq 2s_0 C\|x\|_1$, it yields

$$\mathbb{P}(T(0,x) > \mathbb{E}[T(0,x)] + t) \leq e^{-t^2/4C\|x\|_1}.$$

The result follows.

Chapter 4

Large deviations in first passage percolation

4.1 Large deviations theory

Large deviation theory deals with the study of rare events, such as extreme fluctuations and deviations from typical behavior. The theory provides a framework for understanding the probability of such events. Large deviation theory has broad applications in diverse areas such as statistical physics, insurance, finance. The central question in large deviation theory is about the decay rate of the atypical event.

4.1.1 Sum of IID

Some examples

The simplest context to study large deviations is the case of the sum of IID random variables. Let μ be a distribution that has finite expectation. Let $(X_i)_{i\geq 1}$ be an IID family of random variables distributed according to μ. We want to study the behaviour of the sum $S_n := \sum_{i=1}^n X_i$. Thanks to the strong law of large numbers, we know that

$$\lim_{n\to\infty} \frac{1}{n} S_n = \mathbb{E}[X_1] \quad \text{a.s.}.$$

We are interested in extreme deviations with respect to the mean, that is for $\varepsilon > 0$, we will study $\mathbb{P}(S_n \geq n(\mathbb{E}[X_1] + \varepsilon))$. We can first lower bound the speed of decay. Let $\varepsilon > 0$ such that $\mathbb{P}(X_1 > \mathbb{E}[X_1] + \varepsilon) > 0$. It is easy to check that

$$\mathbb{P}(S_n \geq n(\mathbb{E}[X_1] + \varepsilon)) \geq \mathbb{P}(\forall i \in \{1, \dots, n\} \quad X_i > \mathbb{E}[X_1] + \varepsilon) = \mathbb{P}(X_1 > \mathbb{E}[X_1] + \varepsilon)^n.$$

This inequality indicates that it cannot decay faster than exponentially in n.

Let us now start with an easy example where all computations are explicit: X_1 is distributed as a standard gaussian random variable. Hence, it yields that the sum S_n is distruted as a centererd gaussian random variable with variance n. It follows that

$$\mathbb{P}(S_n > \varepsilon n) = \int_{t \geq \varepsilon n} \frac{1}{\sqrt{2\pi}} e^{-t^2/2n} dt \sim \frac{1}{\varepsilon\sqrt{2\pi n}} e^{-n\varepsilon^2/2}.$$

For general distributions, it is not always possible to get such a precise estimate on large deviations events. Before stating the theorem let us study a last instructive example: the case of a positive random variable where the distribution has heavy tails, that is there exists $\alpha > 0$ such that

$$\exists x_0 > 0 \quad \forall x \geq x_0 \qquad \mathbb{P}(X_1 > x) > x^{-\alpha}.$$

It yields that

$$\mathbb{P}(S_n \geq n(\mathbb{E}[X_1] + \varepsilon)) \geq \mathbb{P}(X_1 > n(\mathbb{E}[X_1] + \varepsilon)) \geq (n(\mathbb{E}[X_1] + \varepsilon))^{-\alpha}.$$

This implies that we need stronger assumption than just the existence of a finite moment to ensure that the large deviation events decay exponentially fast. We need an assumption to make sure the tails are light enough. We will see that when the tails are light, the large deviation events are not induced by shifting only one input but rather by the collective action of all the inputs.

Cramer's Theorem

Define

$$t_0 := \sup\{t \geq 0 : \mathbb{E}[\exp(tX_1)] < \infty\}.$$

We assume that $t_0 > 0$. Define $\varphi(t) := \mathbb{E}[\exp(tX_1)]$. It is easy to check that φ is strictly convex and $\mathcal{C}^\infty$ on $[0, t_0)$. For $a > \mathbb{E}[X_1]$ such that $\mathbb{P}(X_1 > a) > 0$, we define

$$I(a) = \sup_{t \geq 0} at - \log \varphi(t).$$

It is easy to check that the supremum is attained.

Theorem 4.1.1 (Cramer's theorem)**.** *Under the above conditions,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n > an) = -I(a), \tag{4.1.1}$$

that is $\mathbb{P}(S_n > an) = e^{-nI(a)+o(n)}$.

Proof. The upper-bound follows easily by Chernoff bound. We have for $t \in (0, t_0)$

$$\mathbb{P}(S_n > an) = \mathbb{P}(e^{tS_n} > e^{atn}) \leq \mathbb{E}[e^{tS_n}]e^{-atn} = \varphi(t)^n e^{-atn} \leq e^{-I(a)n}. \tag{4.1.2}$$

It yields that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n > an) \leq -I(a). \tag{4.1.3}$$

The lower-bound is the hard part of the proof. To prove the lower-bound, we will do a change of measure to favor large values. Let t^* denote the value achieving the supremum. Let Y be the random variables such that for any continuous function f

$$\mathbb{E}[f(Y)] = \frac{1}{\varphi(t^*)}\mathbb{E}[f(X)e^{t^* X}].$$

In other words, if X has a density P_X, then the density of Y is given by $P_Y(dx) = P_X(dx)e^{t^* x}/\varphi(t^*)$. Note that

$$\mathbb{E}[Y] = \frac{1}{\varphi(t^*)}\mathbb{E}[Xe^{t^* X}] = \frac{\varphi'(t^*)}{\varphi(t^*)}.$$

Besides, since t^* is the value where the supremum is attained, we have $a - \frac{\varphi'(t^*)}{\varphi(t^*)} = 0$. It yields that $\mathbb{E}[Y] = a$. Let $(Y_i)_{i\geq 1}$ be an IID family with the same distribution as Y. We have

$$\begin{aligned}
\mathbb{P}(S_n > na) &= \mathbb{E}[\mathbf{1}_{X_1+\cdots+X_n > na}] \\
&= Z^n \mathbb{E}[\mathbf{1}_{Y_1+\cdots+Y_n > na} e^{-t^*(Y_1+\ldots,Y_n)}] \\
&\geq Z^n \mathbb{E}[\mathbf{1}_{Y_1+\cdots+Y_n \in (na, na+\sqrt{n})} e^{-t^*(Y_1+\ldots,Y_n)}] \\
&\geq Z^n e^{-t^*(na+\sqrt{n})} \mathbb{P}(Y_1 + \cdots + Y_n \in (na, na+\sqrt{n})).
\end{aligned}$$

By using the central limit theorem, we have that

$$\lim_{n\to\infty} \frac{1}{\sigma_Y \sqrt{n}} \left(\sum_{i=1}^n Y_i - an \right) = \mathcal{N}(0,1) \quad \text{in distribution.}$$

In particular, we have

$$\lim_{n\to\infty} \mathbb{P}(Y_1 + \cdots + Y_n \in (na, na + \sqrt{n})) > 0.$$

Combining the previous inequalities, it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n > an) \geq \varphi(t^*) - at^* = -I(a). \tag{4.1.4}$$

The result follows.

□

Exercise Compute I in the case where the distribution is Bernoulli random variable and exponential random variable.

4.1.2 General large deviations principle

In the previous section, we observed that in the case of a sum of independent and identically distributed variables, the occurrence of a large deviation in the upper direction was caused by a joint effort of all inputs, achieved by shifting their values up. In the following discussion, we will explore broader large deviation principles for more general functions that rely on a set of input variables. Through examples, we will observe that in certain cases, only a small fraction of the inputs play a significant role in the large deviation event, leading to a decay rate that may not be directly proportional to the number of inputs.

In the general context, the objects we will study are more general than real number. For that reason, topology will play an important role in the study of large deviation, in particular to quantify how close two objects are. Let us recall some definitions of topology. Let $(\mathcal{X}, \mathcal{O})$ be a regular topological space; regular means that for any $x \in \mathcal{X}$ and any closed subset F of $\mathcal{X}$ such that $x \notin F$, there exist two disjoint open sets U, V such that $x \in U$ and $F \subseteq V$. We suppose that $\mathcal{X}$ is also endowed with a σ-field $\mathcal{B}$ such that each element $x \in \mathcal{X}$ admits a basis of neighbourhoods consisting of open measurable sets:

$$\forall x \in \mathcal{X} \quad \forall O \in \mathcal{O} \quad x \in O \quad \Rightarrow \quad \exists V \in \mathcal{B} \cap \mathcal{O} \quad x \in V \subseteq O. \tag{4.1.5}$$

Definition 4.1.2 (Rate function). *A rate function on $(\mathcal{X}, O)$ is a lower semicontinuous map $I : \mathcal{X} \to \mathbb{R}^+ \cup \{+\infty\}$. A rate function I is said to be good if its level sets $\{x \in \mathcal{X} : I(x) \leq \lambda\}$, $\lambda \in \mathbb{R}^+$, are compact.*

Definition 4.1.3 (Large deviation principle (LDP)). *Let $(\mu_n)_{n\geq 1}$ be a sequence of probability measures defined on the σ-field $\mathcal{B}$. The sequence $(\mu_n)_{n\geq 1}$ satisfies a large deviation principle with speed a_n governed by the rate function I and with respect to the topology $\mathcal{O}$ if for any $A \in \mathcal{B}$,*

$$-\inf\{I(x) : x \in \mathring{A}\} \leq \liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(A) \leq \limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(A) \leq -\inf\{I(x) : x \in \overline{A}\}$$

where $\mathring{A}$ and $\overline{A}$ are respectively the interior and closure of A.

Let $(X_n)_{n\geq 1}$ be a sequence of random variables taking their values in the regular topological space $(\mathcal{X}, \mathcal{O})$, with a σ-field $\mathcal{B}$ satisfying the above hypothesis. We say that $(X_n)_{n\geq 1}$ satisfies a LDP if the sequence of their law does. Rougly speaking, if the sequence $(X_n)_{n\geq 1}$ satisfies a LDP governed by I with speed a_n if $\mathbb{P}(X_n \approx x) = e^{-a_n I(x) + o(a_n)}$.

Remark 4.1.4. *Assume I admits a unique global minimizer x^*, then if X_n satisfies a LDP governed by I, for any neighborhood $U \in \mathcal{B}$ of x, we have*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(X_n \notin U) < 0.$$

Hence,

$$\lim_{n\to\infty} \mathbb{P}(X_n \in U) = 1.$$

This may be interpreted as law of large number. In particular, we must have $I(x^*) = 0$ in order not to contradict the limit above.

How to prove a LDP in practice ?

We now assume that $\mathcal{X}$ is a vector space and we denote by $\mathcal{U}$ a basis of neighbourhood of the origin (as defined in (4.1.5)). To prove a LDP, one need to check the following conditions. A proof of this proposition may be found in section 6 in [8].

Proposition 4.1.5. *Let us assume that the following conditions are satisfied*

- *I is a good rate function;*
- $\forall x \in \mathcal{X} \quad \forall U \in \mathcal{U} \quad \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(X_n \in x + U) \geq -I(x)$;
- *We have I-tightness, that is there exists $c, \lambda_0 > 0$ such that*

$$\forall \lambda \geq \lambda_0 \quad \forall U \in \mathcal{U} \quad \limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(X_n \notin I^{-1}([0, \lambda] + U) \leq -c\lambda;$$

- $\forall x \in \mathcal{X}\ I(x) < \infty \quad \forall \varepsilon > 0 \quad \exists U \in \mathcal{U} \qquad \limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(X_n \in x + U) \leq -I(x)(1 - \varepsilon)$.

Then, the sequence $(X_n)_{n\geq 1}$ satisfies a LDP governed by I with speed a_n.

Remark 4.1.6. *Note that the tightness assumption becomes trivial if with probability 1, X_n takes values in a compact set. Otherwise, we need to ensure that the mass is exponentially concentrated on compact sets.*

Contraction principle

We conclude by mentioning a very useful result of stability of LDP. Let $(\mathcal{X},\mathcal{O}_X)$ and $(\mathcal{Y},\mathcal{O}_\mathcal{Y})$ be two regular topological spaces. We suppose also that $\mathcal{X}$ and $\mathcal{Y}$ are both endowed with two σ–fields $\mathcal{B}_\mathcal{X}$ and $\mathcal{B}_\mathcal{Y}$ satisfying the condition in (4.1.5). Let us assume we know that the sequence $(X_n)_{n\geq 1}$ satisfies a LDP governed by I with speed a_n. Let $f:\mathcal{X}\to\mathcal{Y}$ be a continuous map, measurable with respect to the σ–fields $\mathcal{B}_\mathcal{X}$ and $\mathcal{B}_\mathcal{Y}$. Set $Y_n = f(X_n)$. Then, the sequence $(Y_n)_{n\geq 1}$ also satisfies a LDP.

Proposition 4.1.7. *If the sequence $(X_n)_{n\geq 1}$ satisfies a large deviation principle governed by the good rate function I and with respect to the topology $\mathcal{O}_\mathcal{X}$, then the sequence $(Y_n)_{n\geq 1}$ satisfies a large deviation principle with respect to the topology $\mathcal{O}_\mathcal{Y}$, with the same speed, governed by the good rate function J given by*

$$\forall y\in\mathcal{Y}\qquad J(y):=\inf\{I(x):x\in\mathcal{X},\ y=f(x)\}.$$

Proof. The function J takes its values in $[0,+\infty]$. Let $t\geq 0$ and $y\in\mathcal{Y}$ such that $J(y)\leq t$. Since I has compact level sets, it attains its minimum in the following compact set

$$\{x\in\mathcal{X}:f(x)=y\}\cap\{x\in\mathcal{X}:\ I(x)\leq\lambda\}.$$

Let $x\in\mathcal{X}$ be such that $f(x)=y$ and $J(y)=I(x)$. It follows that

$$\{y\in\mathcal{Y}:J(y)\leq t\}=f(\{x\in\mathcal{X}:I(x)\leq t\})$$

and J also has compact level set. Next we prove the large deviation lower bound. Let $B\in\mathcal{B}_\mathcal{Y}$. Then

$$\liminf_{n\to\infty}\frac{1}{a_n}\log\mathbb{P}(Y_n\in B)=\liminf_{n\to\infty}\frac{1}{a_n}\log\mathbb{P}(X_n\in f^{-1}(B))\geq-\inf\{I(x):x\in \mathring{f^{-1}}(B)\}.$$

Mote that $f^{-1}(\mathring{B})\subseteq \mathring{f^{-1}}(B)$. It yields

$$\begin{aligned}-\inf\{I(x):x\in \mathring{f^{-1}}(B)\}&\geq-\inf\{I(x):x\in f^{-1}(\mathring{B})\}\\&=-\inf\{\inf\{I(x):x\in X,f(x)=y\}:y\in\mathring{B}\}\\&=-\inf\{J(y):y\in\mathring{B}\}.\end{aligned}$$

Finally, let us prove the large deviation upper bound:

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\mathbb{P}(Y_n\in B)=\limsup_{n\to\infty}\frac{1}{a_n}\log\mathbb{P}(X_n\in f^{-1}(B))\ =-\inf\{I(x):x\in\overline{f^{-1}(B)}\}.$$

Note that $\overline{f^{-1}(B)}\subseteq f^{-1}(\overline{B})$, whence

$$\begin{aligned}-\inf\{I(x):x\in\overline{f^{-1}(B)}\}&\leq-\inf\{I(x):x\in f^{-1}(\overline{B})\}\\&=-\inf\{\inf\{I(x):x\in X,f(x)=y\}:y\in\overline{B}\}\\&=-\inf\{J(y):y\in\overline{B}\}.\end{aligned}$$

This concludes the proof. □

Exercise Let $(X_i)_{i\geq 1}$ be an IID family of random variables distributed according to some distribution μ with all exponential moment. Set $S_n:=\sum_{i=1}^n X_i$. Prove that $(\frac{1}{n}S_n)_{n\geq 1}$ satisfies a LDP. Deduce that $((\frac{1}{n}S_n)^2)_{n\geq 1}$ also satisfies a LDP and determine the corresponding rate function.

4.2 Large deviations for FPP

4.2.1 Lower tail large deviations

Let $x \in \mathbb{Z}^d$. We want to study the asymptotic probability that the time is abnormally low, that is the following event $\{T(0, nx) < (1-\varepsilon)\mu(x)n\}$, $\varepsilon > 0$. Intuitively, to create this unlikely event it is sufficient to decrease all the time of the edges along the geodesic. It will induce a speed of large deviation of order n. The following theorem asserts the existence of a rate function and that n is the correct speed of large deviation.

Theorem 4.2.1. *The following limit exists*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(T(0, nx) < (1-\varepsilon)\mu(x)n) := -I_l(x, \varepsilon) \tag{4.2.1}$$

and for $\varepsilon > 0$ *and* $x \in \mathbb{Z}^d$, *we have* $I_l(x, \varepsilon) > 0$

The existence of the limit is an easy consequence of subadditivity and FKG inequality. The second part of the theorem is more involved. To prove this theorem, we will need to use FKG property. We say that an event E is increasing if for any $(t'_e)_{e\in\mathbb{E}^d} \geq (t_e)_{e\in\mathbb{E}^d}$ if $(t_e)_{e\in\mathbb{E}^d} \in E$ then $(t'_e)_{e\in\mathbb{E}^d} \in E$. If two events are increasing, there are positively correlated, hence there have more chance to occur jointly than separately. More formally, we have the following inequality.

Proposition 4.2.2 (FKG inequality)**.** *If* E *and* F *are increasing, then*

$$\mathbb{P}(E \cap F) \geq \mathbb{P}(E)\,\mathbb{P}(F).$$

The same conclusion holds if both events are decreasing.

Define the disjoint occurrence of E and F denoted by $E \circ F$ if for any $(t_e)_e \in E \circ F$, there exists a set of edges G such that the edges in G are sufficient to say that the event E occurs and the edges outside G are sufficient to say that the event F occurs. We will also need the following inequality.

Proposition 4.2.3 (BK inequality)**.** *We have*

$$\mathbb{P}(E \circ F) \leq \mathbb{P}(E)\,\mathbb{P}(F).$$

Proof of Theorem 4.2.1. Note that the event $\{T(0, nx) < (1-\varepsilon)\mu(x)n\}$ is decreasing. By FKG's inequality, we have

$$\begin{aligned}
&\mathbb{P}(T(0,(n+m)x) < (1-\varepsilon)\mu(x)(n+m)) \\
&\qquad \geq \mathbb{P}(T(0,nx) < (1-\varepsilon)\mu(x)n,\, T(nx,(n+m)x) < (1-\varepsilon)\mu(x)m) \\
&\qquad \geq \mathbb{P}(T(0,nx) < (1-\varepsilon)\mu(x)n)\,\mathbb{P}(T(nx,(n+m)x) < (1-\varepsilon)\mu(x)m) \\
&\qquad = \mathbb{P}(T(0,nx) < (1-\varepsilon)\mu(x)n)\,\mathbb{P}(T(0,mx) < (1-\varepsilon)\mu(x)m).
\end{aligned}$$

By setting $u_n := \log \mathbb{P}(T(0,nx) < (1-\varepsilon)\mu(x)n)$. We have that $(u_n)_{n\geq 1}$ is super-additive. We conclude that the limit in (4.2.1) exists by Fekete's lemma.

Let us now turn to the proof of $I_l(x,\varepsilon) > 0$. To simplify, we will assume that the distribution G is compactly supported on $[0, M]$. Let γ be the geodesic between 0 and nx on the event $\{T(0,nx) < (1-\varepsilon)\mu(x)n\}$. Let $K \geq 1$ we will choose later. Let $(x_i)_{i=0,\dots,m}$ be its corresponding K-skeleton as defined in the proof of Proposition 2.3.2. In particular, we have $m \leq |gamma|/K$. By triangular inequality, we have

$$\sum_{i=1}^m \mu(x_i - x_{i-1}) \geq n\mu(x).$$

Besides, we have on the event $\{T(0,nx) < (1-\varepsilon)\mu(x)n\}$

$$\begin{aligned}
(1-\varepsilon)\sum_{i=1}^m \mu(x_i - x_{i-1}) \geq T(0,nx) = \sum_{i=1}^m T(x_{i-1},x_i) &\geq \sum_{i=1}^m T(x_{i-1},x_i)\mathbf{1}_{T(x_{i-1},x_i)>(1-\varepsilon/2)\mu(x_i-x_{i-1})} \\
&\geq \sum_{i=1}^m (1-\varepsilon/2)\mu(x_i-x_{i-1})\mathbf{1}_{T(x_{i-1},x_i)>(1-\varepsilon/2)\mu(x_i-x_{i-1})}.
\end{aligned}$$

It follows that

$$(1-\varepsilon)\sum_{i=1}^{m}\mu(x_i-x_{i-1})\mathbf{1}_{T(x_{i-1},x_i)\le(1-\varepsilon/2)\mu(x_i-x_{i-1})} \ge \frac{\varepsilon}{2}\sum_{i=1}^{m}\mu(x_i-x_{i-1})\mathbf{1}_{T(x_{i-1},x_i)>(1-\varepsilon/2)\mu(x_i-x_{i-1})}$$

Since $\|x_i - x_{i-1}\|_\infty = K$ (except for $i = M$), we have

$$K\mu(\mathbf{e}_1) \le \mu(x_i - x_{i-1}) \le Kd\mu(\mathbf{e}_1).$$

Write $\mathrm{E} := \{i < m : T(x_{i-1}, x_i) > (1-\varepsilon/2)\mu(x_i - x_{i-1})\}$ and $\mathrm{N} := \#\mathrm{E}$. Hence, $\varepsilon\mathrm{N} \le 2(m-\mathrm{N})d$ and $\mathrm{N} \le \frac{2d}{2d+\varepsilon}m$. Let $\delta > 0$. For K large enough we have for all $\|y\|_1 \ge K/2$

$$\mathbb{P}(T(0,y) \le (1-3\varepsilon/4)\mu(y)) \le \delta. \tag{4.2.2}$$

Let $K_0 = \lfloor \varepsilon K/16dM \rfloor$. Denote by $\underline{x} \in \mathbb{Z}^d$ be such that $x \in (2K_0\underline{x} + [-K_0, K_0)^d) := \Lambda_{K_0}(\underline{x})$. Let $\underline{x}, \underline{y} \in \mathbb{Z}^d$. Assume there exists $x \in \Lambda_{K_0}(\underline{x})$ and $y \in \Lambda_{K_0}(\underline{y})$ such that $\|x-y\|_\infty = K$ and

$$T(\Lambda_{K_0}(\underline{x}), \Lambda_{K_0}(\underline{y})) \le (1-\varepsilon/2)\mu(x-y).$$

It follows that

$$\begin{aligned} T(2K_0\underline{x}, 2K_0\underline{y}) &\le dK_0M + T(\Lambda_{K_0}(\underline{x}), \Lambda_{K_0}(\underline{y})) + dK_0M \\ &\le 2(1-\varepsilon/2)K_0\mu(\underline{x}-\underline{y}) + \varepsilon K\mu(\mathbf{e}_1)/16 + 2dK_0\mu(\mathbf{e}_1) \\ &\le 2K_0(1-3\varepsilon/4)\mu(\underline{x}-\underline{y}). \end{aligned}$$

Hence

$$\{T(\Lambda_{K_0}(\underline{x}), \Lambda_{K_0}(\underline{y})) \le (1-\varepsilon/2)\mu(x-y)\} \subset \{T(2K_0\underline{x}, 2K_0\underline{y}) \le 2K_0(1-3\varepsilon/4)\mu(\underline{x}-\underline{y})\}$$

and by inequality (4.2.2)

$$\mathbb{P}(T(\Lambda_{K_0}(\underline{x}), \Lambda_{K_0}(\underline{y})) \le (1-\varepsilon/2)\mu(x-y)) \le \delta.$$

By summing over all possible $\underline{x}_1, \dots, \underline{x}_m$, we get

$$\begin{aligned} &\mathbb{P}(T(0,nx) < (1-\varepsilon)\mu(x)n) \\ &\quad\le \sum_{\underline{x}_1,\dots,\underline{x}_m \text{ admissible}} \mathbb{P}\left(\mathrm{N} \le \frac{2d}{2d+\varepsilon}m, \forall i \quad x_i \in \Lambda_{K_0}(\underline{x}_i)\right) \\ &\quad\le \sum_{\underline{x}_1,\dots,\underline{x}_m \text{ admissible}} \sum_{\substack{E\subset\{1,\dots,m\}:\\ \#E\le\frac{2d}{2d+\varepsilon}m}} \mathbb{P}(\forall i \in \{1,\dots,m\}\setminus E \quad T(\Lambda_{K_0}(\underline{x}_{i-1}), \Lambda_{K_0}(\underline{x}_i)) \le (1-\varepsilon/2)\mu(x_i - x_{i-1})) \\ &\quad\le \sum_{\substack{\underline{x}_1,\dots,\underline{x}_m \text{ admissible}\\ m\le\beta n/K}} \sum_{\substack{E\subset\{1,\dots,m\}:\\ \#E\le\frac{2d}{2d+\varepsilon}m}} \delta^{\frac{\varepsilon}{2d+\varepsilon}m} + \mathbb{P}(|\gamma| \ge \beta n) \\ &\quad\le e^{c\beta\frac{n}{K}\log\varepsilon}2^{\beta\frac{n}{K}}e^{\beta\frac{\varepsilon}{3d}\frac{n}{K}\log\delta} + e^{-cn} \end{aligned}$$

where in the second to last inequality we used that the occurrence of all these events is disjoint and that we can therefore apply BK inequality. We also used Theorem 2.2.4 where we used that $G(\{0\}) < p_c$ since otherwise there is no lower large deviation. The result follows by first choosing δ small enough such that $-\varepsilon\log\delta \ge -\log\varepsilon$ and then K large enough such that (4.2.2) holds. $\square$

4.2.2 Upper tail large deviation

The study of upper large deviation $\{T(0,nx) > (1+\varepsilon)n\mu(x)\}, \varepsilon > 0$ is much more involved. In particular, in the case of bounded passage time, it is not sufficient to increase the passage time along the geodesic to create this event as there will be another path with a passage time close to $n\mu(x)$. In fact, the passage time of a positive fraction of the edges needs to be shifted up to ensure the upper large deviation occurs. When the distribution G has compact support, the speed of large deviation is volumic n^d. When the distribution does not have compact support, the phenomenology is radically different depending on the tail of G. In some cases, the upper large deviation becomes a local event, only the edges nearby the endpoints of the geodesic need to be shifted up to create the upper tail large deviation event. In the case of compactly supported distribution, we have the existence of a rate function.

Theorem 4.2.4 (Basu–Ganguly–Sly [4]). *Assume G has a regular density and compact support. Then, the following limit exists*

$$\lim_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}(T(0,nx) \geq (1+\varepsilon)\mu(x)n) := -I_u(x,\varepsilon).$$

Note that the proof of this theorem is much more involved that its lower tail large deviation counterpart. The proof also relies on sub-additivity but the sub-additivity is much more involved to get. We also have the following theorem for some specific non-compact distributions.

Theorem 4.2.5 (Cosco–Nakajima [10]). *Consider a distribution G satisfying*

$$\mathbb{P}(t_e \geq t) \asymp e^{-\alpha t^r} \text{ as } t \to \infty \text{ with some constants } r \in (1,d],\ \alpha > 0.$$

Then, the rate function for upper tail large deviations exists and is related to the so-called discrete p-capacity and the correct speed is n^r.

We will prove here the following more modest result which confirms that the appropriate speed is volumic in the bounded case.

Theorem 4.2.6. *Assume G has compact support. We have for all $x \in \mathbb{Z}^d$*

$$\limsup_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}(T(0,nx) \geq (1+\varepsilon)\mu(x)n) \leq 0.$$

To prove this theorem, we will first prove the following intermediate proposition. Let $C \subset \mathbb{E}^d$, we denote the time restricted to paths in C by T_C, that is

$$\forall x,y \in C \qquad T_C(x,y) = \inf\{T(\gamma) : \gamma \subset C \text{ path between } x \text{ and } y\}.$$

Proposition 4.2.7. *For all $\varepsilon > 0$, there exists $K \geq 1$ and $c > 0$ such that*

$$\mathbb{P}(T_{\mathrm{cyl}(0,nx,K)}(0,nx) \geq (1+\varepsilon)n\mu(x)) \leq e^{-cn}.$$

Let us postpone the proof of this proposition to the end of the proof of Theorem 4.2.6.

Proof of Theorem 4.2.6. Assume G is supported on the interval $[0,M]$. Let $\varepsilon > 0$. Let K and c be the parameters of Proposition 4.2.7 corresponding to $\varepsilon/2$. Set $\delta = \frac{\varepsilon}{4dM}$. Set $\mathcal{S} := \{0\} \times ((3K\mathbb{Z})^{d-1} \cap [-\delta n, \delta n]^{d-1})$. In particular, thanks to the definition of $\mathcal{S}$, we have that all the cylinders $\mathrm{cyl}(w, w+nx, K), w \in \mathcal{S}$ are disjoint. Now let us assume there exists $w \in \mathcal{S}$ such that $T_{\mathrm{cyl}(w,w+nx,K)}(w, w+nx) \leq (1+\varepsilon/2)\mu(x)n$. It follows that

$$T(0,nx) \leq T(0,w) + T(w,w+nx) + T(w,nx) \leq 2(d-1)M\delta n + (1+\varepsilon/2)\mu(x)n \leq (1+\varepsilon)\mu(x)n.$$

Hence, we have that

$$\{T(0,nx) \geq (1+\varepsilon)\mu(x)n\} \subset \bigcap_{w\in\mathcal{S}} \{T_{\mathrm{cyl}(w,w+nx,K)}(w,w+nx) \geq (1+\varepsilon/2)\mu(x)n\}.$$

Since the cylinders are disjoint, the events in the intersection are independent and

$$\begin{aligned}\mathbb{P}(T(0,nx) \geq (1+\varepsilon)\mu(x)n) &\leq \prod_{w\in\mathcal{S}} \mathbb{P}(T_{\mathrm{cyl}(w,w+nx,K)}(w,w+nx) \geq (1+\varepsilon/2)\mu(x)n) \\ &\leq e^{-cn\#S} \leq e^{-cn^d\left(\frac{\delta}{3K}\right)^{d-1}}\end{aligned}$$

where we used Proposition 4.2.7. The result follows. □

Proof of Proposition 4.2.7. Let $\varepsilon > 0$. Let δ be such that $\delta M \frac{\|x\|_1}{\mu(x)} \leq \frac{\varepsilon}{4}$. Let K_0 be large enough such that

$$\mathbb{P}(T(0,K_0x) \geq (1+\varepsilon/2)\mu(x)K_0) \leq \frac{\delta}{2}$$

and let K be large enough such that $|\gamma(0,K_0x)| \leq K$ with probability at least $1-\delta/2$ (to simplify we assume $G(\{0\}) < p_c$). It follows that

$$\mathbb{P}(T_{\mathrm{cyl}(0,K_0x,K)}(0,K_0x) \geq (1+\varepsilon/2)\mu(x)K_0) \leq \delta.$$

To simplify the notations, let us assume n is a multiple of K_0. Note that

$$\begin{aligned} T_{\mathrm{cyl}(0,nx,K)}(0,nx) &\leq \sum_{k=1}^{n/K_0} T_{\mathrm{cyl}((k-1)K_0x,kK_0x,K)}((k-1)K_0x,kK_0x) \\ &\leq (1+\varepsilon/2)\mu(x)n + M\|x\|_1 K_0 \mathrm{N} \end{aligned}$$

where $\mathrm{N} := \#\{1 \leq k \leq n/K_0 : T_{\mathrm{cyl}((k-1)K_0x,kK_0x,K)}((k-1)K_0x,kK_0x) \geq (1+\varepsilon/2)\mu(x)K_0)\}$. It yields that

$$\{T_{\mathrm{cyl}(0,nx,K)}(0,nx) \geq (1+\varepsilon)\mu(x)n\} \subset \left\{\mathrm{N} \geq \varepsilon\frac{\mu(x)}{2MK_0\|x\|_1}n\right\} \subset \{\mathrm{N} \geq 2\delta\frac{n}{K_0}\}.$$

By looking at $1 \leq k \leq n/K_0$ modulo K, we have

$$\{\mathrm{N} \geq 2\delta\frac{n}{K_0}\} \subset \bigcup_{r=0}^{K}\{\mathrm{N}_r \geq 2\delta\frac{n}{KK_0}\}$$

where

$$\mathrm{N}_r := \#\{1 \leq k \leq \frac{n}{K_0}, k-r \in K\mathbb{Z} : T_{\mathrm{cyl}((k-1)K_0x,kK_0x,K)}((k-1)K_0x,kK_0x) \geq (1+\varepsilon/2)\mu(x)K_0)\}.$$

Using that the cylinders at distance KK_0 are independent, it follows

$$\mathbb{P}(\mathrm{N} \geq 2\delta\frac{n}{K_0}) \leq \sum_{r=0}^{K-1}\mathbb{P}(\mathrm{N}_r \geq 2\delta\frac{n}{KK_0}) \leq \sum_{r=0}^{K-1}\mathbb{P}\left(\sum_{i=1}^{n/KK_0} X_i \geq 2\delta\frac{n}{KK_0}\right)$$

where $(X_i)_{i\geq 1}$ is an IID family of Bernoulli random variables of parameter δ. Using large deviations estimates on Bernoulli random variable, the result follows.

□

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