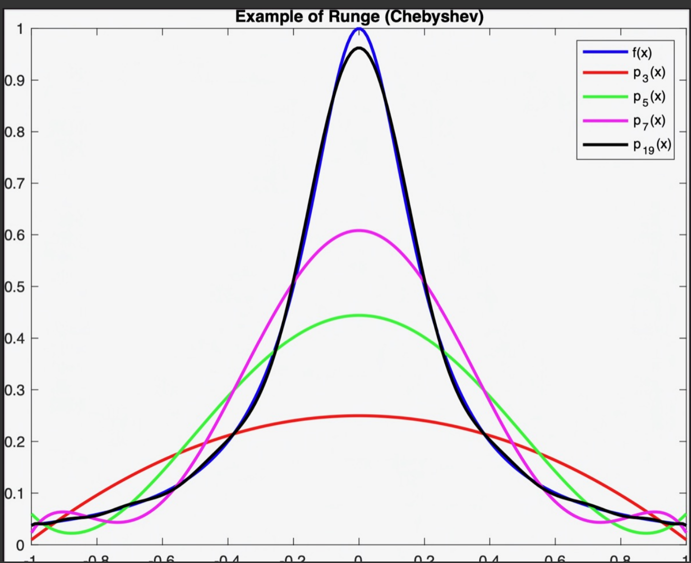
4 MARCH

- Vote
- Tschebyshev
- Quadratur-fehler

ML



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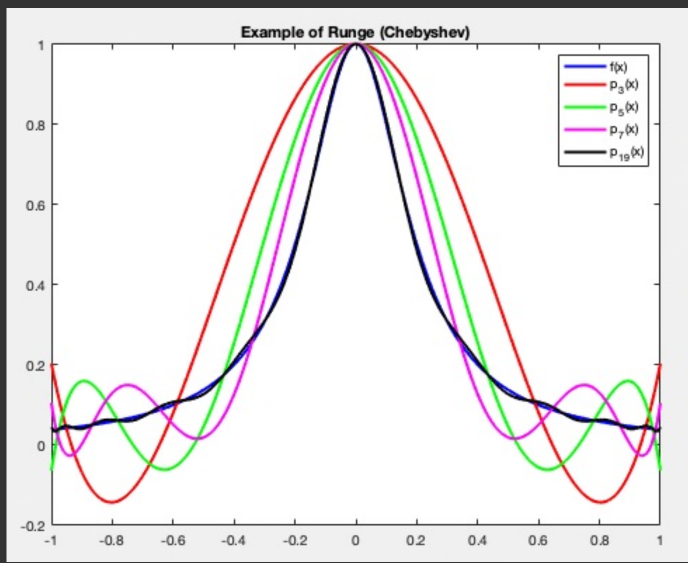

Gauss Quadrature ("easy" question on Quadratur-fehler)

a) Sei $-\infty < a < b < \infty$. Für $f \in C^2([a,b])$ und $\alpha \in \mathbb{R}$ fix we define

$$Q_{[a,b]}[f] := (b-a) f\left(\frac{a+b}{2}\right) + \alpha (b-a)^3 f''\left(\frac{a+b}{2}\right)$$

Finde Konstante $\alpha \in \mathbb{R}$ und $C_0 > 0$ s.d $\forall f \in C^4([a,b])$

$$0 \leq \left| \int_a^b f(x)\,dx - Q_{[a,b]}[f] \right| \leq C_0 (b-a)^5 \| f^{(4)} \|_{\infty,[a,b]}$$

Solutions:

Setze $m = \frac{(a+b)}{2}$

Da $f \in C^4([a,b])$ gilt $f(x) = \sum_{j=0}^{3} \frac{f^{(j)}(m)}{j!}(x-m)^j + \frac{f^{(4)}(\xi_x)}{4!}(x-m)^4$, $(\xi_x \in [a,b])$

Wegen der Ungleichung, wissen wir, dass die Quad. Polynome von grad ≤ 3 exakt integriert.

Eine basis dieser Polynome wäre $\{1, (x-m), (x-m)^2, (x-m)^3\}$.

Wir sehen, dass meine Quadratur $\{1, (x-m), (x-m)^3\}$ exact integriert für alle α.

Setze $g(x) = (x-m)^2$ dann wollen wir:

$$\alpha (b-a)^3 g''(m) \overset{!}{=} \int_a^b g(x)\,dx$$

$$\alpha (b-a)^3\, 2 = \int_a^b (x-m)^2\,dx$$

$Q_{[a,b]}[f] := (b-a) f\left(\frac{a+b}{2}\right) + \alpha (b-a)^3 f''\left(\frac{a+b}{2}\right)$

$\hookrightarrow Q[g] = (b-a)\underbrace{g(m)}_{=(m-m)^2=0} + \alpha (b-a)^3 g''(m)$

$$\Rightarrow \alpha = \int_a^b \frac{(x-m)^2}{2}\,dx \,\frac{1}{(b-a)^3} = \frac{1}{24}$$

Dann: $\int_a^b f(x)\,dx - \left[(b-a) f(m) + \alpha (b-a)^3 f''(m)\right] = \ldots$

$$\ldots = \int_a^b f(m) + f'(m)(x-m) + f''(m)\frac{1}{2}(x-m)^2 + \frac{f^{(3)}(m)}{3!}(x-m)^3 + \frac{f^{(4)}(\xi_x)}{4!}(x-m)^4\,dx$$

$$- (b-a) f(m) - \underbrace{\int_a^b \frac{(x-m)^2}{2}\,dx \,\frac{1}{\cancel{(b-a)^3}}}_{=\alpha} \cancel{(b-a)^3} f''(m)$$

$$= \cancel{(b-a) f(m)} + \underbrace{\int_a^b f'(m)(x-m)\,dx}_{=0} + \cancel{\int_a^b \frac{f''(m)}{2}(x-m)^2\,dx} + \underbrace{\int_a^b \frac{f^{(3)}(m)}{3!}(x-m)^3\,dx}_{=0} + \int_a^b \frac{f^{(4)}(\xi_x)}{4!}(x-m)^4\,dx$$

$$- \cancel{(b-a) f(m)} - \cancel{\int_a^b \frac{(x-m)^2}{2}\,dx\, f''(m)}$$

$$= \int_a^b \underbrace{f^{(4)}(\xi_x)}_{\leq \|f^{(4)}\|_{\infty,[a,b]}} \frac{1}{4!}(x-m)^4\,dx \leq \frac{1}{(2^4 \cdot 4!)} \|f^{(4)}\|_{\infty,[a,b]} (b-a)^5 \quad \square$$

b) Sei $N \in \mathbb{N}$ und C_0, $Q_{[a,b]}$ wie oben. Für $x = (x_j)_{j=0}^{N}$ mit $0 \leq x_0 < \ldots < x_N \leq 1$,

wir betrachten den "composite quadrature rule":

$$Q_X[f] = \sum_{i=0}^{N-1} Q_{[x_i, x_{i+1}]}[f] \qquad \text{auf } [0,1].$$

Für $i = 0, \ldots, N$ setze $y_i = \frac{i}{N}$ und $y = (y_i)_{i=0}^{N}$.

Zeige $\forall f \in C^4([0,1])$:

$$\left| \int_0^1 f(x)\,dx - Q_Y[f] \right| \leq C_0 N^{-4} \|f^{(4)}\|_{\infty,[0,1]}$$

Solution: Sei $k = 4$, $h = \frac{1}{N}$

Δ-Ungl. und $\int_0^1 f(x)\,dx = \sum_{j=0}^{N-1} \int_{y_j}^{y_{j+1}} f(x)\,dx$

$$\left| \int_0^1 f(x)\,dx - Q_Y[f] \right| \leq \sum_{j=0}^{N-1} \left| \int_{y_j}^{y_{j+1}} f(x)\,dx - Q_{[y_j, y_{j+1}]} \right|$$

$$\overset{a)}{\leq} \sum_{j=0}^{N-1} C_0 (\underbrace{y_{j+1} - y_j}_{= h = 1/N})^{k+1} \underbrace{\|f^{(k)}\|_{\infty,[y_j,y_{j+1}]}}_{\leq \|f^{(k)}\|_{\infty,[0,1]}}$$

$$\leq \underbrace{N}_{= h^{-1}} C_0 \|f^{(k)}\|_{\infty,[0,1]} h^{k+1} = C_0 \|f^{(k)}\|_{\infty,[0,1]} N^{-k} \qquad \square$$

c)