Topological Spaces

Def: Given a set X, a topology on X is a collection of subsets of X called open sets such that:

1) $O_\alpha \in \mathcal{O}, \forall \alpha \in A \Rightarrow \bigcup_{\alpha \in A} O_\alpha \in \mathcal{O}$
2) $O_\alpha \in \mathcal{O}, \forall \alpha \in A, A$ finite $\Rightarrow \bigcup_{\alpha \in A} O_\alpha \in \mathcal{O}$
3) $\phi, X \in \mathcal{O}$.

Def given a top. space $(X, \mathcal{O})$ a collection $\mathcal{B}$ of open sets is called basis for $\mathcal{O}$ if every set can be presented as union of open sets in $\mathcal{B}$. ($\forall U \in \mathcal{O}\ \exists \{B_\alpha\}_{\alpha \in A}$ s.t $U = \bigcup_{\alpha \in A} B_\alpha$

Prop: given X and $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying a) and b), then $\exists$ a topology $\mathcal{O}$ on X having $\mathcal{B}$ as a basis.

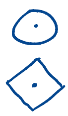
In fact $\mathcal{O} := \{$union of sets in $\mathcal{B}\}$ $\leftarrow$ (giving a topology by means of a basis)

a) $\forall x \in X\ \exists B \in \mathcal{B}$ s.t $x \in B$

b) $\forall B_1, B_2 \in \mathcal{B}$ if $\exists x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ s.t $x \in B_3 \subset B_1 \cap B_2$

Achtung! "different distances may determine different bases both inducing the same topology"

Example: $X = \mathbb{R}^2$, distances $d_2(x,y) = (\sum_i |x_i - y_i|^2)^{1/2}$, $d_1(x,y) = (\sum |x_i - y_i|)$ $\quad \mathcal{B}_2$ $\quad \mathcal{B}_1$



but they both induce Eucl. top.

Lemma: if $\mathcal{B}$ is a basis for $\mathcal{O}$, $\mathcal{B}'$ —— // —— $\mathcal{O}'$ $\}$ top on X and $\forall B \in \mathcal{B}, \forall x \in B\ \exists B' \in \mathcal{B}'$ s.t $x \in B', B' \subset B$

• (symm cond.)

then $\mathcal{O} = \mathcal{O}'$

Def A topo. space $(X, \mathcal{O})$ is called connected if any of the following 3 equivalent cond. holds true.

① X cannot be decomposed as the union of two disjoint non-empty open sets
② X cannot be decomposed as the union of two disjoint non-empty closed sets
③ the only sets $A \subset X$ that are both open and closed are $A = \phi$, $A = X$

Prop $f: X \to Y$ continous and surj. then: X connected (path-con.) $\Rightarrow$ Y connected (path-con)

Prop If a top. space is path-connected then it is connected.

Prop: X connected and locally path connected, then it is path-connected

$\forall x \in X\ \forall \mathcal{U}$ open w/ $x \in \mathcal{U}$, $\exists V \subset \mathcal{U}$ open path-connected

Prop: (baby Tychonoff)

X, Y are compact $\Leftrightarrow$ $X \times Y$ is compact

Def A topo space X is called Hausdorff (T_2) if $\forall x_1 \neq x_2 \in X$ $\exists \begin{cases} O_1 \ni x_1 \\ O_2 \ni x_2 \end{cases}$ open nbhd w/ $O_1 \cap O_2 = \emptyset$

Prop: A compact subspace of a Hausdorff space is closed.

Prop: (homeo Criterion)

$f: X \to Y$ bij. cont. If X compact and Y Hausdorff then f is a homeomorphism

Def: X is normal (or T_4) if it is T_2, plus $\forall A_1, A_2$ closed $A_1 \cap A_2 = \emptyset$ $\exists \begin{cases} U_1 \supset A_1 \\ U_2 \supset A_2 \end{cases}$ open sets

T_2 and not T_4: $X = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$
$\mathcal{T}$ = open balls in $y > 0$ &
for $y=0$: Ball $\cup \{(x,0)\}$
(to show consider $A \cup B$ ← rationals, ← irrationals)

Prop: A compact Hausdorff metric top. space is normal.

Prop: Every open cover of a compact metric space has a Lebesgue number

Def: X, Y metric spaces, $f: X \to Y$ is called unif. cont. if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t
$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$.

Prop: X comp. metric space, Y metric space
then $f: X \to Y$ cont $\Rightarrow$ f unif. cont.

Quotient Spaces

Prop: Let $h: X \to Y$ be a continuous surj. map w/ X compact, Y Hausdorff.
Then it is always true $U \subseteq Y$ is open $\Leftrightarrow$ $h^{-1}(U) \subseteq X$ is open.

Def the quotient topo on $X/\sim$ is defined by declaring $U \subseteq X/\sim$ is open $\Leftrightarrow$ $p^{-1}(U) \subseteq X$ open

Def X, Y be topo. spaces. A surj map $f: X \to Y$ is called a quotient map
if $U \subset Y$ open $\Leftrightarrow$ $f^{-1}(U) \subset X$ open

not open in general: $[-2,2]$, $\{-1\} \sim \{1\}$
$V = (\frac{1}{2}, \frac{3}{2}) \to f(U)$ not open

review: X comp. Y Hausdorff, then any surj. cont map is a quotient map.

Achtung: quotient of Hausdorff top. space may not be Hausdorff.

- We can define an equivalence relation on X denoted by $\sim$

$x_1 \sim x_2 \iff f(x_1)=f(x_2)$: In this case, at the level of sets there is an induced bijection

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \\ & X/_\sim \arrow[u, dashed, "\tilde{f}"'] \end{tikzcd} \qquad \tilde{f}([x]) = f(x)\ ,\ \tilde{f}: X/_\sim \to Y \text{ bijection}$$

Def X, Y sets, $f: X \to Y$ map

given $V \subset X$ subset we call saturation of U, the set $\tilde{V} := f^{-1}(f(V))$

Def. top space X is called topological n-dim manifold if

a) It is Hausdorff

G.Bsp: (loc. Eucl. but not T_2)
$\mathbb{R} \times \{+,-\}$, $(x,-) \sim (x,+)$, $\forall x > 0$

b) any $x \in X$ has an open neighbourhood $U(x)$ that is homeomorphic to $\mathbb{R}^n$

$\hookrightarrow$ loc. path connected, if comp $\Rightarrow$ finitely many connected comp.

Thm: A compact, connected 1-dim manifold is homeo. to S^1

Key points of the proof

Def: Let X be a top space, $A \subset X$ be a top. subspace ; let Y be a top. space.

We say that two cont. functions $f_0, f_1 : X \to Y$ are homotopic (rel. to A)

if $\exists F: X \times I \to Y$ cont. such that i) $F(x,0) = f_0(x)$
$F(x,1) = f_1(x)$, $\forall x \in X$

ii) $F(a,t) = f_0(a) = f_1(a)$, $\forall a \in A$, $\forall t \in [0,1]$

If $A = \emptyset$ then one drops ii) and we just say f_0, f_1 are homotopic.

Def: Two top. spaces X, Y are called homotopic if $\exists f: X \to Y$ and $g: Y \to X$

such that $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$

Rem: $X \overset{\text{homeo}}{\cong} Y \Rightarrow X \simeq Y$

Def: A top. space X is contractible if it is homotopic to a point.

Def We define the fundamental group of a top. space X w.r.t $x_0 \in X$ as the set of all

homotopy classes of loops based at x_0 with operation $[f] \cdot [g] := [f \cdot g]$

Notation : $\pi_1(X, x_0)$ $\quad$ ($\Omega(X, \overset{\text{basepoint}}{x_0})$ space of loops in X)

Def A top. space X is called simply connected if a) it is path connected
b) it has trivial fundamental group.

Prop: Let X, Y be top. spaces and let $f, g: X \to Y$ be homotopically equivalent

Given any $x_0 \in X$ there is an isomorphism $\varphi: \pi_1(Y, f(x_0)) \to \pi_1(Y, g(x_0))$

that makes the following diagram commute:

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[d, dashed, "\varphi"] \\ \pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[r, "g_*"'] & \pi_1(Y, g(x_0)) \end{tikzcd}$$

(i.e $g_* = \varphi \circ f_*$)

Def Let $X, \tilde{X}$ be top. spaces and $p: \tilde{X} \to X$.

We say $U \subseteq X$ open is evenly covered if $p^{-1}(U) = \coprod_{i \in I} U_i$ (disj. union)

and $p_{|U_i}: U_i \to U$ homomorphism

Def Let $X, \tilde{X}$ be top. spaces and let $p: \tilde{X} \to X$. We say that p is a covering map if $\forall x \in X$

$\exists U = U(x)$ open nbhd of x that is evenly covered

Def Let X, Y be top. spaces and $f: X \to Y$. Then $f^{-1}(y)$ is called fiber of y.

Prop: Let X, Y be top. spaces, Y connected.

Assume $f: X \to Y$, then: 1) If f is a covering map, then it is a local homeomorphism and the cardinality of $f^{-1}(y)$, does not depend on y.

2) If f is a local homeomorphism and the fibers $f^{-1}(y)$ have finite cardinality (whose value doesn't depend on y) then f is a covering map

Recall: $f: X \to Y$ local homeo, if $\forall x \in X \ \exists \begin{cases} U = U(x) \text{ open} \\ V = V(f(x)) \text{ open} \end{cases}$ w/ $f_{|U}: U \to V$ homeo.

Def: Let $p: \tilde{X} \to X$ be a covering map and let $f: Y \to X$, $\tilde{f}: Y \to \tilde{X}$ (cont.)

If $f = p \circ \tilde{f}$ we say $\tilde{f}$ is a lift of f

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\equiv"] & X \end{tikzcd}$$

Thm (uniqueness, Prop. 1.34): Let Y be connected. If $\tilde{f}_1, \tilde{f}_2: Y \to \tilde{X}$ are lifts of f.

and $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ for some $y_0 \in Y \Rightarrow \tilde{f}_1 \equiv \tilde{f}_2$

Lift paths

<u>Prop</u>: Let $p: \tilde{X} \to X$ be a covering map. Given a path $\alpha: I \to X$ and $\tilde{x}_0 \in p^{-1}(\alpha(0))$
$\exists!$ lift $\tilde{\alpha}: I \to \tilde{X}$ of α w/ $\tilde{\alpha}(0) = \tilde{x}_0$

Lift homos.

<u>Prop</u>: Let $p: \tilde{X} \to X$ be a covering map. Given $H: I \times I \to X$ homotopy of loops based at x_0 and $\tilde{x}_0 \in p^{-1}(x_0)$ $\exists!$ lift $\tilde{H}: I \times I \to \tilde{X}$ homotopy of paths (all) starting at $\tilde{x}_0$.

<u>Cor</u> (Monodromy theorem)

Let $p: \tilde{X} \to X$ be a covering map and $\tilde{\alpha}, \tilde{\beta}: I \to \tilde{X}$ be paths. Assume that $\alpha := p \circ \tilde{\alpha}, \beta = p \circ \tilde{\beta}$ are homotopic w/ fixed endpoints in X. Then: $\tilde{\alpha}(0) = \tilde{\beta}(0) \Leftrightarrow \tilde{\alpha}(1) = \tilde{\beta}(1)$

W(S): set of all words (finite seq. of letters in S)

<u>Def</u> The free product of two groups H and K is $H * K := W(H \amalg K)/\sim$ where $p \sim p'$ if p' can be obtained from p via a finite chain of moves, which are:

M1) if $p = x_1 \dots x_m$ and $x_i, x_{i+1} \in H$ (or $\in K$) then I replace $x_i x_{i+1}$ by their product in H

M2) if $p = x_1 \dots x_m$ and $x_i = e_H$ (or e_K) then I remove the letter from the string p.

M1)' replace x by $x_i x_{i+1}$ (if $x_i x_{i+1} = x$ in H or in K)

M2)' insert identity element.

Cor $\Rightarrow$ not abelian, $Ab(\mathbb{Z} * \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$

<u>Def</u> We define the free group (of rank two) as $\mathbb{Z} * \mathbb{Z}$ (i.e $H = K = \mathbb{Z}$)

<u>Van Kampen</u>: suppose $X = \bigcup_\alpha A_\alpha$ w/ A_α path connected open in X, $\forall \alpha$.

Suppose $\exists x_0 \in X$ w/ $x_0 \in \bigcap_\alpha A_\alpha$
basepoint for fund. groups.

Then $\exists$ maps induced by the inclusions: $j_\alpha: \pi_1(A_\alpha) \to \pi_1(X)$.
$i_{\alpha\beta}: \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$

By prev. discussion there is a group homo $\Phi: *_\alpha \pi_1(A_\alpha) \to \pi_1$ extending the single maps j_α.

Theorem:

- If X is the union of path-connected open sets each containing the basepoint $x_0 \in X$ and if $A_\alpha \cap B_\alpha$ is path-connected then the homo $\Phi: * \pi_1(A_\alpha) \to \pi_1(X)$ is surjective
- If in addition all triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ are also path connected, then the kernel of the map Φ equals the normal subgroup N generated by all elements of the form $\iota_{\alpha\beta}(\omega)\iota_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Thus there is a group isom. $\pi_1(X) \simeq * \pi_1(A_\alpha)/_N$

Def: Let X be a top. space, let $A \subset X$ and let $\iota: A \to X$ be the inclusion. We say that $\rho: X \to A$ is a retraction if $\rho \circ \iota = id_A$

Key fact: assume that X is path-connected (which implies A is aswell)

Then
$$(\rho \circ \iota)_* = (id_A)_*$$
$$\rho_* \circ \iota_* = id_{\pi_1(A)}$$

$$\Rightarrow \begin{cases} \iota_*: \pi_1(A) \longrightarrow \pi_1(X) & \text{injective} \\ \rho_*: \pi_1(X) \longrightarrow \pi_1(A) & \text{surjective} \end{cases}$$

Def. In the setting of the previous definition, we say that A is a deformation retract if there is a homotopy $R: X \times I \to X$ w/
$$\begin{cases} R(x,0) = x & \\ R(x,1) = \iota \cdot \rho(x) & \forall x \in X \\ R(a,t) = a & \forall a \in A, \forall t \in I \end{cases}$$

Rmks:

1) In part. $\iota \circ \rho: X \circlearrowleft$ is homotopic to the identity map on X.
2) Saying a space X deformation retracts onto one of its points is strictly stronger than saying X is contractible

Key fact: Assume X is path-connected then $(\iota \circ \rho)_* = (id_X)_*$ hence $\iota_* \circ \rho_* = id_{\pi_1(X)}$

so $\iota_*: \pi_1(A) \longrightarrow \pi_1(X)$, $\rho_*: \pi_1(X) \longrightarrow \pi_1(A)$ are isomorphisms

Covering spaces and Fundamental group

Prop.: Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a cover. map Then the induced map $p_*:\pi_1(\tilde{X},\tilde{x}_0)\to\pi_1(X,x_0)$ is injective

The subgroup $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ consists of the equiv. classes of those loops in (X,x_0), whose lifts in $(\tilde{X},\tilde{x}_0)$ are loops.

Prop. 2: The degree of a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ equals the Index of $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$

Def We say that a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ is normal if $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ is a normal subgroup in $\pi_1(X,x_0)$ at least for some choice of $x_0\in X$, $\tilde{x}_0\in p^{-1}(x_0)$

Prop: Let $p:\tilde{X}\to X$ be a covering map. Let $\tilde{x}_0,\tilde{x}_1\in\tilde{X}$, $x_0=p(\tilde{x}_0)$, $x_1=p(\tilde{x}_1)$

Then there exists group Isom. that make this diagram commute

$$\begin{tikzcd}
\pi_1(\tilde{X},\tilde{x}_0) \arrow[r, "\tilde{\varphi}"] \arrow[d, "p_*"] & \pi_1(\tilde{X},\tilde{x}_1) \arrow[d, "p_*"] \\
\pi_1(X,x_0) \arrow[r, "\varphi"] & \pi_1(X,x_1)
\end{tikzcd}$$

Lifting criterion and existence of coverings

Prop: Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space and let $f:(Y,y_0)\to(X,x_0)$ be a cont. map; assume the domain Y is connected and locally path-connected

Then: a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ exists $\iff$ $f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\tilde{X},\tilde{x}_0))$

Prop: Given a covering space $p:\tilde{X}\to X$ and a map $f:Y\to X$ if two lifts $\tilde{f}_1,\tilde{f}_2:Y\to\tilde{X}$ agree at one point $y\in Y$ and Y connected then $\tilde{f}_1\equiv\tilde{f}_2$

$\forall x\in X\ \exists$ p-c nbhd s.t. every loop in U is Nullhomd. in X.

Prop: Suppose X is path connected locally path-connected and semilocally simply connected.

Then for every subgroup $H<\pi_1(X,x_0)$ there is a covering space $p:X_H\to X$ such that $p_*(\pi_1(X_H,\tilde{x}_0))=H$ for suitable $\tilde{x}_0\in X_H$

Uniqueness of covering spaces

Def: Given a top. space X and covering spaces $p_1 : \tilde{X}_1 \longrightarrow X$, $p_2 : \tilde{X}_2 \longrightarrow X$.

We say the first one is isomorphic to the second, if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 f$ (f: Isom. of covering spaces)

Commutative diagram:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & /\!/\!/ & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Fact: this defines an equi. rel. in the class of covering spaces of a given top. space X

$(p_1, \tilde{X}_1) \overset{f}{\sim} (p_1, \tilde{X}_1)$, $f = id$

$(p_1, \tilde{X}_1) \overset{f}{\sim} (p_2, \tilde{X}_2) \overset{f^{-1}}{\Longrightarrow} (p_2, \tilde{X}_2) \overset{f^{-1}}{\sim} (p_1, \tilde{X}_1)$

$(p_1, \tilde{X}_1) \overset{f}{\sim} (p_2, \tilde{X}_2)$, $(p_2, \tilde{X}_2) \overset{g}{\sim} (p_3, \tilde{X}_3) \Longrightarrow (p_1, \tilde{X}_1) \overset{g \circ f}{\sim} (p_3, \tilde{X}_3)$

Prop (uniqueness):

If X is connected, locally path-connected, then two path-connected covering spaces $p_1 : \tilde{X}_1 \longrightarrow X$, $p_2 : \tilde{X}_2 \longrightarrow X$ are isomorphic via an isom. $f : \tilde{X}_1 \longrightarrow \tilde{X}_2$ taking a basepoint $\tilde{x}_1 \in p^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p^{-1}(x_0)$ iff $(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$

Thm (Galois correspondence)

has to do with existence !!!

Let X be path-connected, locally path-connected, semilocally simply connected. Then there is a bijection between the set of basepoint-preserving isom. classes of path-connected pointed covering spaces $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}_1, \tilde{x}_1))$ to the covering space in question.

Def a simply connected covering space of a topological space X is called universal cover

Facts: If X is path connected, locally path-connected, semilocally simply-connected $\Rightarrow$ U.C exists

If X is path connected, locally path-connected $\Rightarrow$ U.C (when it exists) is unique up to isom

Universal, why? the universal cover is actually a cover of any other covering space of the same X in the following sense:

$$\begin{tikzcd} & \tilde{X}' \arrow[d, "p'"] \\ \tilde{X} \arrow[ur, dashed, "q"] \arrow[r, "p"'] & X \end{tikzcd}$$

check: $q: \tilde{X} \to \tilde{X}'$ is itself a covering map!

Deck transformations

<u>Def</u>: Let $p: \tilde{X} \to X$ be a covering space. We call deck transformation (of that covering space) an isom. of the covering itself i.e a homeo $g: \tilde{X} \circlearrowleft$ w/ $pg = p$

Two basic facts: ① the set of deck transformations (of a given covering space) forms a group under composition denoted $G(\tilde{X})$

② Assume $X, \tilde{X}$ are path-connected and locally path-connected. Then a deck transf. has no fixed points (unless it is the identity)

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, dashed, "g"] \arrow[r, "p"'] & X \end{tikzcd}$$

- A deck trans. g is in part a lift of p
- id. is also a lift.

If g has a fixed point, $g(\tilde{x}) = \tilde{x}$ then $g = id$ (uniqueness of lifts)

<u>Differently phrased</u>: a deck transformation is uniquely determined by its value at a point

Examples:

a) $X = S^1$, $\tilde{X} = \mathbb{R}$, $p: \tilde{X} \to X$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$

$G(\tilde{X})$ = integer translations $\cong \mathbb{Z}$.

b) $X = S^1$, $\tilde{X} = S^1$, $p: \tilde{X} \to X$, $z \mapsto z^n$

$G(\tilde{X})$ = rotations of an integer multiple of $2\pi/n \cong \mathbb{Z}/n\mathbb{Z}$

<u>Achtung</u>! In both cases above $\pi_1(X) / p_*(\pi_1(\tilde{X})) \cong G(\tilde{X})$

Fundamental theorem on deck Transformations

Thm:

Let $p:(\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a path-connected covering space of the path-connected, loc. path-connected space X and let H be the subgroup $p_*(\pi_1(\tilde{X},\tilde{x}_0)) < \pi_1(X,x_0)$

Then: (i) this covering space is geom. normal i.e. $\forall x \in X, \forall \tilde{x},\tilde{x}' \in p^{-1}(x)$ there is a deck transformation $f: \tilde{X} \circlearrowleft$ w/ $f(\tilde{x}) = \tilde{x}'$ iff it is alg. normal in the sense that $H \triangleleft G = \pi_1(X,x_0)$

(ii) If the covering is normal then $G(\tilde{X})$ is isom. to $\pi_1(X,x_0)/H$

Comments: • If $\tilde{X}$ is a universal cover (simply connected) then $\underbrace{\pi_1(X,x_0)}_{\text{fund. group of the base}} \cong \underbrace{G(\tilde{X})}_{\text{Automorphisms of the Universal cover}}$

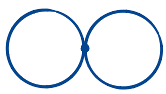
•• If $p:\tilde{X} \to X$ is not normal then the replacement for (ii) is

(ii)': $G(\tilde{X})$ is isom. to the quotient $N(H)/H$, where $N(H)$ normaliser of H inside $\pi_1(X,x_0)$

largest subgroup $K < G$, where H is normal

• $N(H) = G \iff H \triangleleft G$.

An example of a covering space that is not normal



X

$\pi_1(X) \cong \underbrace{\mathbb{Z} * \mathbb{Z}}_{G}$ $\qquad H = \langle ab \rangle \cong \mathbb{Z}$

$a(ab)a^{-1} = a^2ba^{-1} \notin H$.

• H is not normal in G: $aHa^{-1} \not\subset H$

• Invoke "abstract existence result" for X_H.

Lemma (geometric effects of conjugacy)

In the setting above, if $g = [\gamma]$ for $\gamma = p \circ \tilde{\gamma}$, where $\tilde{\gamma}: I \to \tilde{X}$ connects $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ have $g^{-1}Hg = p_*(\pi_1(\tilde{X},\tilde{x}_1))$

Def Given Y top space and group G, an action of G on Y is a group homomorphism $\rho: G \to \text{Homeo}(Y)$

Basic postulate: (properly discontinuous action)

(*) $\forall y \in Y \ \exists U = U(y)$ nbhd. such that the collection $\{g(U)\}_{g \in G}$ consists of pairwise disjoint sets.

$[g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2]$

Thm: If an action of a group G on a top. space Y satisfies (*) then:

(a) the quotient map $p: Y \to Y/G$ is a normal covering map.

(b) if Y is path-connected, then G coincides w/ the group of deck transformations of this covering

(c) If Y is path-connected and locally path-connected then

$$\underset{\text{iso}}{G \cong} \frac{\pi_1(Y/G)}{p_*(\pi_1(Y))}$$

Stuff from exercise classes and problem sets

Def: X is first countable if each point has a countable basis of nbhds

Def: X is second countable if it admits a countable basis for its topology
$\hookrightarrow$ first count. & seperable

Def: $\mathcal{T}_{coc} := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y$ is countable$\}$ is called the cocountable topology

It coincides w/ discrete topo $\Leftrightarrow$ X countable
$\hookrightarrow$ not Hausdorff (if X uncount.) but limits are still unique.

Prop. $f: X \to Y$, X first countable. Suppose whenever $x_n \to x$, $f(x_n) \to f(x)$. Then f is cont.

Topo. Space $\not\Rightarrow$ Metric space: $(\mathbb{R}, \mathcal{T})$ w/ topol. given by $\{\{I_x\}_{x\in\mathbb{R}}, \emptyset, \mathbb{R}\}$, where $I_x := (x, \infty)$ is not Hausdorff thus also not metric space.

iff
- $d(x+v, y+v) = d(x,y)$
- $d(\lambda x, \lambda y) = |\lambda| d(x,y)$

Metric $\not\Rightarrow$ Norm: discrete metric: $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ cant induce a norm cuz: $\|x\| \overset{x\neq 0}{=} d(x,0) = 1$ and $\|\alpha \cdot x\| = |\alpha| \cdot \|x\| = |\alpha| \neq d(\alpha x, 0)$

Def: $\mathcal{F} \subseteq \mathcal{C}(I)$ is called unif. bounded if $\exists M \in \mathbb{R}$ s.t $\underline{|f(x)| \leq M}$, $\forall f \in \mathcal{F}$, $\forall x \in I$

Def: $\mathcal{F} \subseteq \mathcal{C}(I)$ is called unif. equicontinuous if $\forall \varepsilon > 0$ $\exists \delta_\varepsilon$ s.t whenever $|x-y| < \delta_\varepsilon$, we have $|f(x) - f(y)| < \varepsilon$, $\forall f \in \mathcal{F}$

Thm (Ascoli-Arzelá): $\mathcal{F}$ is relatively compact $\Leftrightarrow$ $\mathcal{F}$ is unif. bounded & equicont.

Def: A space X is seq. compact if for every seq. $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ admits a converging subseq.

Def: A space X is totally bounded if $\forall \varepsilon > 0$ $\exists$ finite set of points $x_1, \dots, x_k \in X$ s.t $X \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon)$

Prop. (UE of quotients)

X top space, $\sim$ eq. rel on X, $\pi : X \to X/\sim$ projection

Let $f : X \to Y$ cont. s.t $\forall x_1, x_2 \in X \quad x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

then $\exists! \; \bar{f} : X/\sim \to Y$ s.t $\bar{f} \circ \pi = f$

Prop

$q : X \to Y$ quotient map and $f : X \to Z$ cont. and induces bij. $\hat{f} : Y \to Z$

Then $\hat{f}$ cont but <u>not</u> necessarily a homeo.

$\llcorner$ $X = Y = (\mathbb{R}, \tau_{std})$, $Z = (\mathbb{R}, \tau_{triv})$, $f = q = \hat{f} = id_{\mathbb{R}}$

$$\begin{tikzcd}
(\mathbb{R}, \tau_{std}) \arrow[d] \arrow[dr] & \\
(\mathbb{R}, \tau_{std}) \arrow[r, "id"'] & (\mathbb{R}, \tau_{triv})
\end{tikzcd}$$

$\llcorner$ <u>not</u> homeo!

cuz $(\mathbb{R}, \tau_{triv.})$ not T_2.

Lemma: X connected, $p : \hat{X} \to X$ cont.

1) If p is a covering map, then it is a local homeo and the degree of p is constant.

2) If p is a local homeo <u>and</u> $p^{-1}(x)$ is finite $\forall x$, then p is a covering map.

Def: X any top. space $A \subseteq X$ is a discrete subset if $\forall x \in X \; \exists U$ nbhd of x s.t

$(U \setminus \{x\}) \cap A = \emptyset$