

Control Systems I:

Signals & Systems:

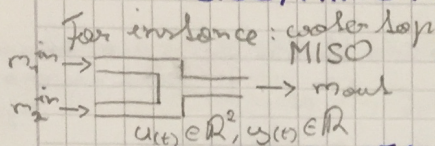
→ Definition:

- Signal: Time dependent information $u(t): \mathbb{T} \rightarrow \mathbb{W}$
- System: Defines a causality between signals, i.e. absorbs input → generates output
- Model: Mathematical description of a system

→ Representation through Blockdiagrams:

→ System Properties:

→ SISO/MIMO: Depends on the dimension of



(input) & Output: SISO: $u, y \in \mathbb{R}$ (scalar)

MIMO: $u \in \mathbb{R}^m, y \in \mathbb{R}^n$ (column vectors)

→ NON-/LINEAR: A system that manipulates an input ($y(t) = \Sigma u(t)$) is linear - and only if: $\Sigma(\alpha u_1 + \beta u_2) = \alpha \Sigma u_1 + \beta \Sigma u_2$

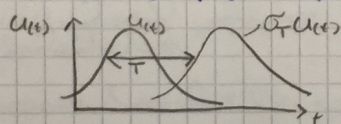
→ (STRICTLY) CAUSAL: The output in an instant t depends

only on the input during $(-\infty, t]$

With $\gamma_T u(t) = \begin{cases} u(t) & t \leq T \\ 0 & t > T \end{cases}$

o system is causal if $\Sigma \gamma_T u(t) = \Sigma u(t)$ (strictly: $(-\infty, t]$)

→ TIME IN-/VARIANT: A system is time invariant, if



$\Sigma \sigma_T u = \sigma_T \Sigma u$ for the time shifting operator σ_T , time variant otherwise

→ STATIC vs. DYNAMIC:

i) A system is static, if it's memoryless, meaning the output only depends on the input at that instant, i.e. there are no reservoirs (for instance: water loop)

ii) A system is dynamic, if there are some reservoirs (inner state variables) and the output depends on both, input as well as those inner state variables (for instance water tank)

→ ORDER: The order of a system is defined by its number of inner state variables $\Leftrightarrow$ order of its ODE

DIMENSION: Smallest possible order to completely describe the system!

For instance: $\dot{x}(t) = A x(t) + B u(t)$ with $x \in \mathbb{R}^n$: n^{th} order

→ DEGREE OF FREEDOM: "Number of control functions"

(for example our typical system - above - DOF = 2, namely F & C)

→ Notes: → Control objectives: "stabilization": The system doesn't blow up

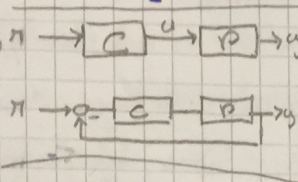
"regulation": Keep control of disturbance

"tracking": Follow a specific reference trajectory

This course: $\mathbb{T} = \mathbb{R}$ ("time")
 $\mathbb{W} = \mathbb{R}^n$ ("Signal Space")
 For SISO-Systems $n=1$

reference trajectory

FEED-FORWARD
 FEED-BACK



TYPICAL SYSTEM OF

THIS COURSE: π controlled input

e error
 u controlled signal
 d disturbance

Modeling a dynamic system:

↳ Definition: (comp. dynamic system) Every causal system depends on all the inputs during $(-\infty, t]$. The STATE of such a system is a set of (for a system of order n at least) n variables, that describe that inputs completely. $x(t) \in \mathbb{R}^n$

THIS SET IS NOT UNIQUE! ← INFINITE "REALIZATIONS"

↳ Remark: We'll only deal with finite-dimensional systems, however there are some example of infinite-dimensional systems (time-delays, continuous state of variables → rivers, ...)

Solution → DISCRETIZE

↳ Continuous-time State Space Model (valid for all finite dimensional, causal, time invariant systems)

$$\frac{d}{dt} z(t) =: \dot{z} = f(z(t), u(t))$$

Inner state z Input u

$$y = g(z(t), u(t))$$

Output y

↳ This State-Space-Model can be obtained by the following general MODELING

GUIDELINES: i) Identify boundaries (what are inputs/outputs)

Recall from Linear Algebra:

If there turns up an ODE of higher order during the modeling process it always possible to rewrite them as a set of first order ODEs!

ii) Identify relevant reservoirs storing energy, mass, charge, ... & their level variables (for instance:

reservoir: energy; level variables: kinetic/potential/internal rotational... energy)

iii) Formulate the ODEs for all relevant reservoirs:

$$\frac{d}{dt} (\text{reservoir content}) = \sum \text{inflows} - \sum \text{outflows}$$

:= INNER STATE

Example: $\ddot{z} = \alpha z + \beta \dot{z} + \gamma$ Equations that relate all relevant reservoirs must be found (expressed in terms of the level variables)

$$x_1 = z, x_2 = \dot{z}$$

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$$

iv) Identify system parameters

vi) Validate the model

PROBABLY NOT WITHIN THIS COURSE

↳ These guidelines yield, as mentioned, the State-Space-Model, which in general is NOT normalized/linear.

↳ NORMALIZATION:

In practice numeric tools (f.i.: MATLAB) will be used → it is good practice to normalize the equations, such that all variables have common values (~ 1). These nominal values are

free of choice, one example could be: $z_{i,0} = z_{i,\max}; u_{i,0} = u_{i,\max}; \dots$

→ Input: $u(t) = \frac{v(t)}{v_0}$

→ Output: $y(t) = \frac{w(t)}{w_0}$

→ State: $x(t) = \frac{z(t)}{z_0}$

→ All different for each i
→ May vary over time

↳ LINEARIZATION:

→ Our analysis will be possible for linear systems only, therefore we will linearize the State-Space-Model at some special operating points → equilibrium points, satisfying

normalized function $f_0(x_e, u_e) = 0$, $(x_e, u_e) \dots$ Equilibrium Points $\begin{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{bmatrix}$
 ↳ Remember: State-Space-ODE: $\dot{x} = f_0(x, u)$
 ↳ If non-normalized w.r. to this point, linearization will be around $x_{0,e}$ $u_{0,e}$

→ Next thing to do is shifting the origin into the obtained equilibrium points, which is equal to Taylor-expansion around that point. Truncating higher order terms yields the Jacobian matrices A, B, C, D .

→ The State-Space-Model becomes to:

$\dot{x} = f_0(x, u)$
 $y = g_0(x, u)$
 LINEARIZATION AT $(x_e, u_e) \Rightarrow \dot{x} = A \cdot x + B \cdot u$
 $y = C \cdot x + D \cdot u$

where: $A = \frac{\partial f_0}{\partial x} \Big|_{x=x_e, u=u_e} = \begin{pmatrix} \frac{\partial f_{0,1}}{\partial x_1} & \dots & \frac{\partial f_{0,1}}{\partial x_n} \\ \frac{\partial f_{0,2}}{\partial x_1} & \dots & \frac{\partial f_{0,2}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{0,n}}{\partial x_1} & \dots & \frac{\partial f_{0,n}}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{n \times n}$
 For TIME-INVARIANCE $A \neq A(t)$
 $B \neq B(t)$
 $C \neq C(t)$
 $D \neq D(t)$
 System's dimension

$B = \frac{\partial f_0}{\partial u} \Big|_{x=x_e, u=u_e} = \begin{pmatrix} \frac{\partial f_{0,1}}{\partial u} \\ \frac{\partial f_{0,2}}{\partial u} \\ \vdots \\ \frac{\partial f_{0,n}}{\partial u} \end{pmatrix} \in \mathbb{R}^{n \times 1}$ (for SISO)

$C = \frac{\partial g_0}{\partial x} \Big|_{x=x_e, u=u_e} = \begin{pmatrix} \frac{\partial g_0}{\partial x_1} & \dots & \frac{\partial g_0}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{1 \times n}$ (again only for SISO)

$D = \frac{\partial g_0}{\partial u} \Big|_{x=x_e, u=u_e} = \frac{dg_0}{du}(u_e) \in \mathbb{R}$

→ Note: $[f_{0,1}; f_{0,2}; f_{0,3}; \dots; f_{0,n}]$ is the matrix obtained from the decomposition of the higher order system's ODE into a system of first order ODEs.

↳ LTI - Continuous State-Space-Model:

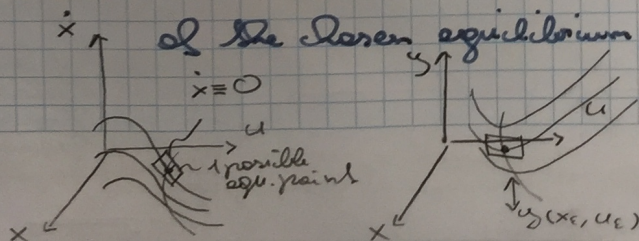
$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$

$y(t) = C \cdot x(t) + D \cdot u(t)$ Feed through

Remark: If the System is 1-dim and linear itself $C=1$, since $g=g(x)$ and $\frac{dg}{dx}=1$, if $D=0 \rightarrow$ strictly proper otherwise $\rightarrow$ proper

↳ A last simplification can be made by summarizing multiple physical variables into a few "control-oriented" parameters

↳ Remark: The coordinates of an LTI-System describe the deviation of the chosen equilibrium point (x_e, u_e)



Time-Response: (Determine the output for given $x_0 = x(0)$ and $u(t)$)

→ Our model is linear → Divide into simple cases and use Superposition

→ Initial condition response $\begin{cases} x_{ic}(0) = x_0 \\ u_{ic}(t) = 0 \end{cases} \Rightarrow y_{ic}$

Forced response $\begin{cases} x_f(0) = 0 \\ u_f(t) = u(t) \end{cases} \Rightarrow y_f$

Since $x = x_{ic} + x_f$, $u = u_{ic} + u_f \Rightarrow y = y_{ic} + y_f$

i) $\begin{cases} \dot{x} = Ax(t) \\ y = Cx(t) \end{cases} \Rightarrow \begin{cases} x(t) = x_0 e^{At} \\ y(t) = Cx_0 e^{At} \end{cases}$

DEFINITION: Use Taylor-series of $e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$

The matrix exponential e^{At} is sometimes called "transition-function" and can be computed as: $e^{At} = T \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) T^{-1}$, where λ_i are the eigenvalues of At and T contains the eigenvectors of At as columns. Important property: $Ae^{At} = e^{At}A$ (commute)

ii) $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, x(0) = 0$

Multiply ODE from left by e^{-At} and integrate over $[0; t]$

(partial integration): $\int_0^t e^{-A\tau} \dot{x}(\tau) d\tau = \int_0^t A e^{-A\tau} x(\tau) d\tau + \int_0^t e^{-A\tau} B u(\tau) d\tau$

$y(t) = C e^{At} B$ is called IMPULSE RESPONSE, since $\delta(t) * \delta(0) = \int_0^t C e^{A(t-\tau)} B \delta(\tau) d\tau = C e^{At} B$ for $u = \delta$, $x_0 = 0$, $d = 0$

$e^{-A\tau} x(\tau) \Big|_0^t + \int_0^t e^{-A\tau} A x(\tau) d\tau = - \dots$
 $x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$
 Convolution Integral

→ Superimpose Results:

$\begin{cases} x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\ y(t) = C e^{At} x_0 + C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t) \end{cases}$

Strictly Proper, 1st-order Systems: (EXAMPLES)

$\begin{cases} \dot{x}(t) = -\frac{1}{\tau} x(t) + \frac{R}{\tau} u(t) \\ y(t) = x(t) \end{cases}$

τ ... Time Constant > 0 , R ... Gain > 0

Determines Speed of Tracking
 otherwise the system would not be stable

IMPULSE RESPONSE	STEP RESPONSE	RAMP RESPONSE	HARMONIC RESPONSE
$y(t) = e^{-\frac{t}{\tau}} (x_0 + \frac{R}{\tau})$ At time τ the output has decreased to a factor of $e^{-1} \approx 0.37$	$y(t) = e^{-\frac{t}{\tau}} x_0 + R(1 - e^{-\frac{t}{\tau}})$ Tangent	$y(t) = e^{-\frac{t}{\tau}} x_0 + R[t + \tau(e^{-\frac{t}{\tau}} - 1)]$ Tangent ramp $R(t)$	$u(t) = \cos(\omega t)$ for $t \geq 0$, else $u(t) = 0$ $y(t) = e^{-\frac{t}{\tau}} x_0 + \frac{R}{1 + \omega^2 \tau^2} (\cos(\omega t) + \omega \tau \sin(\omega t) - e^{-\frac{t}{\tau}})$

QUAL. PLOTS for $x_0 = 0$

Non-zero x_0 → only relevant at beginning

The Steady-State response is also a harmonic function, with the same frequency, but different amplitude m and phase φ
 $m = \frac{R}{\sqrt{1 + \omega^2 \tau^2}}$, $\varphi = \arctan(\omega \tau)$

Linear System: Lyapunov Stability $\Rightarrow$ BIBO-Stability
gilt nur in die eine Richtung

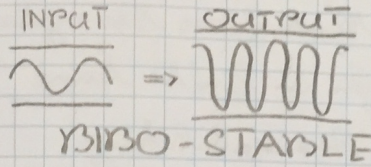
Stability: (DEFINITIONS)

INTERNAL DESC
MUST BE AVAILABLE

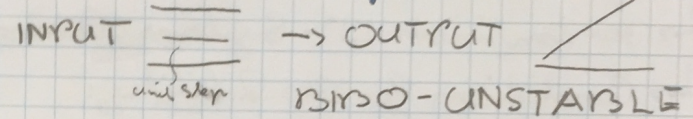
Def.: i) Lyapunov Stability: Analyzes the system behavior near an equilibrium for $u(t) = 0$ & $x(0) \neq 0$

USEFUL IF ONLY
I/O-DESC AVAILABLE

ii) BIBO Stability: Analyzes the output for $x(0) = 0$ and $|u(t)| < \infty \forall t$. Particularly, if a system is BIBO stable $\rightarrow$ Output will be bounded.



VS.



Lyapunov-Stability:

Def.:

VALID FOR
LINEAR
SYSTEMS
ONLY

$\rightarrow$ Asymptotically Stable: $\lim_{t \rightarrow \infty} \|x(t)\| = 0$

$\rightarrow$ Stable: $\lim_{t \rightarrow \infty} \|x(t)\|$ converges, BETTER $\|x(t)\| < \infty \forall t$

$\rightarrow$ Unstable: $\lim_{t \rightarrow \infty} \|x(t)\| = \infty$

Note: The definition of Lyapunov-Stability for non-linear systems is not straight forward, but rather complicated

ALWAYS connected to an equilibrium (for linear systems $\rightarrow$ subspace)

PROCEDURE (model coordinates)

$\rightarrow$ Since $\frac{d}{dt} x = A \cdot x + b \cdot u$, we can use A to model L.-Stab.

$\rightarrow$ Solve Eigenvalue-Problem for A : $\det(A - \lambda I) \stackrel{!}{=} 0$

$\rightarrow$ The Eigenvectors are called modes of the system (The corresponding Eigenvalues define, how they are evolving over time)

$\rightarrow$ Transform into modal coordinates

$\rightarrow$ The general solution in these coord. is $\tilde{x} = \begin{pmatrix} e^{\lambda_1 t} \tilde{x}_{0,1} \\ \vdots \\ e^{\lambda_n t} \tilde{x}_{0,n} \end{pmatrix}$, where λ_i can be complex. Since $\|e^{\lambda t}\| = e^{\text{Re}(\lambda)t}$ following classification can be made:

i) Asympt. Stable: $\forall \lambda \text{ Re}(\lambda_i) < 0 \quad \forall i$

ii) Stable: $\forall \lambda \text{ Re}(\lambda_i) \leq 0$

iii) Unstable: $\exists \lambda, \text{s.t. } \text{Re}(\lambda_i) > 0$

$\left. \begin{matrix} \forall i \\ \forall i \end{matrix} \right\}$ holds also if A is not diagonalizable

Exception Δ : If A is not diagonalizable and has multiple Eigenvalues with real part zero, the system can be unstable

Lyapunov's Stability Principle:

IF the linearization has $\text{Re}(\lambda_i) = 0$, no information of the "real" non-linear model can be obtained !!!

$\rightarrow$ L.-Stability for linear systems is a global concept!

$\rightarrow$ If the linearization of a non-linear model around an equ.-point is asympt. stable (unstable), then this equ. is Δ locally asympt. stable (unstable) for the non-linear system as well!

The Eigenvalues of A are also called poles $\Rightarrow$ See Transfer function

Comes from looking at the response
for $x_0 = 0$! $\rightarrow$ for the form
 $x(t) = e^{At} [B \ A^2 B \ \dots] U$, U cont.
terms of the form $e^{s_i t}$ or $t e^{s_i t}$

Right Eigenvector: $A \cdot v_i = \lambda_i v_i$
Left Eigenvector: $w_i A = \lambda_i w_i$

Makes it possible to subdivide the State-Space into subspaces

$$\begin{bmatrix} R & O \\ O & \bar{R} \end{bmatrix}$$

R... Reachable
O... Observable
 $\bar{R}$... Not Reachable
 $\bar{O}$... Not Observable

A realization containing only modes that are reachable & observable is called minimal

Reachability/Controllability/Observability:

$\rightarrow$ A point $\bar{x} = (x_1, \dots, x_n)$ in the State-Space is called

- "reachable", if there exists an input $u(t)$ that takes the system from the origin ($x(0) = 0$... Equilibrium) to that point in finite time
- "controllable", if there exists an input $u(t)$ that forces the system from the point $x(0) = \bar{x}$ back to the origin in finite time

$\rightarrow$ For continuous LTI-Systems Controllability $\triangleq$ Reachability

$\rightarrow$ Condition for Reachability:

$$\text{rank}([B \ AB \ A^2 B \ A^3 B \ \dots]) \text{ has to be } n, \text{ i.e.}$$

the dimension of the system. $M_c :=$ Controllability matrix

$\rightarrow$ The relation linear $\Leftrightarrow$ non-linear model is similar to the Lyapunov-^(Coccyus-Hamilton Theorem)

$\rightarrow$ If the linearized model is completely controllable, the non-linear system is (locally) completely controllable as well

$\rightarrow$ A system is called "observable", if all initial conditions can be reconstructed, looking at the output only

$\rightarrow$ Condition for Observability: $\rightarrow$ Since if $u(t) = 0$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \end{bmatrix} = M_o \cdot x_0$

$$\text{rank} \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} = n, \text{ i.e. } \text{im}(M_o) \text{ is the whole State-Space}$$

Modal Tests:

$\rightarrow$ Reachability/Observability can also be analyzed using Modal Coord. $\rightarrow$ See Lyapunov-Stability

$\rightarrow$ Find right eigenvectors v_i & left eigenvectors w_i , such that

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} -w_1^T & - \\ \vdots & \\ -w_n^T & - \end{pmatrix}, \text{ then the system } (A, B, C, D)$$

becomes $(D, \tilde{V}^{-1} B, C \tilde{V}, D) \dots \rightarrow$ Substitution $x = V \tilde{x}$ $\rightarrow$ New coordinates

$$\begin{aligned} \text{Component-wise: } \dot{\tilde{x}}_i &= \lambda_i \tilde{x}_i + w_i B u \\ y &= C v_1 \tilde{x}_1 + \dots + C v_n \tilde{x}_n \end{aligned}$$

$\rightarrow$ From these equations it is clear:

- If there is a left eigenvector w_i , such that $w_i B = 0$, then the i -th mode is "uncontrollable"
- If there is a right eigenvector v_i , such that $C v_i = 0$, then the i -th mode is "unobservable"

$\rightarrow$ STABILIZABILITY: All unstable modes are controllable.

$\rightarrow$ DETECTABILITY: All unstable modes are observable.

$\rightarrow$ MINIMAL REALIZATION: IFF it is controllable & observable and decoupled (No Pole/Zero-Cancellation) the complete I/O-Behavior

Laplace Transformation:

→ $\mathcal{L}(x(t))_{(s)} = X(s) = \int_0^{\infty} x(t) e^{-st} dt, s \in \mathbb{C}$

→ Generalization of Fourier-Transform, where $s = j\omega$

→ Properties: i) Linearity: $\mathcal{L}(a \cdot x_1(t) + b \cdot x_2(t)) = a \cdot \mathcal{L}(x_1(t)) + b \cdot \mathcal{L}(x_2(t))$

Notes: ... piecewise-Theory of the signal $x(t)$ is assumed to be 0 for $t < 0$, that is the same as $x(t-a)$

ii) Similarly: $\mathcal{L}(\frac{1}{a} \cdot x(\frac{t}{a})) = \mathcal{L}(x(t))_{(s \cdot a)} = X(s \cdot a)$

iii) T-Shifting: $\mathcal{L}(h(t-a) \cdot x(t-a)) = e^{-as} \mathcal{L}(x(t))$

iv) S-Shifting: $\mathcal{L}(x(t) \cdot e^{at}) = \mathcal{L}(x(t))_{(s-a)} = X(s-a)$

v) Derivative s: $\mathcal{L}(t \cdot x(t)) = -\frac{d}{ds} \mathcal{L}(x(t))_{(s)}$

vi) Derivative t: $\mathcal{L}(\frac{d^n x(t)}{(dt)^n}) = s^n \cdot \mathcal{L}(x(t))_{(s)} - \sum_{i=1}^n s^{n-i} \cdot \frac{d^{i-1} x(t)}{(dt)^{i-1}} \Big|_{t=0}$

TO/OOE → TO/Sol
L ↓

↑
FO/TF → FO/Sol
Y(s) = G(s)U

Control-Design in frequency-domain possible

→ Special Cases: $\mathcal{L}(\dot{x}(t)) = s \cdot \mathcal{L}(x(t)) - x(0)$

$$\mathcal{L}(\ddot{x}(t)) = s^2 \mathcal{L}(x(t)) - s x(0) - \dot{x}(0)$$

vii) Integration t: $\mathcal{L}(\int_0^t x(\tau) d\tau) = \frac{1}{s} \mathcal{L}(x(t))$

viii) Integration s: $\mathcal{L}(\frac{1}{t} x(t)) = \int_s^{\infty} \mathcal{L}(x(t))_{(s)} ds$

ix) Convolution t: $\mathcal{L}(x_1(t) * x_2(t)) = \mathcal{L}(x_1) \cdot \mathcal{L}(x_2)$

x) Convolution s: $\mathcal{L}(x_1 \cdot x_2) = \mathcal{L}(x_1) * \mathcal{L}(x_2)$

→ Initial Value Theorem: $\lim_{t \rightarrow 0^+} x(t) = \lim_{s \rightarrow \infty} s \cdot \mathcal{L}(x(t))_{(s)}$

→ Final Value Theorem: $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0^+} s \cdot \mathcal{L}(x(t))_{(s)}$

→ Standard-Signals:

Impulse: $\delta_0(t)$	→ 1	$\delta(t-t_0)$	→ e^{-st_0}
Step: $h(t)$	→ $\frac{1}{s}$	$h(t-a)$	→ $\frac{1}{s} e^{-as}$
$h(t) \cdot t^n$	→ $\frac{n!}{s^{n+1}}$	$h(t) \cdot \sin(\omega t)$	→ $\frac{\omega}{s^2 + \omega^2}$
$h(t) \cdot e^{at}$	→ $\frac{1}{s-a}$	$h(t) \cdot \cos(\omega t)$	→ $\frac{s}{s^2 + \omega^2}$
$h(t) \cdot \sinh(\omega t)$	→ $\frac{\omega}{s^2 - \omega^2}$	$h(t) \cdot \cosh(\omega t)$	→ $\frac{s}{s^2 - \omega^2}$

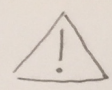
→ Inverse Laplace Transform: $\mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} f(s) ds$

→ The Laplace-Transformation is a tool to express functions as a sum of complex exponentials.

→ Example:

$$\begin{aligned} f(t) &= h(t) e^{2t} \cdot [\cos(2t) + \sin(2t)] + \delta(\frac{1}{2}t - 1) + (\frac{1}{2}t - 1) \cdot h(t-a) \\ \mathcal{L}(f(t)) &= \mathcal{L}(h(t) \cdot e^{2t} \cdot \cos(2t)) + \mathcal{L}(h(t) e^{2t} \sin(2t)) + \mathcal{L}(\delta(\frac{1}{2}(t-2))) + \mathcal{L}(\frac{1}{2}(t-2)h(t-2)) \\ &= \frac{s-2}{(s-2)^2 + 4} + \frac{2}{(s-2)^2 + 4} + \mathcal{L}(\frac{1}{2}t) e^{-2s} + \frac{1}{2} e^{-2s} \mathcal{L}(t), \text{ Similarly} \\ &= \frac{s}{s^2 - 4s + 8} + 2e^{-2s} + \frac{1}{2} e^{-2s} \cdot \frac{1}{s^2} \\ &= \frac{s}{s^2 - 4s + 8} + e^{-2s} (2 + \frac{1}{2s^2}) \end{aligned}$$

If the input is $u(t) = \sum_i e^{s_i t} U_i$, then because of the systems linearity: $y(t) = \sum_i y_i = \sum_i g(s_i) e^{s_i t} U_i$



Transfer-Functions: (ALWAYS RATIONAL for LTI)

(ALWAYS A DESCRIPTION OF A LINEAR SYSTEM)

VS. $g(s) = e^{s \cdot \text{delay}}$ $s \in \mathbb{C}$

- ↳ We know how to write a Input-Signal as the (infinite) sum of e^{st} .
- ↳ The Output for that signal is: $y(t) = C e^{At} X_0 + C \int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau + D e^{st}$
- $y(t) = \dots = \underbrace{C e^{At} X_0}_{\text{Transient Response}} + \underbrace{C (sI - A)^{-1} e^{At} B + (C (sI - A)^{-1} B + D) e^{st}}_{\text{Steady-State Response}}$

- ↳ We call $g(s) = C (sI - A)^{-1} B + D$ "Transfer Function" → Does NOT change if different coordinates are used → SYSTEM PROPERTY
- ↳ Note: If the System is asymptotically stable $e^{At} \rightarrow 0$ and there-fore the "Transient Response" will vanish.

- ↳ If we take the Laplace-Transform of $y(t) = TR + SSR$ we get:
 $Y(s) = g(s) U(s) + C (sI - A)^{-1} X_0 \Rightarrow g(s)$ maps the Output to the Input
- ↳ Compute STEADY-STATE-RESPONSE:

I. VARIETY: If the input can be written directly as $\sum_i e^{s_i t} U_i$, just use above formula; for instance $u(t) = \cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$
 $y_1 = \frac{1}{2} g(j\omega) e^{j\omega t}$
 $y_2 = \frac{1}{2} g(-j\omega) e^{-j\omega t} \Rightarrow y(t) = y_1 + y_2$

II. VARIETY: Laplace-Transform the Input-Signal, then multiply by $g(s)$ and transform back! nominator

↳ The Transfer Function can also be computed as $g(s) = \frac{C \cdot \text{adj}(sI - A) B}{\det(sI - A)} + D$
 where $\text{adj} = \text{cof}(\cdot)^T$, $\text{cof}(\cdot) \dots$ Vorzeichen behaltene Minoren, i.e. $\text{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ denominator

- ↳ (The denominator roots = POLES are the Eigenvalues of A) ONLY IF DESCRIPTION IS MINIMAL
- ↳ The Transfer Function is equal to the Laplace Transform of the systems impulse response: $\mathcal{L}(C e^{At} B) = C (sI - A)^{-1} B$ (A, B, C, D) not necessarily
- ↳ Going from the Transfer-Function to a State-Space-Realisation: $g(s) = C (sI - A)^{-1} B + D$

i) Let the TF be $g(s) = \frac{c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \dots + c_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} + d$

ii) One possible State-Space-Realisation (out of infinitely many) is

CONTROLLER CANONICAL FORM $A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, C = (c_0 \ c_1 \ \dots \ c_{n-1}), D = d$

iii) If all Pole-Zero-Cancellations are done before transforming this realisation is MINIMAL (no uncontrollable/unobservable modes)

↳ POLE-ZERO-Cancellation: When computing $g(s) = C (sI - A)^{-1} B + D$ and some factors cancel → These modes were uncontrollable/unobservable.
 If $A = \text{diag}(\lambda_1, \dots, \lambda_n) \rightarrow g(s) = \frac{c_1 b_1}{s - \lambda_1} + \dots + \frac{c_n b_n}{s - \lambda_n} \Rightarrow \text{if } b_i = 0 \rightarrow \text{mode } i \text{ is not}$

g(s) describes Input/Output-behavior and NOT internal behavior

EASY

BIBO-Stability: (Assessment using the Transfer Function)

↳ A system is BIBO-Stable, IFF all finite inputs result in finite outputs

↳ For linear systems: BIBO-Stability $\Leftrightarrow \int_0^{\infty} |\sigma(t)| dt < \infty$ $\xrightarrow{\text{Impulse Response}} \mathcal{L}^{-1}(G(s))$

↳ A System represented by its Transfer Function is:

i) BIBO-Stable, IFF $\text{Re}(\text{POLES}) < 0$

ii) NOT BIBO-Stable, else

↳ Difference between Lyapunov- & BIBO-Stability

i) When there are stable modes, which are observable & controllable, the system is NOT BIBO-Stable. (That is: Poles on Imaginary-axis, but no zeros on Im-axis)

ii) Unstable modes are uncontrollable & unobservable $\Rightarrow$ System is BIBO-Stable

Poles: (Roots of the Transfer-Function's Denominator)

↳ Poles π define the oscillation behavior of the system.

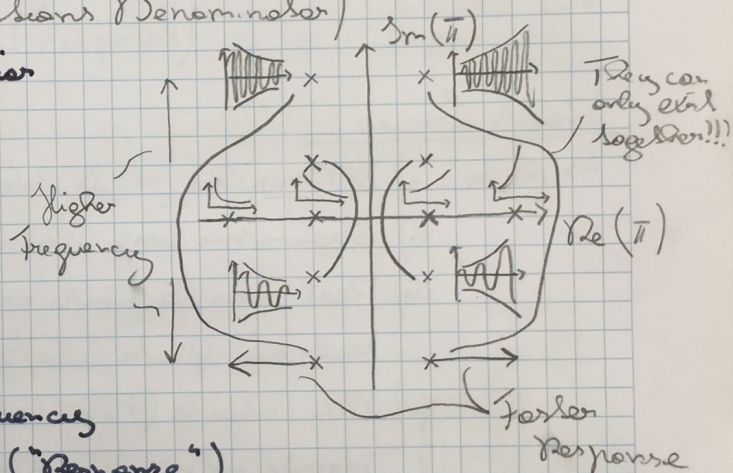
↳ Example: Second-order-system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\delta\omega_0 s + \omega_0^2} \quad \delta \in [0, 1]$$

$$\text{Poles: } \bar{\pi}_{1,2} = \frac{-\delta\omega_0 \pm j\omega_0\sqrt{1-\delta^2}}{1} = \underbrace{-\delta\omega_0}_{=: \sigma} \pm j\underbrace{\omega_0\sqrt{1-\delta^2}}_{=: \omega}$$

$\omega = \text{Im}(\bar{\pi})$... Oscillation frequency

$\sigma = \text{Re}(\bar{\pi})$... Rate of decay ("Response")



Zeros: (Roots of the Transfer-Function's Numerator)

↳ Minimum-Phase Zero: $\text{Re}(z_i) < 0$

Non-minimum-Phase Zero: $\text{Re}(z_i) > 0$
(Often caused by poor sensor placement)

↳ Overshoot: $u(t) = p(t)$

↳ Undershoot: $u(t) = p(t)$

↳ 2nd-order-System:

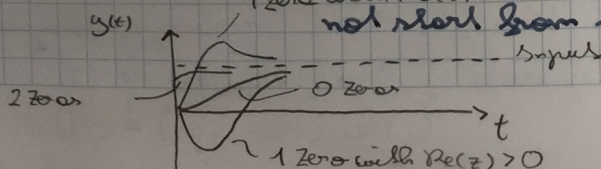
→ No Zero present: Slope at origin is zero

→ One Zero: Slope at origin $\neq 0$: Direction depends

Input affects 2nd & 1st-Derivative of the Output directly on the sign of that zero (Over-/Undershoot)

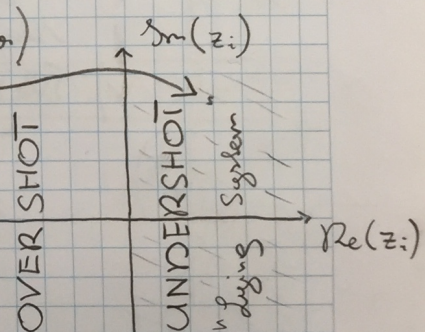
→ Two Zeros: The output is directly affected by the input $\rightarrow$ It will

not start from the origin!



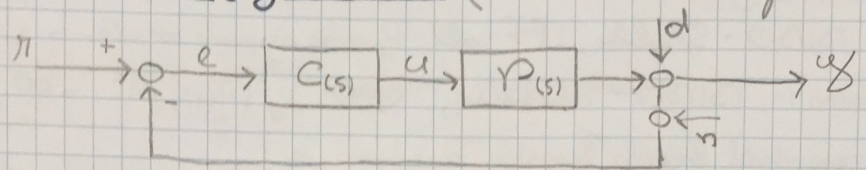
(For high oscillations, $\text{Im}(\pi) \gg 1$)
More in an overshoot

A zero close to a pole will reduce the impact of that pole



Since our systems are LTI, in the frequency domain they are represented by rational functions $\rightarrow$ we can write $\frac{1}{1+L}$ or $\frac{L}{1+L}$

Feedback-Configuration: (Closed-Loop-Transfer-Functions)



$$T + S \stackrel{!}{=} 1$$

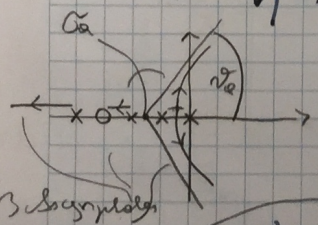
- L> (Open-) Loop Gain: $L(s) = P(s)C(s)$ ($e \rightarrow y$)
- L> Sensitivity: $S(s) = \frac{1}{1+L(s)}$ (From $r \rightarrow e$) & ($d \rightarrow y$)
- L> Complementary Sensitivity: $T(s) = \frac{L(s)}{1+L(s)}$ ($r \rightarrow y$) ($r \rightarrow y: T(s)$)
- L> Well-posedness: Let C & P be gains $\rightarrow$ if $L = CP = (-1)$ the interconnection makes no sense $\rightarrow$ it's NOT well-posed

L> Internal Stability: ALL Closed-Loop Transfer Functions between NEVER cancel an unstable any two signals must be stable.
 (Example: $L = \frac{s+1}{(s-1)(s+2)} \Rightarrow T = \frac{1}{s+2}$)
 In the frequency sense
 Seems to be fine

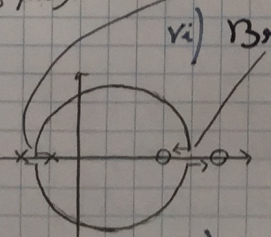
Root Locus: ($R > 0$, i.e. positive root locus, since $R < 0$ is physical non-sense)

- L> Rules: i) ~~Branches~~ = ~~Poles~~
 ii) Symmetric wrt. the real axis
 iii) Real-axis Segments are to the left of an odd number of real-axis finite poles/zeros. (Complex-conjugate Poles DO count double)
 iv) RL starts ($\hat{=} R=0$) at poles & ends ($\hat{=} R \rightarrow \infty$) at zeros if degree of numerator $m =$ degree of denominator n . If $n \neq m$ there will be $(m-n)$ poles approaching infinity! (See asymptotes)
 v) The RL has $(n-m)$ asymptotes:

(Guide)
POLES
ZEROS
x
o



"Center of mass" $\rightarrow$ Intersection of asymptotes on the real axis at $\sigma_a = \frac{\sum \text{Re}(p_i) - \sum \text{Re}(z_i)}{n-m}$
 Angles of asymptotes: $\varphi_a = \frac{(2i+1)\pi}{n-m}, i = \{0, \pm 1, \pm 2, \dots\}$



- vi) Break-On & Break-Away-Points:
 When two branches come together, they will "break away" (in)
 These points are found by $\frac{dL(s)}{ds} \stackrel{!}{=} 0 = \frac{d}{ds} \left(\frac{1}{L(s)} \right)$

- vii) Imaginary axis crossing: (That is when $KL(j\omega) = (-1)$)
 Solve the equation $KL(j\omega) = -1$, using a compare of coefficients $\rightarrow$ Separate $\begin{cases} \text{Im}(KL(j\omega)) \stackrel{!}{=} 0 \\ \text{Re}(KL(j\omega)) \stackrel{!}{=} (-1) \end{cases}$

PROOF: $u(t) = \cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$

Assumption:
 $|G(j\omega)| = |G(-j\omega)|$
 $\angle(G(j\omega)) = -\angle(G(-j\omega))$

$$y(t) = y_1 + y_2 = \frac{1}{2}G(j\omega)e^{j\omega t} + \frac{1}{2}G(-j\omega)e^{-j\omega t}$$

$$y(t) = |G(j\omega)| \cdot \frac{1}{2}(e^{j(\omega t + \angle(G(j\omega)))} + e^{-j(\omega t - \angle(G(j\omega)))})$$

$$y(t) = |G(j\omega)| \cos(\omega t + \angle(G(j\omega)))$$

The Frequency Response:

↳ Definition: For the input $u(t) = \cos(\omega t)$, we call the corresponding **STEADY-STATE** output the frequency response.
 (This is a function of ω) ↳ Really it is the mapping $\mathbb{R} \rightarrow \mathbb{C}$

↳ It can be shown: The frequency response for every system is of the form $y_{ss}(t) = |G(j\omega)| \cos(\omega t + \angle(G(j\omega)))$

↳ Interpretation: i) Amplitude of the Output is $|G(j\omega)|$ times amplitude input
 ii) Phase of the Output lags the input phase by $\angle(G(j\omega))$

↳ In the following we try to visualize the Frequency Response!

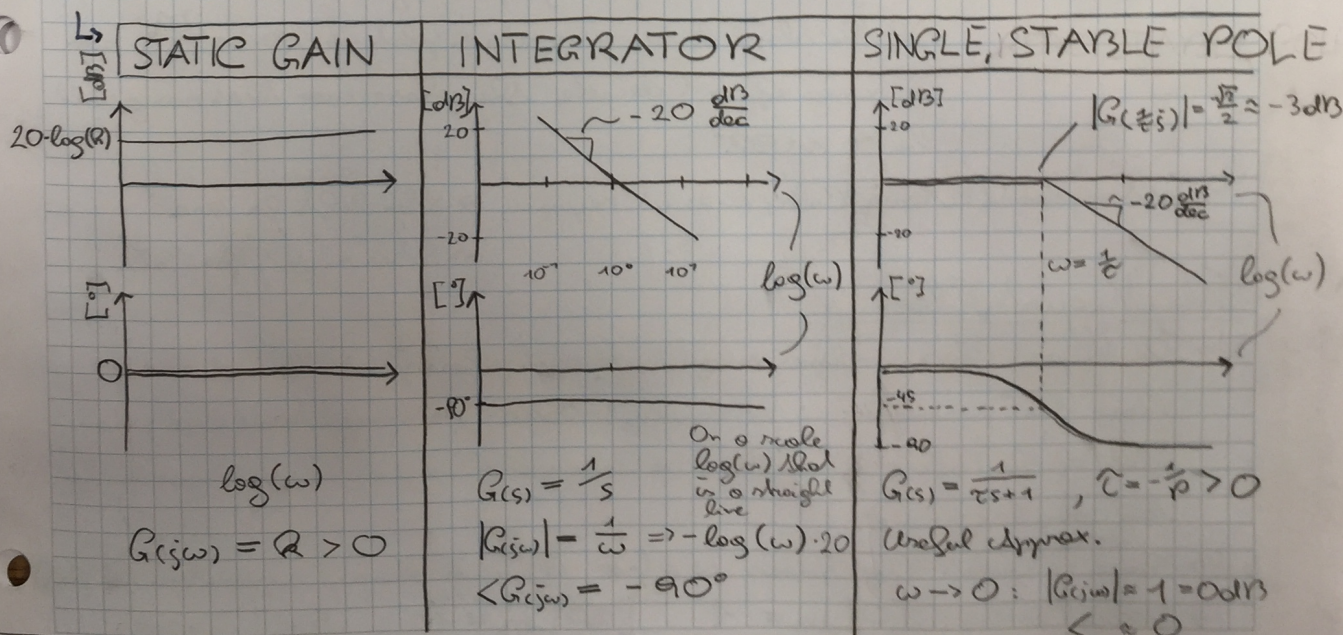
Bode - Plot:

↳ 2 Plots: i) Magnitude Plot $|G(j\omega)|$
 ii) Phase Plot $\angle(G(j\omega))$ } Both as an explicit function of ω

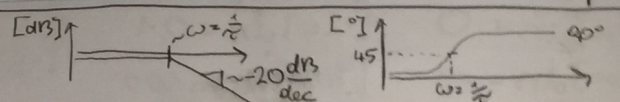
↳ Scaling: i) $10 \log(\omega) \rightarrow |G(j\omega)| [\text{dB}]$, where $|G(j\omega)| [\text{dB}] = 20 \cdot \log(|G(j\omega)|)$
 ii) $10 \log(\omega) \rightarrow \angle(G(j\omega)) [^\circ]$

↳ From the log-scaling it follows:

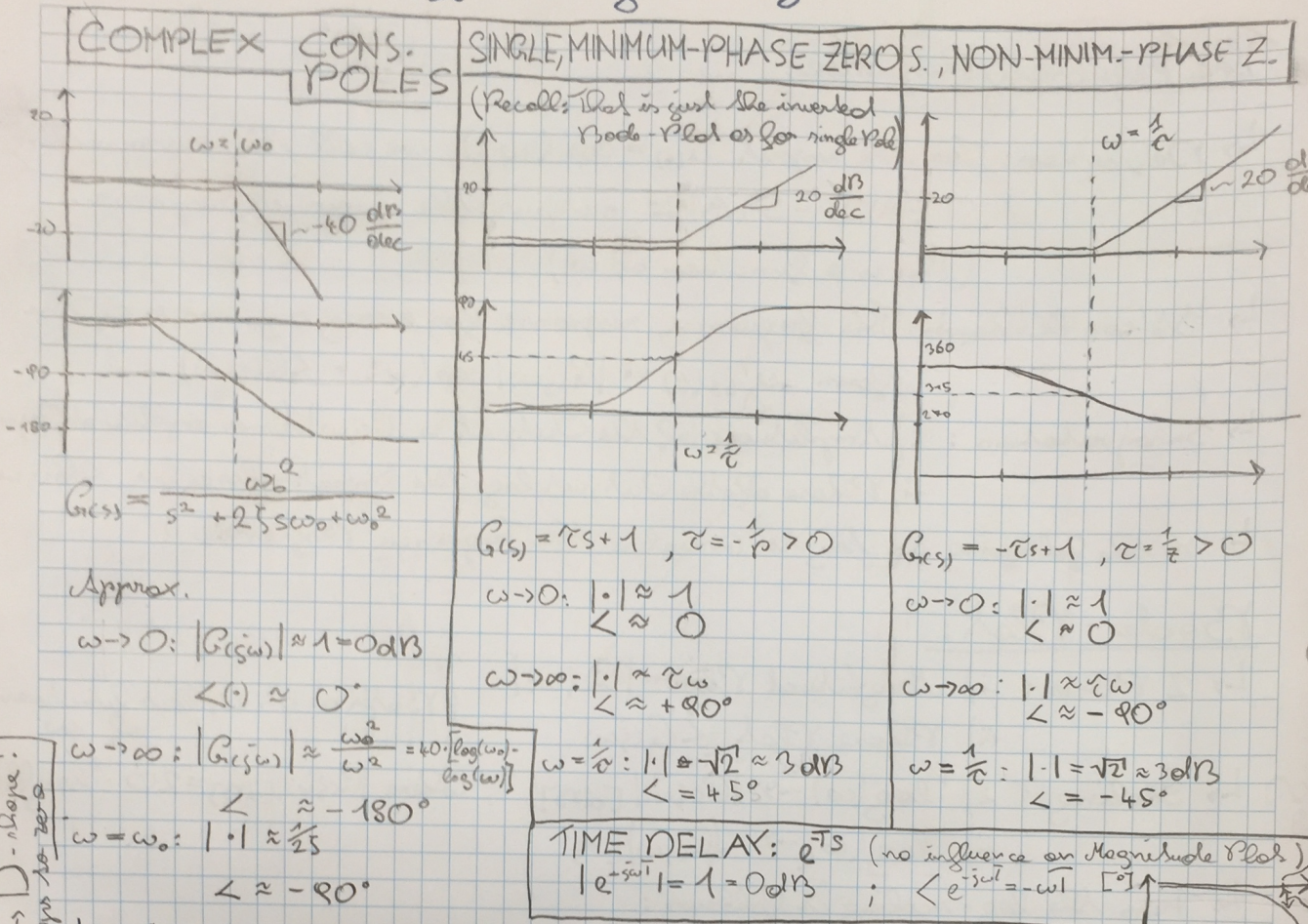
- Multiplying two transfer function (i.e. $L(s) = P(s)C(s)$) means **ADDING** their Bode-Plots (use exp()-description: $|P(s)| \cdot |C(s)| e^{j(\angle(P(s)) + \angle(C(s)))}$ angles add anyway, while magnitude add because of logarithmic scale)
- Inserting a transfer function means **reflecting** its Bode Plot about the horizontal axis



UNSTABLE
 POLE: $G(s) = \frac{1}{-\tau s + 1}$



NOTE: If Phase in $\angle L(j\omega) = -180$ ($\omega \rightarrow 0$),
the static gain is negative



NOTE: A shift of the Phase-Plot by 360° is irrelevant!!!

The Nyquist D-contour: The Nyquist plot is the contour $L(s)$, where s goes over the complete imaginary axis + a arc with infinite radius $\rightarrow$ D-shape. The arc usually goes to ∞ in the right half plane.

When sketching the Bode-Plot of a complicated Transfer-Function, it is useful to write it in "Bode-Term", i.e. make standard-elements visible:

$$G(s) = \frac{1}{2} \frac{(s+2)(s+10)}{(s^2+s+1)(s+5)} \Rightarrow 2 \frac{(\frac{s}{2}+1)(\frac{s}{10}+1)}{(s^2+s+1)(\frac{s}{5}+1)}$$

Bode's Law: Magnitude & frequency of a system are NOT independent! If the slope of the magnitude plot is $20 \cdot \kappa \frac{\text{dB}}{\text{dec}}$ over a range of ~ 1 decade the phase over that range will be $\sim \kappa \cdot 90^\circ$

Test approach: Mark Pole/Zeros at frequency $\rightarrow$ Shift left for Zero $\rightarrow$ Shift right for Pole

The Nyquist Plot:

The Nyquist Plot combines magnitude and phase plot, but doesn't show the "parameter" ω explicitly! Idea: From magnitude and phase plot of the Bode-Diagrams, we have polar coordinates for a curve in the complex plane.

Since Nyquist-Plot shows $\omega \in (-\infty; \infty)$ it is always symmetric to $\text{Re}()$

Useful Values:

$[0]_{\text{dB}} = 1$	$[20]_{\text{dB}} = 10$	$[-40]_{\text{dB}} = 0.01$
$[-\infty]_{\text{dB}} = 0$	$[-20]_{\text{dB}} = 0.1$	$[5]_{\text{dB}} = 1.778$
$[\infty]_{\text{dB}} = \infty$	$[40]_{\text{dB}} = 100$	$[-5]_{\text{dB}} = 0.562$

APPENDICUM: Dealing with uncertainty: Define a nominal model and a "real" system, such that $\tilde{P} = P(1 + W_2 \Delta)$, where P ... nominal plant, $\|\Delta\| < 1$ and $\|W_2\| > \frac{1}{\epsilon} - 1$, i.e. W_2 describes how much the ratio of real and nominal TF is away from being equal to one. **NYQUIST:** $\|L_{nom} + 1\| > \|W_2 L_{nom}\|$

The Nyquist Criterion:

i.e. NO crossovers

Distance from L to (-1) point

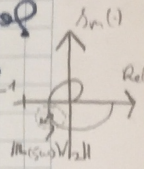
Remark: If s travels a simply connected region of the complex plane (Γ)

then $L(s)$ describes another closed curve! The net variation of

$\angle L(s)$ counts the number of poles & zeros the path of s

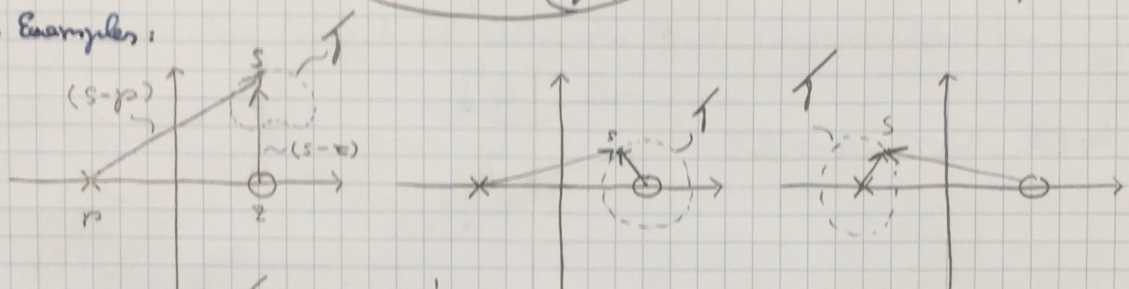
encircles! WHY? $\rightarrow$ $\Im \arg: L(s) = \frac{s-z}{s-p}$, then

$\angle L = \angle(s-z) - \angle(s-p)$ $\rightarrow$ VECTOR FROM z/p to s



GUZZELLA VERSION	
$n_c = \frac{1}{2} n_o + n_+$	
n_c ... number of closed-loop poles	
n_o ... number of open-loop poles	
n_+ ... number of unstable open-loop poles	
n_- ... number of unstable closed-loop poles	

3 Examples:



As s moves on Γ , both vectors will just wiggle around $\Rightarrow$ The vector $(s-z)$ will make a whole circle $\angle L(s) = 0$ $\angle L(s) = 2\pi$

$\angle L(s) = -2\pi$

Summarised: $N = Z - P$, where: N ... Encirclements of the closed curve

of $L(s)$ of the origin (counted pos.

if in the same direction as s

Z ... Zeros of $L(s)$ in $\mathcal{D}$

travels on $\partial\mathcal{D}$

P ... Poles of $L(s)$ in $\mathcal{D}$

Reminder: For closed-loop stability we require the complementary sensitivity to have poles with $\Re(\pi_i) < 0 \forall i$!

Comp. Sensitivity: $T(s) = \frac{L(s)}{1+L(s)}$

Closed-Loop Stability $\iff$ Roots of $(1+L(s))$ have negative $\Re()$

$1+L(s)$ is in general a ratio of two polynomials $\frac{N(s)}{D(s)} \Rightarrow$ The poles are the same as for $L(s)$! Zeros shall have negative Real-Part!

Let s travel along the $\mathcal{D}$ -contour, then curve $1+L$ satisfies

$N = Z - P$, Z ... # of unstable CLOSED-LOOP-POLES

P ... # of unstable OPEN-LOOP-POLES

CONCLUSION:

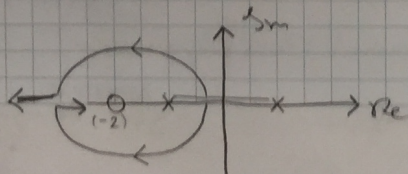
The number of times N , that $L(s)$ encircles $-\frac{1}{R}$ satisfies $N = Z - P$

Z ... # unstable closed loop poles

P ... # unstable open loop poles

Example: Open-loop unstable system $L(s) = \frac{s+2}{s^2-1}$

Root-Locus:

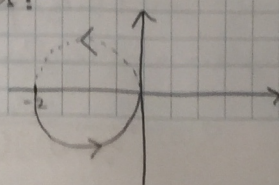


Nyquist-Plot:

$P = 1$
 $Z \neq 0$ (Stability)

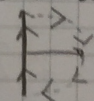
$N = -1$

$\rightarrow -\frac{1}{R}$ has to be within the circle! This is the case for $R > \frac{1}{2}$



Counted positive clockwise

As s moves along $\mathcal{D}$ -contour



Encirclements of the origin (Clockwise)

REMEMBER: If the open loop is stable, then we want NO encirclement of (-1)

Specifications:

Definitions: (The following values give information on the system's

i) Critical Distance μ_{min} : Minimal Distance between ROBUSTNESS

$$\mu_{min} = \min |1 + L(j\omega)| \quad \text{Return Difference} = |1 + L(j\omega)|$$

$$= \max |S(j\omega)|$$

the (-1) point and the Nyquist-Plot

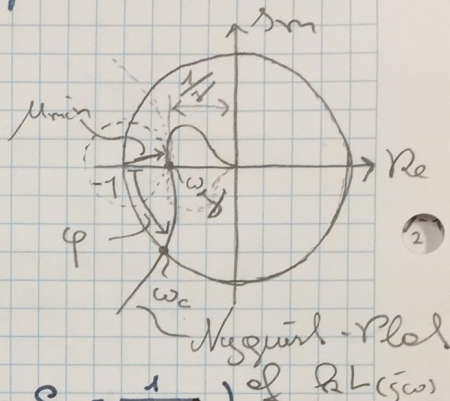
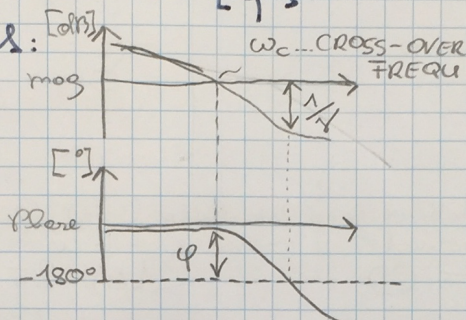
ii) Gain margin: Value for the space left, when crossing the imaginary axis wrt the (-1) point! [y]

iii) Phase margin: Angle at which the Nyquist plot intersects the unit circle! [φ]

On the Bode-Plot:

VALID only if open-loop is stable!

Otherwise → use Nyquist-Plot to check stability



Steady-State Error: (Recall: $(s \rightarrow 0) \cdot (d \rightarrow e)$: $S(s) = \frac{1}{1 + L(s)}$)

Final value Theorem: $\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} S(s) R(s)$

Type of a system: # Integrator (That is the number of poles of $L(s)$ at $s=0$)

Recall $R(s)$ can be read off the TF, when writing in the "BODE" form: $L(s) = R(s) \cdot \frac{(s+1)^m}{(s+1)^n}$

Special Case: Sinusoidal Input:

$$e_{ss} = \pi_{ss} - y_{ss}, \text{ NEG. if } y_{ss} > \pi$$

The Static Error will be a function of time! $e_{ss}(t) = \|S(j\omega)\| \sin(\omega t + \angle S(j\omega))$

If, for instance, e_{ss} for sinusoidal inputs up to a frequency of $\omega = 10 \frac{\text{rad}}{\text{s}}$

the static error shall be smaller than 5% $\Rightarrow \|S(j\omega)\| \leq 0,05 \quad \forall \omega \leq 10 \frac{\text{rad}}{\text{s}}$

Time-Domain-Specifications:

Usually given for a 2nd-order system

$$L(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega = \sqrt{\omega_n^2 - \zeta^2\omega_n^2} \quad \zeta = \frac{\omega_n}{\omega_0}$$

i) Rise Time $T_{100} = \frac{\pi}{\omega} (\zeta < 1) \rightarrow \frac{1}{\text{Rise Time}} \propto \text{Bandwidth} \propto \text{Crossover Frequency}$

$$T_{100} \approx (0,14 + 0,4\zeta) \frac{\pi}{\omega_0}$$

ii) % Overshoot: $M_p \approx -\frac{\sigma\pi}{\omega}$

iii) Settling-Time: e.g. 1 to 2% $\rightarrow T_s = -\frac{\ln 2\%}{\sigma}$ Pole

If the System is of higher order: Use Dominant Pole Approx.

Example:

$$G(s) = \frac{13}{(s+0,5)(s+1+5j)(s+1-5j)} \Rightarrow \text{From the } s: \text{Approx } G(s) = \frac{0,5}{(s+5)}$$

Relation for:

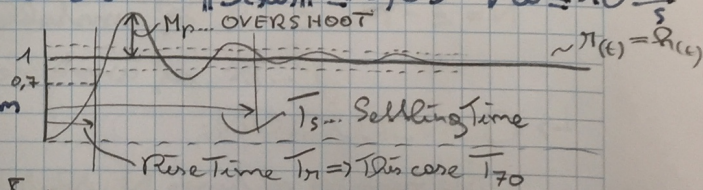
TIME-DOMAIN CLOSED-LOOP SYSTEM $\Leftrightarrow$ FREQU-DOM OPEN-LOOP GAIN

Decays as $e^{-\frac{1}{2}t}$

Decays as e^{-t} (faster!)

Decays as $e^{-\frac{1}{2}t}$

APPROX. $M_p \approx 71\%$ $T_{100} \approx 11\%$



↳ Constraints coming from "Time-Domain Spec's":

- Settling Time: Real Part of dominant Poles less than some maximum
- Peak Overshoot: Damping Ratio more than some minimum
- Rise-Time: Dominant Poles have some minimum distance from origin & $\zeta \omega_n > \text{some minimum}$

↳ Frequency Domain Specifications:

- Disturbances: Low Frequency ($< 10 \text{ Hz}$) $\rightarrow S(s)$
- Noise: High Frequency ($> 100 \text{ Hz}$) $\rightarrow T(s)$



$$\|S(j\omega)\| \ll 1 \text{ at low frequ.} \iff \|L(j\omega)\| \gg 1$$

$$\|T(j\omega)\| \ll 1 \text{ at high frequ.} \iff \|L(j\omega)\| \ll 1$$

↳ Definition: Bandwidth: Maximum frequency for which

$$\|T(j\omega)\| > \frac{1}{\sqrt{2}} \text{ (Command tracking with } \approx 0.7 \text{)}$$

↳ Thumb Rule: Bandwidth $\sim \omega_c$

$$\max\{10 \cdot \omega_d; 2 \cdot \omega_z\} < \omega_c < \min\{0.5 \cdot \omega_z; 0.5 \omega_{\text{delay}}; 0.5 \omega_z; 0.1 \omega_n\}$$

where: ω_d ... Relevant disturbance frequ

ω_z ... Max. frequ. of unstable poles

$$\omega_z \dots |W_2(j\omega)| = 1$$

$\omega_{\text{delay}} \dots \frac{1}{T_{\text{delay}}} \text{ (} e^{-0.01t}, T_0 = 0.01 \text{)}$

ω_n ... Min. frequ. of non-min. zeros

ω_n ... Noise frequency

Loop Shaping: (Lead/Lag)

↳ I. Proportional Control: Shift the magnitude plot up & down (w/ gain to control ω_c)

↳ II. Lead Compensator: $C_{\text{lead}} = K \frac{s+a}{s+b}$, $0 < a < b$

→ Increase Magnitude at high frequ. by $\frac{b}{a}$

→ Increase Slope of Mag. between a, b by $20 \frac{dB}{dec}$

→ Increase Phase around $\sqrt{ab}$
(val ... geometric midpoint)

→ Increase GAIN MARGIN (by $\hat{\varphi}$):

i) Pick $\sqrt{ab} = \omega_c$ (Cross-over frequ)

ii) Solve: $\hat{\varphi} = 2 \cdot \arctan(\sqrt{\frac{b}{a}}) - 90^\circ$

iii) Adjust K , such that $\|K \cdot C_{\text{lead}} \cdot P\| = 1$ at ω_c

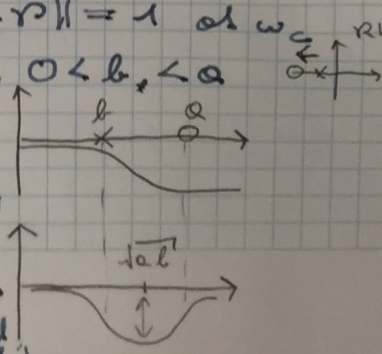
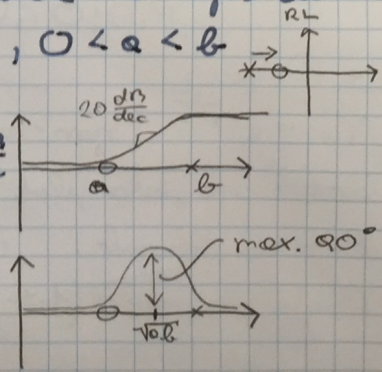
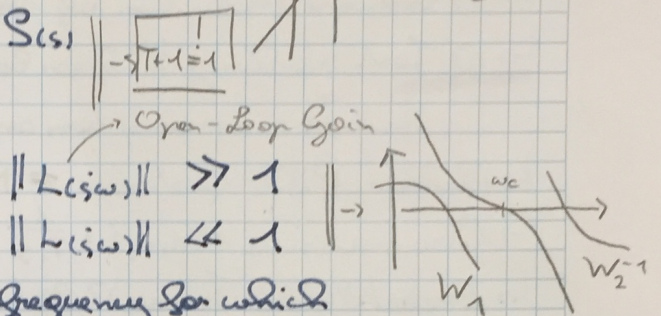
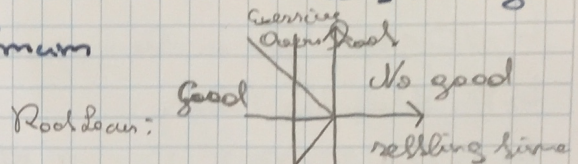
↳ III. Lag Compensator: $C_{\text{lag}} = K \frac{s+b}{s+a}$, $0 < b < a$

→ Decrease Mag. at high frequ. by $\frac{b}{a}$...

→ IMPROVE COMMAND TRACKING & DISTURBANCE REJECTION

i) Rule of thumb: Choose it's zero a decade below crossover

ii) Choose K & b , such that ω_c is not affected



Remark:
i) NMP Zeros limit ω_c (have bandwidth)
ii) Open-loop unstable Poles require a Dig ω_c

PID-Control:

↳ I/O-Relation with Error feedback in the Time Domain:

$$u = \underbrace{K_p e}_{\text{Proportional}} + \underbrace{K_i \int_0^t e(\tau) d\tau}_{\text{Integral}} + \underbrace{K_d \frac{d}{dt} e(t)}_{\text{Derivative}} = K_p \left(e + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{d}{dt} e(t) \right)$$

↳ I/O-Relation in the frequency domain: $C(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s \right)$ → increases response behavior

↳ P: Stabilize Plants ↳ fiddle errors

↳ I: Complete Reduction of the static error (lowers Robustness, i.e. phase margin) ($\angle_s = -90^\circ$)

↳ D: Fast Tracking! "Linear Extrapolation" predicts the error at time t

Acts as a damping term on $\dot{e}$ (improving transient behavior)

↳ Problem: PID-Controller = NON-CAUSAL (not implementable) → Roll-off-term

↳ $C_{PID-PRACTICAL} = K_p \left(1 + \frac{1}{T_i s} + T_d s \right) \cdot \frac{1}{(\tau s + 1)^2}$, where $\tau \dots$ small (lowers noise)

↳ Ziegler/Nichols Tuning Rules (Approximation) (if there's no model available)

→ Approximate the given plant with $P(s) = \frac{K}{\tau s + 1} e^{-T_s}$ ($\frac{T}{T+\tau} \dots$ small)

→ Find critical values, where: $\omega^* \dots < \angle P(j\omega^*) = -180^\circ$

The gain $K_p^* P(j\omega^*)$ should be -1 on the Complex Plane

be -1 on the Complex Plane $\Rightarrow$ such that $\|K_p^* P(j\omega^*)\| = 1$

→ Then:

	K_p	T_i	T_d
P	$0,5 K_p^*$	$\infty \cdot T^*$	$0 \cdot T^*$
PI	$0,45 K_p^*$	$0,85 T^*$	$0 \cdot T^*$
PD	$0,55 K_p^*$	$\infty \cdot T^*$	$0,15 T^*$
PID	$0,6 K_p^*$	$0,5 T^*$	$0,125 T^*$

Recall

$$C_{PID} = K_p \left(1 + \frac{1}{T_i s} + T_d s \right)$$

Time Delay:

↳ Time Domain: $y(t) = u(t - T)$

↳ $Y(s) = e^{-sT}$ (non-rational, however: Time-Delay → Linear Operator)

↳ "Smear" the poles plot of a ideal (non-delayed TF) infinite times over Re-axis

↳ Rule of Thumb: $\phi_T = \phi_0 - \omega_c T$ → PHASE MARGIN with delay

↳ Poole Approx.: (1st-order) $e^{-sT} \approx \frac{2 - Ts}{2 + Ts} = \frac{2 - Ts}{2 + Ts}$ From the presence of a NMP-zero we see that the GAIN may not be infinitely large!!

Integrator Anti-Windup:

The feed-back loop is effectively OPENED.

↳ If the input is physically limited ("cannot push for throttle arbitrarily large")

the linear model will not behave like the non-linear (Integral accumulation error)

↳ Solve it by: i) Compare actual input & commanded input

ii) If the same → proceed as usual

iii) If not → Reduce integral error by a constant time Δu

In order to use the Root Locus Method for TF's with time delay (RL requires rational TF)