

MRI in Medicine

Can be derived using operators - see

I. Physical Basis - NMR

↳ Need from QM - Angular momentum of e^- :

2 contributions to angular momentum:

i) Orbital: " e^- moves around nucleus"

△

Depends on energy state of e^- !

$$||L|| = \hbar \sqrt{L(L+1)}$$

$L_z = m_l \hbar$

Can be arbitrarily large!

ii) Spin:

Sometimes compared to ball rotating around its own axis

Refers: "intrinsic angular momentum"

QM

$$||S|| = \hbar \sqrt{S(S+1)}, S_z = m_s \hbar$$

$$I = \frac{1}{2} \hbar \sqrt{I(I+1)}, I_z = m_I \hbar$$

↳ For MRI we don't consider e^- , but rather full nuclei - in simplified case $H \hat{=} \text{"Proton"}$

$\Rightarrow$ Only Spin angular momentum! (No orbital)

↳ Charged Particles: Angular momentum proportional

to magnetic dipole moments (classical physics):

$$\vec{\mu} = \gamma \vec{S}, \gamma = \frac{e}{2m_p} \dots \text{gyromagnetic ratio}$$

↳ Consider $I = \frac{1}{2}$ nuclei (e.g. $^1H, ^{13}C, ^{31}P$)

$$I = \frac{1}{2} \Rightarrow m_I = \pm \frac{1}{2} \Rightarrow S_z = \pm \frac{1}{2} \hbar \Rightarrow \mu_z = \pm \frac{1}{2} \gamma \hbar$$

↳ For an external magnetic field $\vec{B} = B_0 \cdot \vec{e}_z$

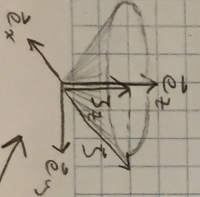
$$E = -\vec{\mu} \cdot \vec{B} = -\mu_z B_0 = \pm \frac{1}{2} \gamma \hbar B_0$$

$$E_{\pm \frac{1}{2}} = \pm \frac{1}{2} \gamma \hbar B_0$$

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NOTE: Dipole moment / Spin ratio of 1H is $\gamma = 2.675 \times 10^8 \text{ rad T}^{-1} \text{ s}^{-1}$



↳ Not all protons in low-energy state (Tend away)

Boltzmann statistics: $\frac{n_{+1/2}}{n_{-1/2}} = \exp(-\frac{\Delta E}{k_B T})$

$$\exp(-x) \approx 1 - x \Rightarrow \frac{n_{+1/2}}{n_{-1/2}} \approx 1 - \frac{\Delta E}{k_B T} \Rightarrow n_{+1/2} = (1 - \frac{\Delta E}{k_B T}) n_{-1/2}$$

$$\Rightarrow \Delta n = n_{+1/2} - n_{-1/2} = \frac{\Delta E}{k_B T} n_{+1/2}$$

$$n = n_{+1/2} + n_{-1/2} = (2 - \frac{\Delta E}{k_B T}) n_{+1/2}$$

↳ Population difference given by the macroscopic magnetization

$$M_z = \sum \mu_z = \Delta n \mu_z$$

$$\Delta E = \gamma \hbar B_0 = \hbar \omega$$

$$\omega = \gamma B_0 \dots \text{Larmor frequency}$$

$$\Rightarrow \text{Problem: } \frac{\Delta E}{k_B T} \ll 1 \Rightarrow \Delta n \approx \frac{\Delta E}{k_B T} n$$

$$\Rightarrow \text{MRI suffers from low sensitivity!}$$

(We try to detect single molecules)

↳ Dynamics of macroscopic magnetization:

Side Note - QM:

$M_0 \dots$ "Ensemble of individual protons" $\rightarrow$ Should consider

averaging the expected value of a single proton:

$$\text{Consider a proton at rest } \rightarrow \text{Two possible spin states } |S, m_s\rangle = |\frac{1}{2}, \pm \frac{1}{2}\rangle$$

$$\text{"Use eigenvalues of } \hat{S}_z \text{-operator: } |\frac{1}{2}, \pm \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Ansatz for general wave function: } \Psi = a |\frac{1}{2}, \pm \frac{1}{2}\rangle + b |\frac{1}{2}, \mp \frac{1}{2}\rangle$$

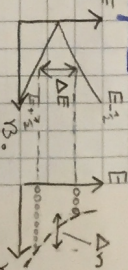
$$\Psi = (a \exp(i \frac{1}{2} \gamma B_0 t) + b \exp(-i \frac{1}{2} \gamma B_0 t)) \cdot e^{-i \frac{E_0}{\hbar} t}$$

$$\text{Ansatz can be solved by Schrödinger equation: } \hat{H} \Psi = E \Psi$$

$$\text{When } \hat{H}_0 \text{ Hamiltonian for a nucleus in a magnetic field}$$

$$\hat{H}_0 = \gamma \hbar B_0 \hat{S}_z$$

$$\text{For a single nucleus/proton: } S_x, S_y \text{ components are "not defined", mixed with } S_z, S_y$$



MRI only works because there are a lot of 1H atoms

MORE
LATER =

Macroscopic Magnetization
(Ensemble of protons)

See Griffiths QM p. 179

↳ ad Dynamics of macroscopic magnetization:
ad QM - Side Note:

For a 2-spin state system of 2 spins = Eigenstate of $\hat{S}_z$
 $\hat{S}_z^2 = \frac{3}{4} \hbar^2$, $\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Writing initial condition of wave function as
 $\alpha = \cos(\frac{\theta}{2})$, $\beta = \sin(\frac{\theta}{2}) \Rightarrow \Psi = \begin{pmatrix} \cos(\frac{\theta}{2}) \exp(i\omega_0 t) \\ \sin(\frac{\theta}{2}) \exp(i\omega_0 t) \end{pmatrix}$

Can use ad expected values for $\langle x, y, z \rangle$ - comp.

ad spin (= magnetization):

$$\langle S_x \rangle = \langle \Psi | \hat{S}_x | \Psi \rangle = \frac{\hbar}{2} \alpha \sin \alpha \cos(\omega_0 t)$$

$$\langle S_y \rangle = \langle \Psi | \hat{S}_y | \Psi \rangle = \frac{\hbar}{2} \alpha \sin \alpha \sin(\omega_0 t)$$

$$\langle S_z \rangle = \langle \Psi | \hat{S}_z | \Psi \rangle = \frac{\hbar}{2} \alpha \cos \alpha$$

Spin vector (magnetization) precesses around $\hat{e}_z$, where

α is determined from init. cond.

Remark: Should compare to dynamics of "macroscopic ensemble"

IN EXPECTATION

↳ Above: Very simple set-up (no relaxation)! Can

more generally derive dynamics of macroscopic magnetization $\rightarrow$ BLOCH EQUATIONS:

$$\vec{M} = \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix}, \quad \frac{d}{dt} \vec{M} = \gamma (\vec{M} \times \vec{B}) - \begin{pmatrix} \frac{1}{T_2} & 0 & 0 \\ 0 & \frac{1}{T_2} & 0 \\ 0 & 0 & \frac{1}{T_1} \end{pmatrix} (\vec{M} - \vec{M}_0)$$

↳ T_2 ... Transversal Relaxation

↳ T_1 ... Longitudinal Relaxation

↳ Accurate phenomenological equations for return to equilibrium

$$\vec{M}_0 = \begin{pmatrix} 0 \\ 0 \\ M_0 \end{pmatrix} \rightarrow \text{direction of } \vec{B} = B_0 \hat{e}_z$$

↳ General trends of relaxation times:

$$T_1 = \rho(B_0), \quad T_1 \ll 1000 \text{ ms}, \quad T_2 \neq \rho(B_0), \quad T_2 \ll 100 \text{ ms}$$

• Fluids (e.g. CSF) very long relaxation times

• $B_0 \uparrow : T_1 \uparrow$ (Rudin: $T_1 \propto \omega_0^2 \propto (B_0)^2$)

↳ Solving the Bloch equation for 2 important configurations

Prequisite: Time derivatives in rotating reference frames

$$[\vec{M}]_{\text{LAB}} = \sum M_i(t) \cdot \hat{e}_i$$

$$[\vec{M}]_{\text{ROT}} = \sum M_i(t) \cdot \hat{e}_i'(t)$$

Both representations should describe the same vector!

$$[\vec{M}]_{\text{LAB}} = [\vec{M}]_{\text{ROT}} = \vec{M}$$

We're interested in the (of course unique) derivative (NOTE: derivative of $\vec{M} \rightarrow \frac{d}{dt} \vec{M} = \frac{d}{dt} [\vec{M}]_{\text{LAB}} = \frac{d}{dt} [\vec{M}]_{\text{ROT}}$)

(but expressed in terms of the rotating frame):

$$\frac{d}{dt} \vec{M} = \frac{d}{dt} \sum M_i(t) \cdot \hat{e}_i'(t) = \sum \left(\frac{d}{dt} M_i(t) \cdot \hat{e}_i'(t) + M_i(t) \cdot \frac{d}{dt} \hat{e}_i'(t) \right)$$

$$(i) \sum \left(\frac{d}{dt} M_i(t) \right) \hat{e}_i' = \left[\frac{d}{dt} \vec{M} \right]_{\text{ROT}} \rightarrow \text{NOTE: Now } \frac{d}{dt} \text{ is inside } [\]_{\text{ROT}}$$

$$(ii) \frac{d}{dt} \hat{e}_i' \dots \text{Velocity vectors of rotating frame}$$

$$\Rightarrow \text{Of course } \frac{d}{dt} \hat{e}_i' = \vec{\omega}_0 \times \hat{e}_i'$$

$$\text{So: } \sum M_i' \cdot \frac{d}{dt} \hat{e}_i' = \sum M_i' (\vec{\omega}_0 \times \hat{e}_i') = \vec{\omega}_0 \times \sum M_i' \hat{e}_i'$$

$$= \vec{\omega}_0 \times [\vec{M}]_{\text{ROT}} = \vec{\omega}_0 \times \vec{M}$$

$$\text{We derived: } \frac{d}{dt} \vec{M} = \left[\frac{d}{dt} \vec{M} \right]_{\text{ROT}} + \vec{\omega}_0 \times \vec{M}$$

$$(*) \quad \frac{d}{dt} \hat{e}_x' = \cos(\omega_0 t) \hat{e}_x + \sin(\omega_0 t) \hat{e}_y \quad \left| \frac{d}{dt} \hat{e}_y' = -\sin(\omega_0 t) \hat{e}_x + \cos(\omega_0 t) \hat{e}_y \right.$$

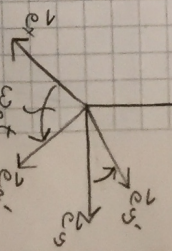
$$\Rightarrow \frac{d}{dt} \hat{e}_x' = -\omega_0 \sin(\omega_0 t) \hat{e}_x + \omega_0 \cos(\omega_0 t) \hat{e}_y = \omega_0 \cdot \hat{e}_y'$$

$$\Rightarrow \frac{d}{dt} \hat{e}_y' = -\omega_0 \cos(\omega_0 t) \hat{e}_x - \omega_0 \sin(\omega_0 t) \hat{e}_y = -\omega_0 \cdot \hat{e}_x'$$

$$\Rightarrow \frac{d}{dt} \hat{e}_z' = 0$$

$$\text{Same for } \hat{e}_i'$$

Exception: Fluid



DECOUPLED SYSTEM

↳ Now for rotating frame equation:

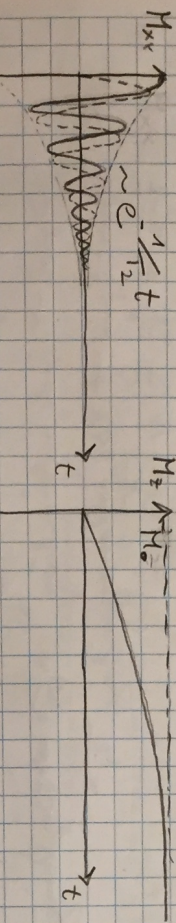
① Static magnetic field $\vec{B} = B_0 \hat{e}_z$, with incl. coord. $\vec{M}(0) = (M_{x0}, M_{y0}, M_{z0})^T$

Transform into rotating frame ($\vec{\omega} = -\gamma \vec{B}$)

$$\frac{d}{dt} \vec{M} = \left[\frac{d}{dt} \vec{M} \right]_{rot} + \vec{\omega} \times \vec{M} = \gamma (\vec{M} \times \vec{B}) - \text{diag}(\frac{\gamma}{2}, \frac{\gamma}{2}, 0) [\vec{M} - \vec{M}_{eq}]$$

$$\left[\frac{d}{dt} \vec{M} \right]_{rot} = -\text{diag}(\frac{\gamma}{2}, \frac{\gamma}{2}, 0) [\vec{M} - \vec{M}_{eq}]$$

$$\begin{aligned} M_x' &= M_{x0}' \cdot \exp(-\frac{\gamma}{2} t) \\ M_y' &= M_{y0}' \cdot \exp(-\frac{\gamma}{2} t) \\ M_z' &= M_{z0}' + (M_{z0} - M_{z0}') \cdot \exp(-\frac{\gamma}{2} t) \end{aligned}$$



100% - frame change:

For disturbed initial condition ($\vec{M}_0 \neq \vec{M}_{eq}$), $\vec{M}$ precesses around static magnetic field vector!

$\vec{\omega} = -\gamma \vec{B} \Rightarrow$ Rotating frame component induces voltage in pick-up coil $\Rightarrow$ NMR-signal!

② How do disturb magnetization out of equilibrium?

$\rightarrow$ Oscillating B_z -field perpendicular to M_{eq}

$$\vec{B} = \begin{pmatrix} B_1 \cos(\omega t) \\ 0 \\ B_0 \end{pmatrix}$$

NOTE: linearly polarized field can always be decomposed in 2 counter-rotating circles

$$B_1 \cos(\omega t) \hat{e}_x = \frac{1}{2} (B_1 e^{i\omega t} \hat{e}_x + B_1 e^{-i\omega t} \hat{e}_x)$$

and ②: ignore relaxation terms in Bloch: $\dot{M} = \gamma \vec{M} \times (\vec{B}_0 + \vec{B}_1)$

We place ourselves again in a rotating frame - one that moves with one of the circular polarizations - will hardly see diff. one...

$$\vec{B}_1 = B_1 [\cos(\omega t) \hat{e}_x + \sin(\omega t) \hat{e}_y], \vec{B}_0 = B_0 \hat{e}_z$$

$$\frac{d}{dt} \vec{M} = \gamma (\vec{M} \times \vec{B}) = \gamma (\vec{M} \times (\vec{B}_0 + \vec{B}_1)) = \gamma (\vec{M} \times \vec{B}_0) + \gamma (\vec{M} \times \vec{B}_1)$$

$$= \gamma \vec{M} \times (\vec{B}_0 + \vec{B}_1) = \gamma \vec{M} \times \left[(B_0 + \frac{\omega}{\gamma}) \hat{e}_z + B_1 \hat{e}_x' \right]$$

$$\Rightarrow \left[\frac{d}{dt} \vec{M} \right]_{rot} = \gamma \vec{M} \times \vec{B}_{eff}$$

$\vec{B}_{eff}$ is exactly the same as in ①, i.e. magnetization will precess around $\vec{B}_{eff}$!

Now we can choose $\omega = -\gamma B_0$ (i.e. we choose to rotate with B_0 - same direction as $\vec{M}$ would precess if there is only $B_0 \hat{e}_z$ alone) $\Rightarrow B_{eff} = B_1 \hat{e}_x'$

$\Rightarrow$ In rotating frame of reference when M describes a very simple motion:

Notes:

(i) When switching into rotating frame we

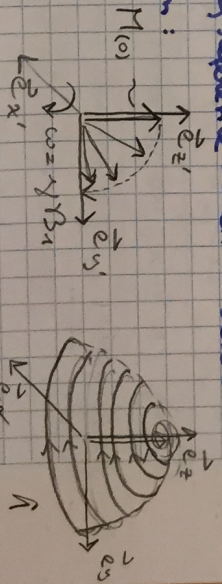
$$\text{simplified: } \vec{B}_1 = 2B_1 \cos(\omega t) \hat{e}_x = \vec{B}_1 + \vec{B}_1 \approx 2B_1 \hat{e}_x' \text{ FRAME}$$

i.e. we only have $B_1 \hat{e}_x'$, but ignored B_0

diff. rotates with $\omega = 2\omega$ in rotating frame!

Rotating M wiggles around the above discussed "average"

(ii) Since $B_0 \gg B_1 \rightarrow$ Resonance $\omega = \omega_0$ is very "sharp"



No surprise nobody ever used
 $\Delta E \propto \hbar \gamma B_0$ for imaging before
 MRI $\Rightarrow$ terrible resolution $\Delta x \propto \frac{1}{\gamma}$

$\hookrightarrow$ Take some message from the above:

Given a magnetization in equilibrium $\vec{M} = \vec{M}_0 = M_0 \vec{e}_z$,
 can be tipped over by an angle θ , if we switch
 on $B_1 = 2\gamma B_1 \cos(\omega t)$ for a time τ (pulsing)

$\theta = \omega \tau = \gamma B_1 \tau$
 $\omega = \gamma B_0$

$\hookrightarrow$ Make an relaxation:

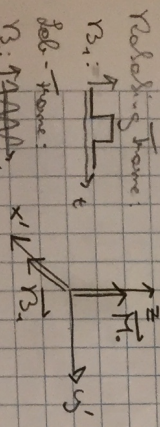
(i) Spin-lattice relaxation T_1
 "loss of energy"

STOCHASTIC $\Rightarrow$ (ii) Spin-spin relaxation T_2
 "spin-spin interaction" $\propto \frac{1}{M_0} \Rightarrow$ depending

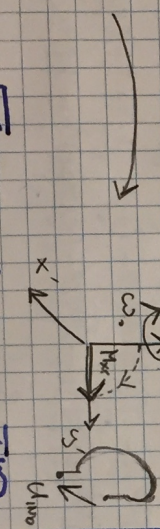
DETERMINISTIC (iii) Spin-spin + B_0 inhomogeneity T_2^*
 $\frac{1}{T_2^*} = \frac{1}{T_2} + \gamma \cdot \Delta B_0$
 $\Delta B_0 \neq 0$ (field distribution μ_B)

$\hookrightarrow$ Basic MRI - Experiments:

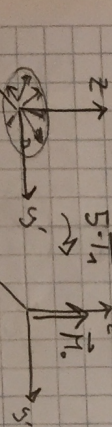
I. Excitation:



II. Free induction decay



III. Relaxation:



Free induction decay (induced voltage)
 $V(t) = V_0 e^{-t/T_2^*}$
 $T_2^* = T_2 + \gamma \cdot \Delta B_0$

II. Image formation:

$\hookrightarrow$ FID or dephased before only given info. about total magnetization/spin density of whole sample!

$\hookrightarrow$ How to encode information spatially? $\rightarrow$ Field gradients

$\hookrightarrow$ Make some frequency dependent on (x, y, z) by introducing

$G_x = \frac{\partial}{\partial x} B_z$ $G_y = \frac{\partial}{\partial y} B_z$ $G_z = \frac{\partial}{\partial z} B_z$

(NOTE: $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ direction)

$\hookrightarrow$ 3 methods to spatially encode information:

(i) Selective excitation & induction of spin echo:

(ii) Slice selective excitation (90° -pulse)

(iii) Application of G_x for line $2\pi\tau$

(iv) All time T_1 relative inversion (180° -pulse)

only acting on a band $\Delta\omega$

(v) Spin echo form only for $\Delta\omega$ -band

$\Rightarrow$ acquisition under influence of G_x

(vi) Fourier - Transform gives spin density along a line in the plane

$\Rightarrow$ rxn nra region in cycles read yield acquired area!

(2) Projection Reconstruction Technique:

(i) Slice selective excitation (90° pulse)

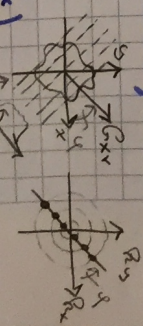
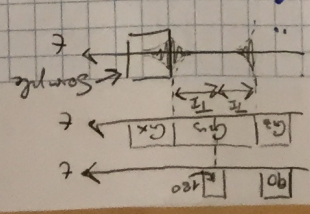
(ii) Record under influence of $G_x = (G_x)$

(iii) Can use 2D projection technique (A) lower frequency!

(iv) rxn-nra: in cycles, but Rigid SNR (measured)

$\hookrightarrow$ Spatial encoding
 "Spin" = frequency spectrum of FID = measure for magnetization $M_0 \rightarrow$ density

SOME NUMBERS: Larmor freq. in tissue ~ 42.57 MHz/T, i.e. $\gamma = 42.57$ MHz/T
 $\Rightarrow$ Larmor: $\mu_B \sim 1, \gamma \sim 80 \Rightarrow \text{Crisse} \sim \frac{1}{\mu_B} = \frac{C_0}{\mu_B \hbar}$
 $\Rightarrow \lambda_{\text{tissue}} = \frac{C_0}{\gamma \hbar \gamma} \approx 30 \text{ cm}$



(*) Note: ΔQ_x determines ΔT_x
 [R_{xx}] = $\frac{1}{m} \dots$ Spatial Res. "

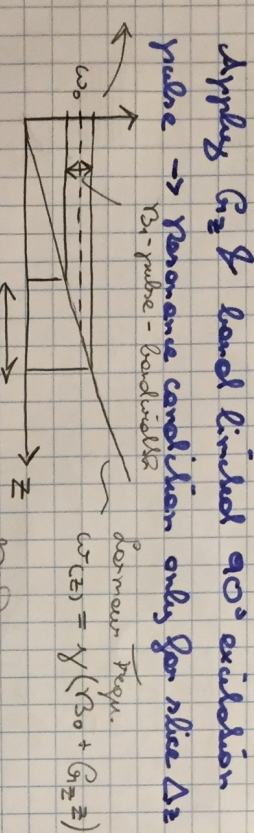
NOW THE DETAILS

Let's refresh the previously encoded information:

③ Fourier Imaging (R. Ernst - Chemist @ ETH)

(i) Slice selection: ("Encoding" z-axis)

How to band-limit? Probably Gaussian envelope for $B_1(t)$



Apply G_z & band limited 90° excitation pulse $\rightarrow$ resonance condition only for slice Δz

(ii) Frequency - encoding: ("Encoding" x-axis)

During sampling with an $G_x \Rightarrow$ frequency of FID is a function of x: $\omega(x) = \gamma(B_0 + G_x x)$

(iii) Phase - encoding: ("Encoding" y-axis)

Before sampling with an G_x for line T_x

Recorded signal: $S(x,y) = \int \int G(x,y) \cdot e^{i(\omega(x)t + \phi(y))} dx dy$

Demodulation: $\omega(x) \rightarrow \gamma G_x x$ & $\phi(y) \rightarrow \gamma G_y y$

iv) Sample the signal - "Traverse R-space"

Do n imaging cycles with phase encoding gradients $G_y^{(e)}$ ($e=1, \dots, n$) & for each have n samples of the signal at times $t = T_x^{(a)}$ ($a=1, \dots, n$)
 $\Rightarrow S_{ae} = \int \int G(x,y) \cdot \exp(i[\gamma G_x x T_x^{(a)} + \gamma G_y y T_x^{(e)}])$

Let's refresh the previously encoded information:

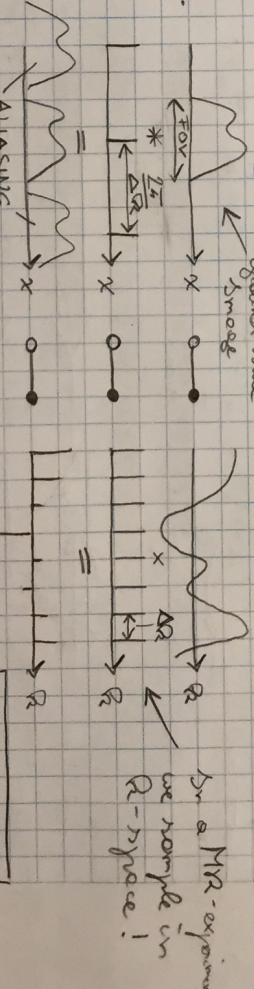
③ Fourier Imaging: Define: $R_x^{(a)} = \gamma G_x T_x^{(a)}$, $R_y^{(e)} = \gamma G_y T_x^{(e)}$

$\Rightarrow S_{ae} = \int \int G(x,y) \cdot \exp(i[R_x^{(a)} x + R_y^{(e)} y]) dx dy$

(v) Reconstruction - Simple Fourier Transform

For each sample, we convert $T_x^{(a)}$ & $T_x^{(e)}$ to sample $R_x^{(a)}$ & $R_y^{(e)} \Rightarrow$ Place recorded sample S_{ae} in $(n \times n)$ data matrix $S(R_x, R_y)$ at position $(R_x^{(a)}, R_y^{(e)}) \Rightarrow S(x,y) = \mathcal{F}[S(R_x, R_y)]$

Aliasing & Resolution in Fourier - Imaging: (*)



Avoid Aliasing: $FOV < \frac{2\pi}{\Delta x} \Rightarrow \Delta R < \frac{2\pi}{FOV}$

Resolution: Determined from FOV & # samples N

Take in: $\mathcal{F}$ in x-direction: $FOV_x = \Delta x \cdot N_x$

i) $\gamma G_x \Delta x = \frac{2\pi}{FOV}$ Use relations $FOV_x = \frac{2\pi}{\Delta R_x}$ & $\gamma B_1^2 = \Delta R_x \cdot N_x \Rightarrow \Delta R_x = \Delta x \cdot N_x \Rightarrow \Delta x = \frac{2\pi}{\gamma B_1^2 N_x}$

ii) $\gamma G_y \Delta y = \frac{2\pi}{FOV}$ Use relations $FOV_y = \frac{2\pi}{\Delta R_y}$ & $\gamma B_1^2 = \Delta R_y \cdot N_y \Rightarrow \Delta R_y = \Delta y \cdot N_y \Rightarrow \Delta y = \frac{2\pi}{\gamma B_1^2 N_y}$

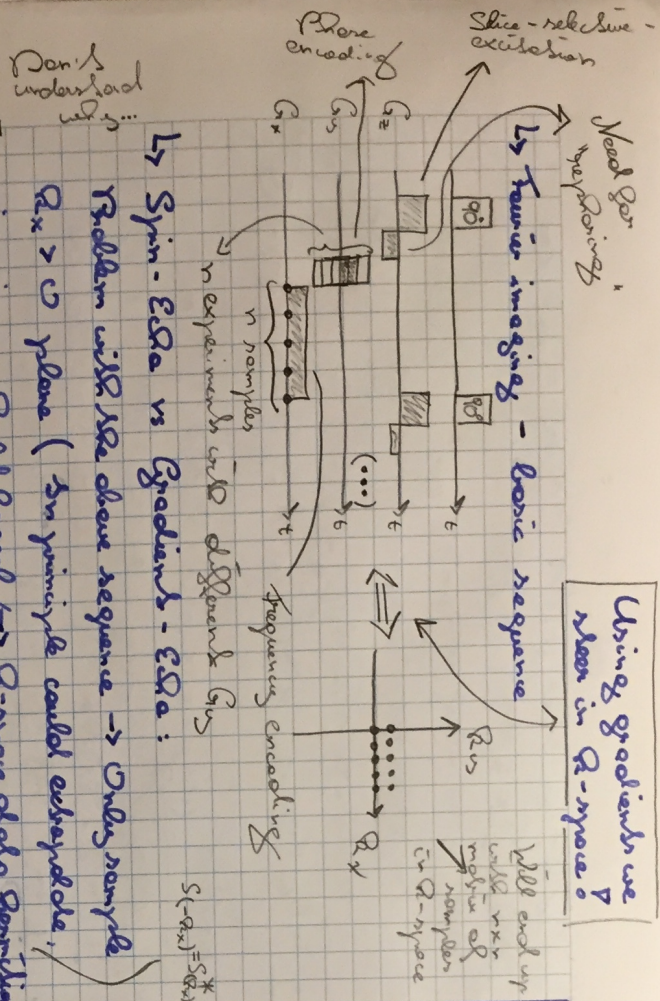
FOV of Fourier - Imaging: (APPROX. MRI Q & LSI - Single Scan: $FOV \approx 2 \cdot \pi \cdot H \cdot \gamma \cdot B_1 \cdot \Delta x \cdot \Delta y$ can get H by $\Delta x \cdot \Delta y$ which we do in R-space)

• x, y : i) We band-limit $\rightarrow$ Truncate R-space (rect() in MATLAB)
 ii) Relaxation $\rightarrow$ Signal in R-space decays exponentially (exp(-1/T1) or "T1-weighting")

• z : Truncate RF-pulse in time domain

Need for
rephasing

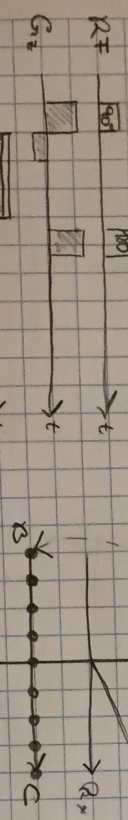
Using gradients we
move in k -space



Spin-Echo vs Gradient-Echo:

Problem with SE above sequence $\rightarrow$ Only sample $k_x \rightarrow 0$ plane (In principle could extrapolate, since image should be real $\leftrightarrow$ k -space data Real & Imag. BUT: Can introduce artifacts due to motion!)
2 methods to sample $k_x \neq 0$ plane. See:

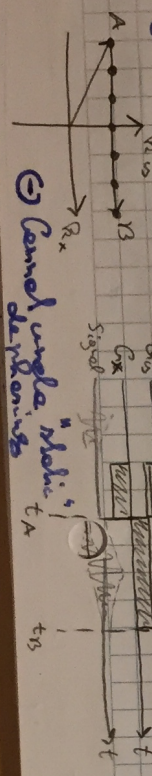
(1) Spin-Echo:



(+) Can undo dephasing due to B_0 -inhomogeneity

(-) Long time & intense 180° pulse

(2) Gradient-Echo:



(-) Can't undo 'static' dephasing

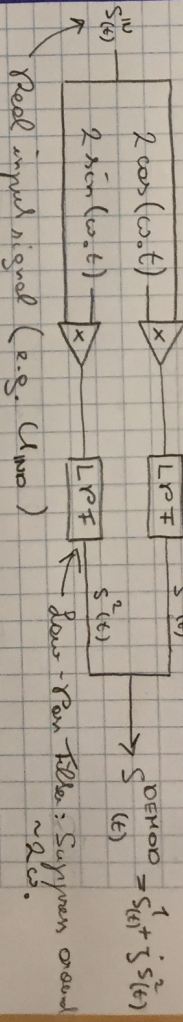
Note: Demodulation of induced signal

We derived SE recorded signal at time t or $s_{SE} = \iint G(x, y, z) \cdot \exp(i[Q_x^{(SE)}x + Q_y^{(SE)}y]) dx dy$

For $Q_x^{(SE)} = \gamma G_x t$ & $Q_y^{(SE)} = \gamma G_y t$

Then we can compare of perfect demodulation of carrier frequency (carrier frequency) $\omega_c = \gamma B_0$.

How does this work in practice? E.g. $1/(2 - \text{Demod.})$



Consider only frequency encoding. How

$s^{(1)}(t) = m_0 \cdot \cos(\omega_c t + \Delta\omega t)$

$s^{(2)}(t) = \text{LPF}\{2 \cos(\omega_c t) \cdot s^{(1)}(t)\}$

$= \text{LPF}\{2 m_0 \cos(\omega_c t + \Delta\omega t) \cos(\omega_c t)\}$

$= \text{LPF}\{m_0 [2 \cos(2\omega_c t + \Delta\omega t) + \cos(\Delta\omega t)]\}$

$= m_0 \cdot \cos(\Delta\omega t)$

$s^{(2)}(t) = \text{LPF}\{2 \sin(\omega_c t) \cdot s^{(1)}(t)\}$

$= \text{LPF}\{2 m_0 \cos(\omega_c t + \Delta\omega t) \cdot \sin(\omega_c t)\}$

$= \text{LPF}\{m_0 [2 \sin(2\omega_c t + \Delta\omega t) - \sin(\Delta\omega t)]\}$

$= -m_0 \cdot \sin(\Delta\omega t)$

$\Rightarrow s^{\text{DEMOD}}(t) = s^{(1)}(t) + i s^{(2)}(t) = m_0 \cdot e^{-i\Delta\omega t}$

Trig. Id. $\cos(\alpha) + \cos(\beta) = 2 \cos(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$
 $\sin(\alpha) - \sin(\beta) = 2 \sin(\frac{\alpha+\beta}{2}) \sin(\frac{\alpha-\beta}{2})$

1. 'In-phase' (see cos)

2. 'Quadrature' (meaning $\frac{\pi}{2}$ phase shift)

Goal Demodulation:

Try explain also the origin of complex-valued data in MRI-images:

We get the image as $S(x,y) = \mathcal{F}(S_{AE})$

From properties of the Fourier Transform we know that

Real Signal $\Leftrightarrow$ Hermitian FT $|\mathcal{F}(x-y) = \mathcal{F}(x)^*$

Indeed our demodulated signal is Hermitian, as

$S(t) = m \cdot e^{-i\Delta\omega t} = m \cdot e^{-i\Delta\omega_x \cdot x} \Rightarrow S(-\Delta\omega_x) = S^*(\Delta\omega_x)$

For the ideal case, where the only source of $\Delta\omega$ is the magnetic field gradients!

However: With some many effects (B₀-inhomogeneity, motion,...) disturbing MRI Hermitian property $\Rightarrow$ give complex image

Why is signal of EPI increasing in time?
 Here we ignore relaxations & only consider the dominant gradient echoes!
 Rephasing due to short G_y
 Rephasing due to long G_x

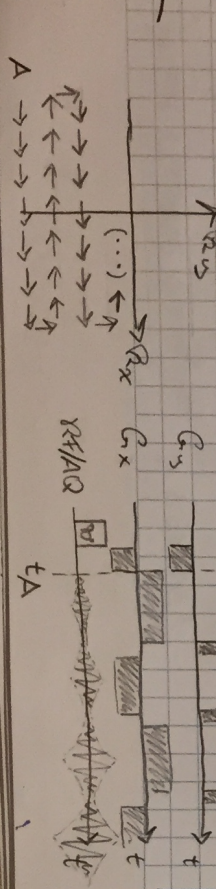
III. Fast Imaging:

$\hookrightarrow$ Reduce acquisition time via optimal R-rates
 Acquisition!

i) Gradient-Echo parallel MRI (pMRI) (see above)

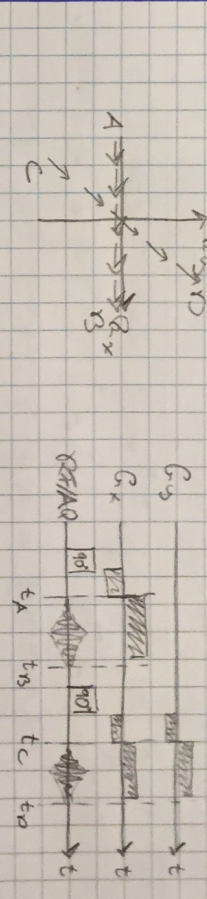
ii) Echo-planar imaging (EPI)

The unique (in principle) acquire whole slice with a single excitation: G_x

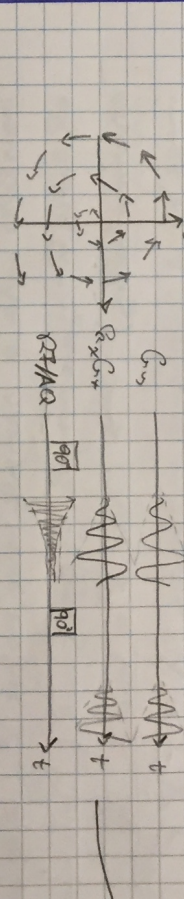


$\hookrightarrow$ optimal R-rates Acquisition:

iii) Radial imaging



iv) Spiral imaging



$\hookrightarrow$ Collection speed is limited by strength of gradients!

(Safety limit on how fast we may build up gradients! Stronger gradients need to be switched slower!)

$\hookrightarrow$ Parallel Imaging: (on the example of SENSE)

2 ways of multi-resolution resolution { Sparse Sensitivity

Parallel Imaging methods combine both encoding

principles: $S_{yx} = \int S(x_2) \cdot G_y(x_2) \cdot e^{i\Delta\omega_x \cdot x_2} dx_2$

Signal in coil y when sampling R-rates pairs G_x (using a corresponding gradient region) "High of encoding"

Because $G_y(x_2)$ is not determined by digital phase, can use multiple coils during same experiment, different encodings of same line = "Parallel imaging"

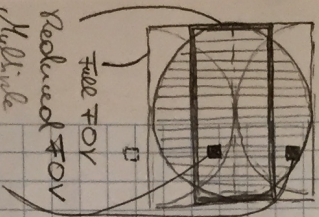
COVER MORE SAMPLES AFTER A SINGLE EXCITATION

UNDER SAMPLING for acquisition speed up only possible in PHASE ENCODING DIRECTION (y-direction)

↳ Parallel Imaging - SENSE:

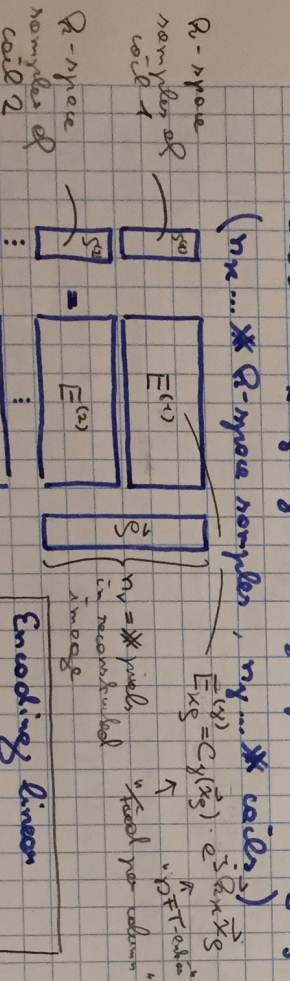
How do we multiple coils for fast imaging?
Can undersampling (k-space) from undersampling!

⇒ Undersampling with R-nyes resolution (more R-nyes)
But need to do aliasing & reduced FOV. SENSE:
"Unfold convolution from all (aliasing) pixels in full FOV, to a pixel in reduced FOV!"



Weights in different coils!
TRA model: Using a convolutional kernel
Sample R-nyes position $\vec{R}_k$ with coil y_i :
 $S_{y_i, k} = \sum_i S_i(\vec{x}_g) \cdot C_{y_i}(\vec{x}_g) \cdot e^{i\vec{R}_k \cdot \vec{x}_g}$

Obtain $m \times m$ equations for pixel values $S_i(\vec{x}_g)$
($m \dots R$ -nyes samples, $m \dots R$ coils)
 $\vec{E} \cdot \vec{S} = \vec{y}$
 $\vec{E}_{k \times g} = C_{y_i}(\vec{R}_k) \cdot e^{i\vec{R}_k \cdot \vec{x}_g}$
↑
"Fast Fourier Transform" (FFT) for columns
↑
"Fast Fourier Transform" (FFT) for rows



Encoding linear
⇒ linear system for decoding
 $\vec{y} = \vec{E} \vec{S}$

ENCODING MATRIX

RECONSTRUCTION MATRIX

↳ Parallel Imaging - SENSE - old Model

Encoding $\vec{S} = \vec{E} \vec{S}$
Decoding $\vec{S} = \vec{F} \vec{y}$
Autotune constraints
 $\vec{F} \vec{E} = \vec{I}_{m \times m}$

Usually have $m \times m \dots$ over determined system
Two solutions:
i) Ordinary least squares: $\vec{F} = (\vec{E}^H \vec{E})^{-1} \vec{E}^H$
ii) Optimal SNR: $\vec{F} = (\vec{E}^H \vec{W} \vec{E})^{-1} \vec{E}^H \vec{W}$

Note: Optimal SNR - solution:
Consider noise spectral from independent processes within the sample/imaging system/...
Each pixel up the noise power (only with different resolution) ⇒ "Residual noise is correlated"

Translates into a sample noise vector $\vec{w} \in (m \times m) \dots$ covariance of every R-nyes sample of every coil (determined experimentally).
Once $\vec{W}$ is known, the image noise covariance is simply $\vec{X} = \vec{F} \vec{W} \vec{F}^H$

Obtain reconstruction matrix as minimizer of X
under the constraint $\vec{F} \vec{E} = \vec{I}_{m \times m}$:
 $\vec{F} = \arg \min (\vec{F} \vec{W} \vec{F}^H + \lambda [\vec{F} \vec{E} - \vec{I}])$
⇒ ... ⇒ $\vec{F} = (\vec{E}^H \vec{W} \vec{E})^{-1} \vec{E}^H \vec{W}$

TRA - R-nyes message: Variance (R-nyes SNR) is correlated across image & dependent on location of coils!

NOTE:
+ $\vec{E} \dots$ Spatial response
↑
"Fast Fourier Transform" (FFT) for rows
↑
"Fast Fourier Transform" (FFT) for columns

Get rid of noise for obvious reason $\vec{F} = \vec{F} \vec{S} = \vec{F} \vec{E} \vec{S}$

Good Recall Smearing - SENSE:

Specially dependent SNR specific to the

$$g_{\text{sm}} = \sqrt{\frac{(E^H E)_{11}}{(E^H E)_{ii}^{-1}}}$$

$$SNR_{\text{SENSE}} = \frac{SNR_{\text{Full}}}{\sqrt{R_2} \cdot g_{\text{sm}}}$$

"Adding the negative peaks superimposed during encoding" (Rigley smearing dependent)

Especially in very Rigid R₂ > 4T can

Note: Coil nonuniformity

Reciprocity: $C_{\text{sm}} \propto R_2(x^2)$ where R_2 in

the field, when driving coil with current I !

$$R_{\text{ind}} - \text{Savart: } R_2 = \frac{\mu_0}{4\pi} \oint \frac{I d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

IV. Image Contrast:

Reminders: Relaxation

T_1 : Spin-Lattice-Interaction (Triple-Triples, chemical exchange, ...)

T_2 : "Spin-Spin"-Interaction (Triple-Triples, any motion, e.g.: Brownian motion results in dephasing (* through R₂-incoherence))

In R₂: T_1 increases with R_2

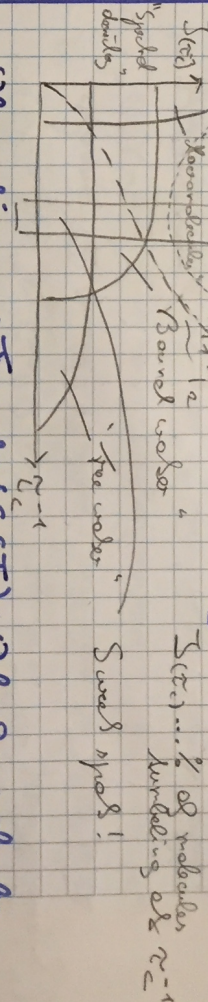
T_2 is constant in R_2 .

Good Reminder: Relaxation

R_2 in T_1 increasing with R_2 ?

T_1 is robust when the molecular correlation time

τ_c is comparable to Larmor frequency! ($\propto \tau_c \cdot \gamma B_0$)



Observations: $\rightarrow$ Free water ($C \approx 5 \text{ T}$): Only free molecular

of water spin $\rightarrow$ long T_1 . Also

relatively independent, since $\tau_c \propto \gamma B_0$

$\rightarrow$ Bound water: Smaller $R_2 \rightarrow$ R₂ R₁

much spin density $\rightarrow$ $\tau_c \propto \gamma B_0$ molecules

i.e. R₂ R₁ - times

$\rightarrow$ Until now: Assumed long repulsion time, and that

magnetization is in thermal equilibrium before every

excitation ($T_R \gg T_1$, e.g. $T_R \approx 2000 \text{ ms}$)

$\Rightarrow$ Obtain spin density map $g_{\text{sm}}(x,y)$

$\rightarrow$ You have some parameters T_1 & T_2 for image

contrast? Don't wait until thermal equilibrium!

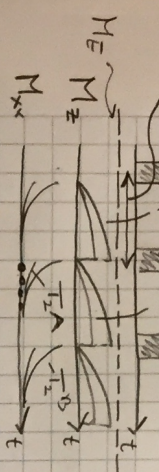
Recall: "Recovery of longitudinal magnetization after

on excitation of angle α " $\rightarrow M_z(t=0) = M_z \cdot \cos \alpha$

$$R_2 \text{ R}_1: \frac{d}{dt} M_z = \frac{1}{T_1} (M_z - M_z) \Rightarrow M_z(t) = M_z + C \cdot \exp(-\frac{t}{T_1})$$

$$= M_z \cos \alpha - M_z = M_z \left[1 - \exp(-\frac{t}{T_1}) \right] + M_z \cos \alpha \cdot \exp(-\frac{t}{T_1})$$

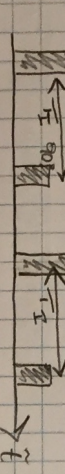
Partial Saturation Method - T_1 weighting



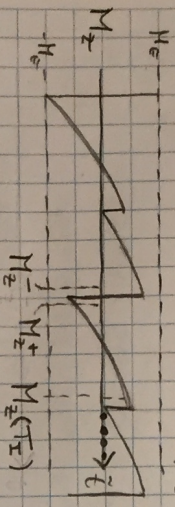
$T_1^A < T_1^B$
Time A appears brighter!
 $T_1 \approx 500$ ms

Signal proportional to $(1 - \exp(-\frac{T_R}{T_1}))$: $\alpha = 90^\circ$
 $\sigma \propto G(x,y) \cdot (1 - \exp(-\frac{T_R}{T_1}))$
 - longitudinal mag. at end of cycle
 - will be smaller for acquisition

Spin Echo Method - T_2 weighting



Can use Spin Echo.
No global magnetization will given T_1



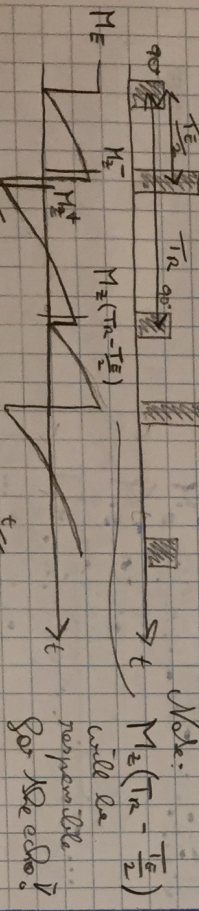
Example values:
 $T_1 \approx 500$ ms
 $T_2 \approx 200$ ms

$$M_z^+ \propto G(x,y) [1 - \exp(-\frac{T_R}{T_1})]$$

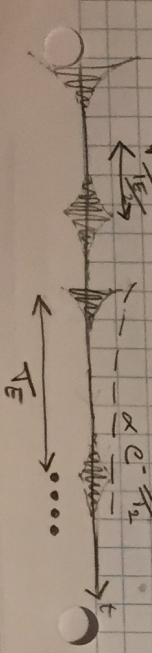
$$M_z^+ = -M_z^-$$

$$M_z(T_1/2) \propto G(x,y) [1 - \exp(-\frac{T_1}{2T_1})] + M_z^- \exp(-\frac{T_1}{2T_1})$$

Spin Echo Method - T_2 weighting



Note:
 $M_z(T_1 - \frac{T_1}{2})$
 will be reproducible for the echo!



Spin Echo Method - T_2 weighting

$$M_z^- \propto G(x,y) [1 - \exp(-\frac{T_R}{2T_1})], \quad M_z^+ = -M_z^-$$

$$M_z(T_1 - \frac{T_1}{2}) \propto G(x,y) [1 - \exp(-\frac{T_1}{2T_1})] + M_z^+ \exp(-\frac{T_1 - T_1/2}{T_1})$$

$$M_z(T_1 - \frac{T_1}{2}) \propto G(x,y) [1 - 2 \exp(-\frac{T_1 - T_1/2}{T_1}) + \exp(-\frac{T_1 - T_1/2}{T_1})]$$

Recovery of $M_z \propto M_z(T_1 - \frac{T_1}{2}) \cdot \exp(-\frac{T_1}{T_1})$

Differs contrast methods with Spin Echo:

- "Long T_R " $\rightarrow$ T_1 weighting ($T_R > T_1$)
- "Short T_R " $\rightarrow$ T_2 weighting ($T_R < T_2$)
- Example:
 - Spin density: $T_R \approx 2000$ ms, $T_2 \approx 100$ ms
 - T_1 weighted: $T_R \approx 500$ ms, $T_2 \approx 100$ ms
 - T_2 weighted: $T_R \approx 2000$ ms, $T_2 \approx 100$ ms

More methods are omitted $T_R \gg T_2$, i.e. dynamic magnetization before excitation pulses completely lost!

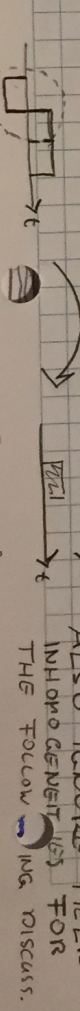
Now: Another method for speed up scan times (also denoted "R-space trajectory" & "parallel imaging")

Same T_R model! Problem: Will cause the data with "off-axis" k-space magnification... (off-axis M_{xy})

Let's look at some examples of what might happen due

to off-axis M_{xy} while applying RF-pulses with short T_R

Note: Many RF pulses & short $T_R \rightarrow$ Transverse R-space using gradient echo (den RF required) = k-space (den RF required)



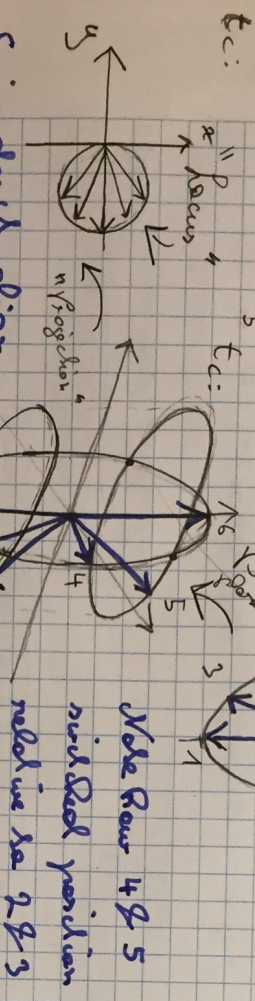
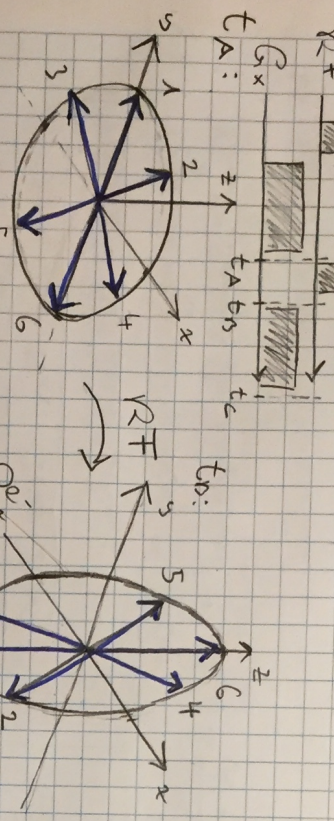
the net effect on phase of k-space mag.

ALSO INDICATES FIELD INHOMOGENEITIES FOR THE FOLLOWING DISCUSSION.

2 subregions 90° -pulses needed?

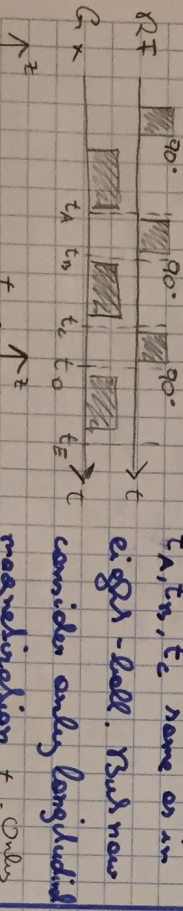
↳ goal: "Forming shell copying with echo-aver H_{xy} "
 (1) Eighs - ball echo (maximum spin-echo)

Somehow H_{xy} - echo is used for some type echo

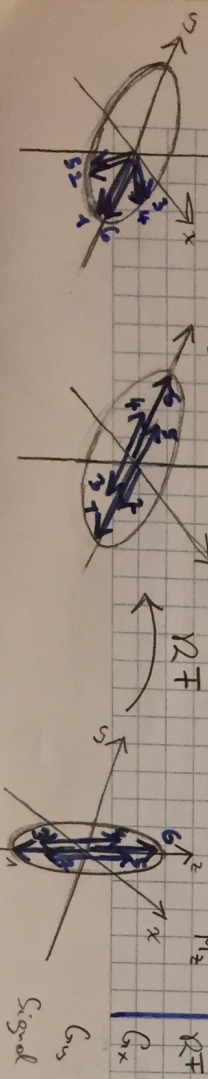


Spin don't flip regularly $\rightarrow$ eighs ball echo $\approx \frac{1}{2}$ of original signal strength

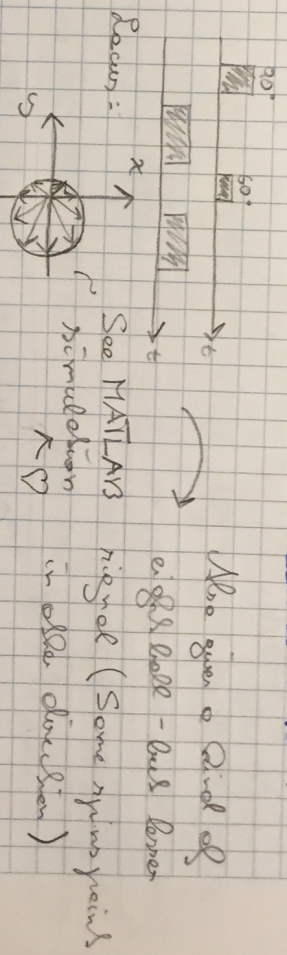
(2) Stimulated echo



consider only longitudinal magnetization $t_C = M_z$



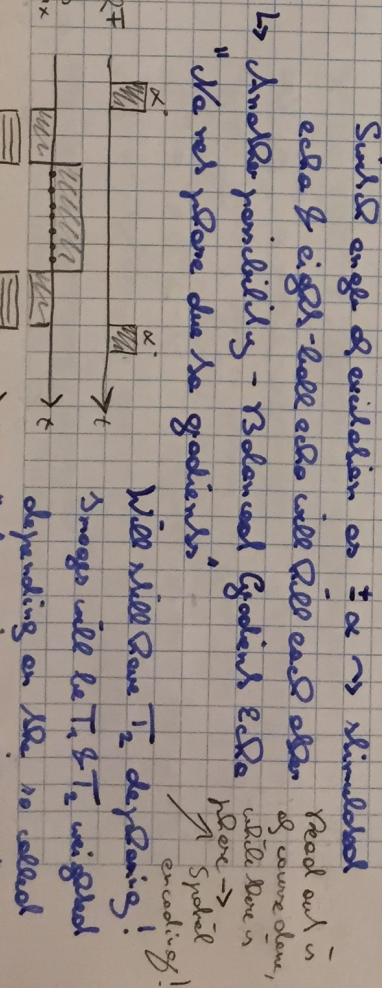
(3) Eighs - ball echo with excitation $\alpha < 90^\circ$:



↳ If we apply a pulse train with constant T_R , it is also from examples (1)-(3) that an echo will always occur at some time on excitation $\rightarrow$ not homogeneous magnetization will be tipped in longitudinal direction $\rightarrow$ "complicated signals"

↳ How to overcome this issue? **SPOILING**

- Gradient Spinning: "Drive out of R-space after read out"
- RF Spinning: "Switch angle of excitation as $\pm \alpha \rightarrow$ stimulated echo & eighs-ball echo with T_R read out of course slow, will be in phase $\rightarrow$ spoiled exciting!"



Will still have T_2 decay! Strongly will be T_1 & T_2 weighted depending on M_z no signal "dynamic equilibrium" ∇ No net gradient over ∇

② Important imaging characteristics:

- Resolution → See later on timing ⇒ $\Delta x = \frac{2\pi}{\gamma B_1}$
- More sophisticated description via PSF! Main contribution: i) "Band-limiting" the signal

Overall PSF given as convolution of sinc() * sinc()

- ii) Relaxation (homogeneous)

Signal decays as $\propto \exp(-\frac{t}{T_2^*})$

Always keep in mind from previous discussion: higher resolution means smaller Δx

Useful resolution limits: (ASSUME INFINITE SNR)

i) Relaxation → Need $T_{ACQ} \ll T_2^*$

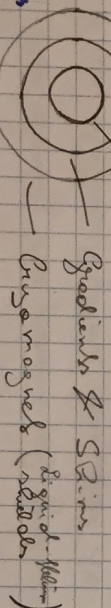
ii) Diffusion → H^+ more during acquisition

lines! Need $T_{ACQ} \ll \frac{\langle x^2 \rangle}{\sigma^2}$

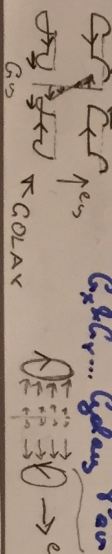
$T_{ACQ} = 100ms$
 $\sigma^2 = 10^{-3} \frac{mm^2}{s}$
 $\Rightarrow \langle x^2 \rangle = 8 \mu m$

$V_{(k)} - MRI$ hardware

Setup: RF - Coils



Gradients & SGRs



VI. For Smearing

① Method - Amplitude Contrast / Signal Contrast:

Can be also of Δz under negative excitation! Static tissue → no motion

Moving blood → "fuzzy" blurred blood always available (no more smearing)

Amplitude contrast with Gradient Echo

i) $\frac{\gamma B_1 T_2}{\Delta z}$ "Good" blood

ii) $(1 - \frac{\gamma B_1 T_2}{\Delta z})$ "Bad" blood

The corresponding signal is:

$$\sigma_{GRE}^{FLOW} \propto S_{tissue} \cdot \Delta z \cdot (1 - \frac{\gamma B_1 T_2}{\Delta z}) + S_{tissue} \cdot \Delta z \cdot \frac{\gamma B_1 T_2}{\Delta z}$$

Note valid for $\gamma B_1 T_2 < \Delta z$, otherwise $\sigma^2 = \sigma_{GRE}^2$

Amplitude contrast with Spin Echo

Up to time t_A : assume $\frac{T_2}{2} \ll T_R$, we can approximate the available M_z from compartment ① & ② as before!

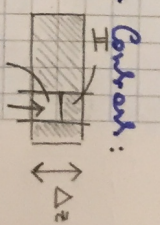
System is not under gain with a 90° pulse, regarding the homogeneous magnetization we will use for the spin-echo! But between 90° & 180° pulse will have some of M_z transverse mag. (outflow)

Only $(1 - \frac{\gamma B_1 T_2}{2 \Delta z})$ will be available for SE

$$\sigma_{SP}^{FLOW} \propto \sigma_{GRE}^{FLOW} \cdot (1 - \frac{\gamma B_1 T_2}{2 \Delta z}) \cdot \exp(-\frac{T_2}{T_2^*})$$

QUANTITATIVE ASSESSMENT

IMPOSSIBLE to produce...



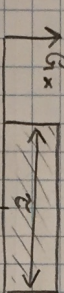
(*) 1st moment actually independent of reference time t_0 ($\int_{-T}^T M_0 = 0$)

Notes: "When we read out the phase of the oscillating M_0 gradient, we are actually measuring its velocity at time t_0 "

Position in $\vec{e}_x$ -direction!
Expansion points to determines of which "lines we measure V_0/a_0 "!

② Maxwell - Phase Contrast:

- When spins move during the presence of gradients, they will pick up different phase compared to static spin! Why? They exhibit different total changes of different lines!
- This may be used to "encode velocity". Let's analytically describe this in a simple example!
- Consider a spin moving in x -direction, with a gradient in x -direction is present.
- Tougher - expand the spin position ("equation of motion")

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2} a_0(t - t_0)^2 + \dots$$
- Simple gradient: 
- The phase we pick up is:

$$\varphi = \int \omega(t) dt = \int \gamma \cdot B_x(t) dt = \gamma \cdot \int G_x(t) x(t) dt$$

$$\varphi = \gamma \cdot \int_{-T}^{+T} G_x(t) \cdot x_0 dt + \gamma v_0 \int_{-T}^{+T} G_x(t) \cdot (t - t_0) dt + \frac{1}{2} \gamma a_0 \int_{-T}^{+T} G_x(t) \cdot (t - t_0)^2 dt + \dots$$

$\left. \begin{array}{l} \text{System} \\ \text{contribution} \end{array} \right\}$ $\left. \begin{array}{l} \text{Velocity} \text{ & } \\ \text{acceleration} \end{array} \right\}$ $\left. \begin{array}{l} \text{contribution} \end{array} \right\}$
- Can define "moments of G_x cont. t_0 ":

$$M_n = \int_{-T}^{+T} G_x(t - t_0)^n dt$$
- Then the phase becomes:

$$\varphi = \gamma x_0 M_0 + \gamma v_0 M_1 + \gamma \frac{1}{2} a_0 M_2 + \dots$$

(*)

- Good: Want a gradient that only results in phase due to velocity!

How? "Bipolar Gradient" & choose $t_0 = T$

$$M_0 = \int_{-T}^{+T} G_x(t) dt = 0$$

$$M_1 = \int_{-T}^{+T} G_x(t - T) dt$$

$$= \int_{-T}^{+T} G_x(t') dt' \quad (t' = t - T)$$

$$= G_x \cdot \frac{1}{2} (t')^2 \Big|_{-T}^{+T} + (-G_x) \cdot \frac{1}{2} (t')^2 \Big|_0^{+T} = G_x(t') \cdot t'^2 \Big|_0^{+T}$$

$$= -G_x \cdot \left(\frac{T}{2}\right)^2 = -A \frac{T}{2}$$

$$M_2 = \int_{-T}^{+T} G_x(t - T)^2 dt = \int_{-T}^{+T} G_x(t')^2 dt' = 0$$

- Note: More general for a bipolar gradient for any

1st & 2nd-order moments with its "center of gravity" φ

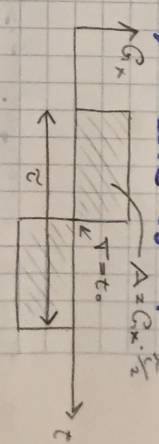
- Then the phase is only velocity dependent:

$$\varphi = \gamma v_0 M_1 \rightarrow \text{NICE!}$$

- Note: Once the spin was "brought" into this phase, it will move along & maybe we encode it's position at some later time - but it will still carry this phase, corresponding to the velocity v_0 at Root of time t_0

- We encode velocity via phase $\rightarrow$ OR we can use

related energy might Riskin: ALLASING $\rightarrow \varphi \in [-\pi, \pi]$
 "Maximum velocity we can uniquely encode using a given gradient: $\pi = \gamma V_{enc} M_1$ "



Center of gravity of gradient

This works for $\vec{e}_x$ & $\vec{e}_z$ direction, since for there our usual sequence for zero gradient case ($M_0 = 0$) does not work as easy for $\vec{e}_y$ $\Rightarrow$ Will see LATER

ed (2) Phase Control:

$\Rightarrow$ We can describe the needed signal as

$$\sigma \propto \int S(\vec{x}) e^{i\vec{q}\vec{x}} e^{i\vec{q}\vec{x}} d\vec{x}$$

$$\text{where } \vec{q} = \gamma \vec{M}_0 \text{ \& } \vec{q}_1 = \gamma \vec{M}_1$$

$\Rightarrow$ We can get the complex-valued signal intensity $\propto \text{gr}(e^{i\vec{q}\vec{x}})$ via a sample Fourier Transform!

Information of velocity per pixel can then be obtained from the phase of the complex number!

$\Rightarrow$ Typical Procedure:

Acquire 4 scans! 3 scans with each being sensitive to velocity in one direction, while the other direction one velocity insensitive & 1 scan getting velocity insensitive! This way it is possible to compensate for additional phase due to the field inhomogeneity via "phase differences"

$\Rightarrow$ How to make a non-insensitive to velocity?

Can make a 2 bipolar gradients together with different "polarity", then perpendicular 1st order moments cancel!

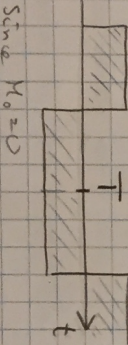
For any t_0 :

$$M_1 = \int_{T-2\tau}^{T+2\tau} G(t-t_0) dt$$

$$= \int_{T-2\tau}^{T+2\tau} G \cdot t \cdot dt$$

$$= G \cdot \left[\frac{1}{2} t^2 \right]_{T-2\tau}^{T+2\tau} = \frac{1}{2} t^2 \Big|_{T-2\tau}^{T+2\tau}$$

$$= G \cdot \left[(T+2\tau)^2 - (T-2\tau)^2 \right] = G \cdot [4T\tau + 4\tau^2] = 0$$



Because $\vec{q}_1$ & $\vec{q}_2$ are not collinear, we can write

$$S(\vec{x}) = e^{i\vec{q}\vec{x}}$$

$\Rightarrow$ Let's work out how flow might influence our signal:

(1) GRE - sequence (no modifications) & constant flow

assume flow in x -dir

with constant velocity

Choose T_0 as origin

$$\text{points for } \vec{r}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} (t - T_0)$$

We assumed $v_y = 0$!

Note: $\vec{e}_x$ - direction already in velocity sensitive!

(Since our gradients echo is bipolar)

We want to find the phase of our spin at time TE:

$$\varphi(TE) = \gamma \int_{T_0}^{TE} G(t) \cdot \vec{r}(t) dt$$

$$= \gamma \cdot \int_{T_0}^{TE} G(t) \cdot (x_1 + v_x [t - T_0]) dt +$$

$$+ \gamma \cdot \int_{T_0}^{TE} G(t) \cdot (-v_y [t - T_0]) dt +$$

$$+ \gamma \cdot \int_{T_0}^{TE} G(t) \cdot (x_2 + v_y [t - T_0]) dt$$

$$= \gamma G \Delta G_0 \gamma_0 + \gamma G \Delta G_0 v_x \cdot \int_{T_0}^{TE} (t - T_0) dt +$$

$$+ \gamma (-G_x) v_x \cdot \frac{1}{2} (t - T_0)^2 \Big|_{T_0}^{TE} +$$

$$+ \gamma G_x v_x \cdot \frac{1}{2} (t - T_0)^2 \Big|_{T_0}^{TE}$$

$$= \gamma G \Delta G_0 \gamma_0 + \gamma G \Delta G_0 v_x \cdot \int_{T_0}^{TE} (t - T_0) dt +$$

$$+ \gamma G_x v_x \cdot \frac{1}{2} [(T_x - T_0)^2 - (T_x - T_0)^2]$$

$$= \gamma G \Delta G_0 \gamma_0 + \gamma G_x v_x \cdot \frac{1}{2} [(T_x - T_0)^2 - (T_x - T_0)^2]$$

$$= \gamma G \Delta G_0 \gamma_0 + \gamma G_x v_x \cdot \frac{1}{2} [(T_x - T_0)^2 - (T_x - T_0)^2]$$

$$= \gamma G \Delta G_0 \gamma_0 + \gamma G_x v_x \cdot \frac{1}{2} [(T_x - T_0)^2 - (T_x - T_0)^2]$$

$$= \gamma G \Delta G_0 \gamma_0 + \gamma G_x v_x \cdot \frac{1}{2} [(T_x - T_0)^2 - (T_x - T_0)^2]$$

12

VERY MUCH AS EXPECTED

(*) So the numerator:

Misregistration arises from the fact, that spins move between the line when they are

Currents phase encoding gradient is $R \cdot \Delta G_{xy}$ & we have a sample at current line $T_x(e)$

od (1) GRE - sequence & constant phase

So we record the phase of our moving spin at the line we start the read out under influence of the frequency-encoding gradient

$(t = TE)$ is given as $\varphi(TE) = R_{xy}^{(e)} \cdot \gamma_{xy} + R_{rx}^{(e)} \cdot \gamma_{rx}$

The signal we record then is:

$$\sigma_{ea} \propto \iint \rho(r_1^{(e)}) \cdot \exp(i \int \omega_{T_x}(e) + \varphi) = \iint \rho(r_1^{(e)}) \cdot \exp(i \int [R_{xy}^{(e)} \gamma_{xy}(T_x^{(e)}) + R_{rx}^{(e)} \gamma_{rx}(T_x^{(e)}) + R_{xy}^{(a)} \gamma_{xy} + R_{rx}^{(a)} \gamma_{rx}])$$

From $T_x^{(e)} \equiv TE$ (Maybe TE would be better for some reason) So (as it's quickly ignore the n_0 -minimized) we can simplify Fourier inverse transform our signal not to

the get the usual values: $G(\vec{r}) \cdot e^{i R_{xy}^{(e)} \gamma_{xy}(\vec{r})}$

Since we know $R_{xy} = \gamma M_x^y$ we get $\gamma_{xy}(\vec{r})$ by adding the phase of R_{xy} complex-valued signal!

But: The minimization of γ_{xy} & $\gamma_{xy}(TE^{(e)})$ leads to MISREGISTRATION - ARTIFACT.

What we really should do is take the line TE : $\gamma_{xy}(TE) = \gamma_{xy}(TE) - \gamma_{xy}(TE - T_0)$

Plug into integral $\rightarrow \sigma \propto e^{i R_{xy} \gamma_{xy}(TE - T_0)} \cdot \iint \rho(\vec{r}) d\vec{r}$

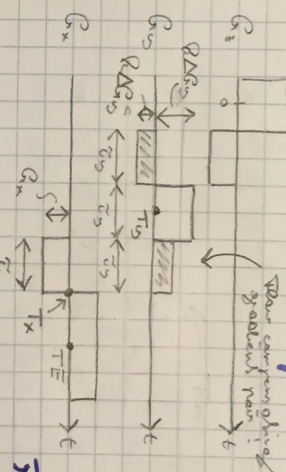
Fourier SRTT Transform $\rightarrow$ Moving spin displaced in $\hat{e}_{xy}$ -dir

(*)

THIS IS A LEGIT FT

2) GRE with compensation for phase in $\hat{e}_{xy}$ -dir.

As good as the phase for the $\hat{e}_{xy}$ -direction & the line when the signal is recorded? $(T_x - T_0) \neq 0$



Some assumptions as above: Velocity is constant & only in x_1y_1 -plane!

Expected overall T_0 :

$$\vec{r}(t) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} (t - T_0)$$

It's not what compensation gradients $R \Delta G_{xy}^c$ are required for a given phase encoding gradients $R \Delta G_{xy}$! The phase of line $t = TE$ is

$$\varphi(TE) = \gamma M_0^y \gamma_y + \gamma M_1^y \gamma_y + \gamma M_1^x \gamma_x$$

($\gamma_y, \gamma_x, \gamma_z$) are always the values of the respective "coordinates" of the gradient

Where:

$$\begin{aligned} M_0^y &= R \Delta G_{xy} \cdot \vec{v}_y \\ M_1^y &= R \Delta G_{xy} \cdot 2 \vec{v}_y^2 \\ M_1^x &= G_x \vec{v}_x^2 \end{aligned}$$

Then previous discussion we know that the misregistration arises since the phase in phase-encoding direction does not match!

That is: We know $R_{xy}^{(a)}$ γ_y , but want $R_{xy}^{(e)}$ $\gamma_{xy}(TE)$

Can we the velocity term be canceled for R_{xy} by reading: $\gamma_{xy}(TE) = \gamma_y + \gamma_y (TE - T_0)$

Answer: $\gamma M_1^y \gamma_y = \gamma M_0^y \gamma_y (TE - T_0)$

$$\begin{aligned} M_1^y &= M_0^y (TE - T_0) \\ \Rightarrow 2 R \Delta G_{xy} \vec{v}_y^2 &= R \Delta G_{xy} \vec{v}_y (TE - T_0) \end{aligned}$$

THIS IS A LEGIT FT

ad ② Still More Context:

↳ Influence of field inhomogeneities $\varphi_{\text{par}} = \gamma \delta B_z \cdot t$ cannot be ignored for more context images!

Solution: More Difference images

(already mentioned above)

Velocity sensitive scan: $S(\vec{x}) \cdot e^{i\vec{R}_V \cdot \vec{r}} \cdot e^{i\varphi_{\text{par}}}$

Velocity compensated scan: $S(\vec{x}) \cdot e^{i\vec{R}_V \cdot \vec{r}}$

More Difference: $S(\vec{x}) \cdot e^{i\vec{R}_V \cdot \vec{r}}$

Recall: One we determined the more in the image, the velocity is determined

$$\varphi = \vec{R}_V \cdot \vec{r}$$

$$\varphi_{\text{max}} = \vec{r} \cdot \vec{R}_V \cdot \vec{v}_{\text{enc}} \quad \left\{ \begin{array}{l} v = \frac{\varphi}{\vec{r} \cdot \vec{R}_V} \end{array} \right.$$

↳ Generalized Velocity Encoding ("Homon")

Until now assumed a single velocity per pixel!

Real actually there is a whole distribution of velocities per pixel!

(DISCRETE)

$$\text{The signal is } S \propto \sum_i \sum_j S(\vec{x}_i) e^{i\vec{R}_V \cdot \vec{r}_i} e^{i\vec{R}_V \cdot \vec{r}_j}$$

Seems very similar → Can additionally

take samples for different $\vec{R}_V \rightarrow$ get an

image with spatial dimension & velocity

dimension (meaning for each pixel we now

get a whole distribution of velocities!)

③ Method - Context Agents (CA)

↳ For on-site agents can increase signal of blood pool true contrast agents!

(i) Parametric CA - Mainly $T_1 \downarrow$

Most commonly gadolinium based (reduce T_1 relaxation time) → Gad-DTPA / Gad-DOTA

How does it work?

→ Gad is labeled & most of its coordinating e^-

are bound to the chelator - but also have

one exchangeable H_2O which interacts with

a free e^- (real R_1 paramagnetic relax)

→ Signal increase / T_1 decrease depends on

R_{ex} (how fast H_2O are exchanged)

τ_c (rotational correlation time - tumbling time)

(ii) Superparamagnetic CA - Mainly $T_2^* \downarrow$

Monopoles: Iron-oxide (Fe_3O_4) nanoparticles 100

or even monocrystal (MION), or poly-crystal (PIO)

These are compatible with coating with dextran (= macromolecule)

Note - Quantitative MRI for flow measurements:

Using more contrast we can get a lot of flow info!

→ Reminder: $\Delta P = Q \cdot R$, $R \propto \frac{\mu \cdot L}{\Delta r^4}$

$\Delta P = P_{\text{in}} - P_{\text{out}}$, Q ... flow

→ Turbulent flow: Additional ΔP can be due to turbulence...

$$\text{for } Re = \frac{v \cdot \rho \cdot D}{\eta} > 2300$$

→ Some $\Delta P \Rightarrow$ smaller Q

(*) Recall Fourier - Parseval: $V_x(x) = V_{max} (1 - (\frac{x}{a})^2)$

VII. Motion Artifacts:

↳ MR experiments duration often on some time scale

on physiological motion! leads to spins with different phase/frequency or the overall phase....

Can distinguish Suba-TR & Suba-TR motion

① Suba-TR motion (Blood flow)

↳ "Spins undergo significant displacement during acquisition of one R-phase profile (Read R_{xy})"

↳ Constant motion along frequency-encode direction:

Spins will pick up phase due to linear gradient of GFE - negative (see previous section)

The magnitude image, phase would not play significant role - RSET: Sp read dimensions

are too big → Multiple spins in voxel therefore, resulting in signal loss!

Side Note - Quantifying the signal loss:

$y_i \dots$ Position of voxel center
line area: $V_x(y) = V_x(y_i) + V(y - y_i)$

Average magnitude of all the spins of the voxel:

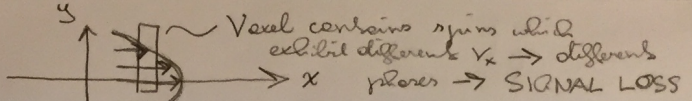
$$S_{avg}(x, y_i) = \frac{1}{\Delta y} \int_{y_i - \Delta y/2}^{y_i + \Delta y/2} S(x, y) \cdot \exp(i \gamma M_z V_x(y)) dy$$

$$= \frac{1}{\Delta y} \int_{y_i - \Delta y/2}^{y_i + \Delta y/2} S_{avg}(x, y) \cdot \exp(i \gamma M_z V_x(y)) dy$$

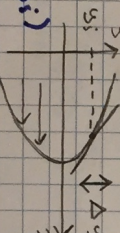
Recognize Fourier Transform of a Read (Δy) - function:

$$F[Read(\Delta y, y)] = \Delta y \cdot \text{rect}(\Delta y, R_y)$$

$$S_{avg}(x, y_i) = S(x, y_i) \cdot \exp(i \gamma M_z V_x(y_i)) \cdot \text{sinc}(\gamma M_z \Delta y)$$



On effective spin density over pixel

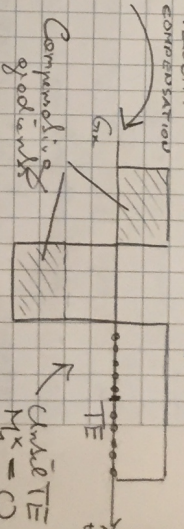
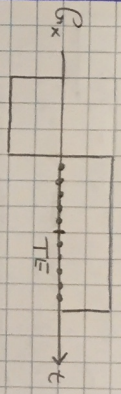


For Blood flow - RSET (Read) gradients are "overlapped" → "Read TR"

and ① Suba-TR motion

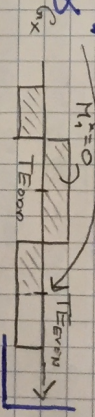
↳ and motion along frequency-encode direction

Compensation gradients: "RSET - to RSET RSET"



Note: Similar effect for even echo during EPI

"Even echo refocusing"

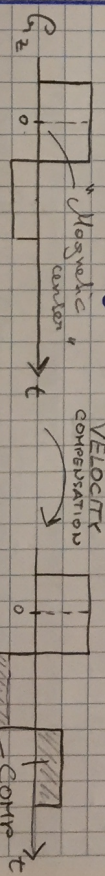


↳ Constant motion along slice-selection direction

Very similar to frequency-encoding but → Spins will pick up phase & if voxel contains spins with different velocity → Refocusing & Signal loss!

(Note: Also signal loss due to inflow/outflow effects of multiple excitations)

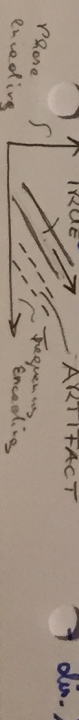
Compensation gradients:



Recall: Transverse magnetization is in phase of line $t=0$ (magnetic center)

↳ Constant motion along phase-encoding direction

See above → distinguish! (Can predict flow dir.)



Do discussed?

Read $M_z=0$

Under TE $M_z=0$

Where can blood flow? selection gradient R_{xy} $T_z = 0$ $\rightarrow M_z \neq 0$ → "under refocusing"

$$T_{\text{acq}} = N_y \cdot T_R \gg N_x \Delta t$$

(Even $T_R > N_x \Delta t$ usually)

→ Multiple excitations to record a single profile!

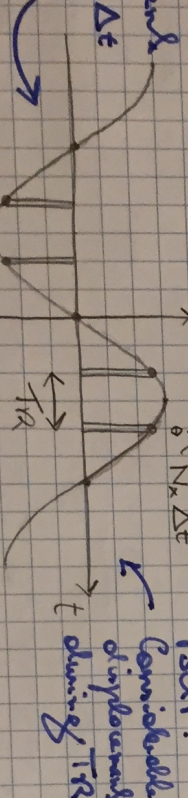
→ T0R - gamma memory (5x TR - motion):

- Readout flow in frequency-encoding direction
⇒ Signal low (if used dim. large)
- Readout flow in phase-encoding direction
⇒ Signal low (if used dim. large)
- Readout flow in phase-encoding direction
⇒ Magnification artifact

② Under - TR motion:

- In our balanced 2D Fourier imaging method a single "profile" (i.e. $R_y = \text{const}$ / a line in frequency direction) is acquired very fast ($N_x \Delta t$)
- In contrast "per line in phase-encode direction" we effectively need to acquire the whole image!
- Order of magnitude longer acquisition time!
- A lot of physiological movements are negligible during "freq-enc acquisition" but NOT during "phase-enc acquisition"

Displacement during $N_x \Delta t$ negligible!



BUT:

Comickels displacement during TR

Note: There TR motion for line between acquisition of two profiles! Can include multiple excitations, or multiples a long recovery time before making 2D profiles

→ From the above it should be clear:

Any motion (an line node $\odot (N \cdot TR)$), in motion with its dominant direction $\vec{u}$, has most effect on phase-encoding direction!

Let's work out an example - "Periodic motion in $\vec{u}$ - direction"

Consider an object moving with $\vec{u}(t) = \vec{u}_0 + A \sin(\omega t)$

When acquiring the sample with $T_R^{(e)} \gg T_{R_0}$ it

$$\sigma_{EA} \propto \int \int G(x, y) \exp(i y G_x x \cdot T_R^{(e)}) \cdot \exp(i y G_y y \cdot T_R^{(e)}) \cdot \exp(i y G_z z \cdot T_R^{(e)}) \cdot A \sin(\omega t) dx dy$$

$$\text{for } \sigma_{EA} = y G_x T_R^{(e)} \quad \& \quad Q_y^{(e)} = y G_y T_R^{(e)}$$

$$\sigma_{EA} \propto \int \int G(x, y) \exp(i [Q_x^{(e)} x + Q_y^{(e)} y] \cdot T_R^{(e)}) \cdot A \sin(\omega t) dx dy$$

Note we are identifying in our 2D Fourier function of the 1st Q and: $\exp(i y G_y y \cdot T_R^{(e)}) = \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \exp(i y G_y y \cdot T_R^{(e)})$

$$\sigma_{EA} \propto \int \int G(x, y) \exp(i [Q_x^{(e)} x + Q_y^{(e)} y] \cdot T_R^{(e)}) \cdot A \sin(\omega t) dx dy$$

$$\sigma_{EA} \propto \left[\sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \exp(i y G_y y \cdot T_R^{(e)}) \cdot \exp(i y G_z z \cdot T_R^{(e)}) \right] \cdot \int \int G(x, y) \exp(i [Q_x^{(e)} x + Q_y^{(e)} y] \cdot T_R^{(e)}) dx dy$$

So our "Q-space" signal of the non-moving object is multiplied by the term in brackets!

Some multiple σ_{TR}

FOR SAMPLES WE CAN EFFECTIVELY WRITE $t = n \cdot T_R$ (Same multiple of R_0)

little blurring

ad ② Sub-TR motion

ad 1: "Periodic motion in x -direction"

Multiplication in Q -space with: $\text{exp}(i \cdot x \cdot \Delta k)$

$$\left[\sum_p \sum_q (Q_x^{(p)} A) \exp(i p \cdot x \cdot \Delta k) \right] \cdot \sigma_{\text{static}}^{(q)}$$

$\Leftrightarrow$ • In y -direction: Convolution with

sum of Dirac deltas

• In x -direction: "Hermite-Derivate"

But usually the $\sum_p (Q_x^{(p)} A)$ only

has very little effect - small blurring

in x -direction

$\hookrightarrow$ What about random motion?

Of course also noise present in phase-enc.

direction! Maybe can lead down in sum of

periodic motion?

$\hookrightarrow$ Cardiac Motion - How to avoid artifacts?

"ECG-Triggings": Measure cardiac cycle

and ECG & only acquire during a specified

imaging window

Could also continuously acquire over multiple

cardiac cycles & afterwards use ECG to

sort Q -space profiles $\rightarrow$ ECG multiplies images,

each of a different phase of cardiac cycle

ad ② Sub-TR motion

$\hookrightarrow$ Respiratory Motion - Avoid Artifacts:

(i) Breath-holding: First acquisition time

(ii) Magnetic ECG:

Can track current state / phase of respiration

cycle via position of diaphragm!

How? Between different acquisitions of

R_{12} = center (of our coronal slice, i.e. the largest

image), acquire a single profile through center of

a sagittal plane (see pic) $\Rightarrow$ corresponds to a

position of all spins $\Rightarrow$ Can read of diaphragm pos.

Can use for:

• Gating: "Only acquire during certain cycle phase"

• Volume Imaging: "Adapt position of image slice"

(Combine with motion model)

$\hookrightarrow$ Self-Gating: (Motion-est. via ECG / Magn. ECG)

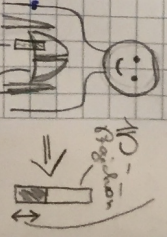
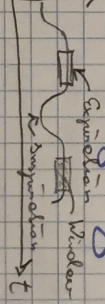
• Main Idea: Q -space center comes into an echo

cardiac cycle & respiration cycle!

Signal of $(Q_x = 0, Q_y = 0)$: $\sigma_{(0,0)} \propto \iint S(x,y) dx dy$

Also called: DC-Signal

Diaphragm Pos.



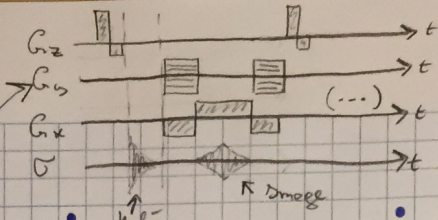
NO respiration

Determines
GRID SPACING:

$$\Delta G_{xTAG} = \frac{1}{N}$$

→ Tagging:

- "Instead of compensating motion - study it!"
- Overlay grid of rotated spins on motion & study its deformation!
- Steps:
 - i) N RF-pulses with (90°) while gradients is on (Gins G_x , G_y , G_z). Between RF pulses, spins dephase due to gradients! Only spins picking up a phase of 2π between pulses, will be flipped 90°



ad (2) Sub-TR motion
→ ad Self-Gating:

- Why does DC-Signal contain info on motion: Cardiac: R-R-interval → Heart beat in ECG → more spins in FOV → bigger DC

Respiratory: Expiration → Diaphragm moves up → large structures move into FOV (Liver) → more spins in FOV → bigger DC

- Use sampling strategies to sample $(R_z=0, R_y=0)$ with every profile (e.g. radial, spiral, or modified SSFP)

• Need to given recorded signal to distinguish cardiac & respiratory motion!

ad (2) Sub-TR motion

→ ad Tagging:

- ad Steps:

ad i) In contrast, spins picking up phase of $(2\pi + \epsilon)$ will stay unaffected by pulse train!

ii) Spreading:

Contrast between magnetization of spins, which were flipped down → don't wait for T1-recovery → Trunc Rate Max $\ll T_1$ & $T_2 \ll T_1$ → Spins are "rotated" → the signal is picked up again!
iii) Imaging Sequence: e.g. EPI

