

# STUDY - SUMMARY

## BMI

### Signals & Systems:

↳ Good imaging system? → LSI

i) Linear:  $S[\sum \alpha_i f_i] = \sum \alpha_i S[f_i]$

ii) Shift-Independent:

$$S[f(x+\Delta x, y+\Delta y)](x, y) = S[f(x, y)](x+\Delta x, y+\Delta y)$$

↳ LSI-Systems: Complex Exp. = Eigenfunc.

$$f(x, y) = e^{i(\alpha_x x + \alpha_y y)} \rightarrow S[f] = H(\alpha_x, \alpha_y) \cdot f$$

↳ What is this good for? We know how to write every signal as sum of infinite complex exponentials → FOURIER

↳ LSI-Systems then perform a convolution

$$S[f] = S[f] * \delta, \forall \delta$$

where  $S[\delta]$ ... Impulse Response ("Point-Spread-Function")

↳ Convolution: ("COPY & PASTE")

$$f * g = \iint f(x-x', y-y') g(x', y') dx' dy'$$

$$F(f * g) = F[f] \cdot F[g]$$

↳ Fourier-Transform:

i) Cont.:  $f(x, y) = \frac{1}{(2\pi)^2} \iint F(\alpha_x, \alpha_y) \cdot e^{i(\alpha_x x + \alpha_y y)} d\alpha_x d\alpha_y$

$$F(f)(\alpha_x, \alpha_y) = \iint f(x, y) \cdot e^{-i(\alpha_x x + \alpha_y y)} dx dy$$

Pairs:

Gaussians

$\delta(\omega)$

$\delta(\omega - \omega_0)$

$\frac{1}{2}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$

$\cos(\omega_0 t)$

$\Delta \omega = \frac{2\pi}{\Delta t}$

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↳ Sampling: How dense & which range?

i)  $\Delta x = \frac{2\pi}{\Delta \omega}$

Prevents aliasing:  $\frac{2\pi}{\Delta x} > BW$

ii)  $\Delta x = \frac{2\pi}{\Delta \omega}$

Reduced resolution

$\Delta R = \frac{2\pi}{N \Delta x}$ , Instead of  $e^{i\alpha x}$  → SWAP

↳ DFT:

$$F[f](\alpha_n) = e^{-i\alpha_n x_0} \sum_{n=0}^{N-1} f(x_n) e^{-i\alpha_n \frac{2\pi}{N} n}$$

ADD MANUALLY

$\Delta R = \frac{2\pi}{N \Delta x}$ , Instead of  $e^{i\alpha x}$  → SWAP

X-RAY: ( $\lambda_{X-RAY} = 10\text{nm} - 10\text{pm}$ )

↳ Instruments: i) X-Ray-Tube

ii) Collimator ("Blende")

iii) Anti-Scatter grid

iv) Detector

① Voltage divider

electrons → hit target, produce  $\gamma$  (Param.: U, I, target)

② Reduce Noise & Scattering (Made from lead)

③ Contrast-Noise-Ratio: ↑

Overall signal intensity ↓

↳ Meaning: Right dose for some sensitivity

④ i) Computed Radiography (Digital)-CR

When X-Ray hits plate,  $e^-$  go from valence band into conduction band!

→ Trapped  $e^-$  → Read Plate to "reader"

Reader: Laser-Diode → Make  $e^-$  return to ground state →  $\Delta E$  is emitted as light

→ Photodiode convert light to voltage

ii) Digital Radiography (Digital)-DR

Distance to CR: No read-out device

CsI layer convert  $\gamma$  → light (green and blue)

→ Read slope → light emitted at bottom

→ TFT array → Voltage

↳ Generation of X-Ray:

i) "Bremsstrahlung"

↳ Continuous Spectrum

ii) "Characteristic radiation"

↳ Peaks

①  $E_{\gamma}$  depends on

$E_{\gamma} = E_{\text{anode}} - E_{\text{cathode}}$

→ Distance to nucleus

→ Z-number →  $Z \uparrow$ :  $\Phi \uparrow$  (Probability, Real  $e^-$  is deflected)

Electron decelerates when being deflected by nucleus from original path

→ Difference in kinetic energy is emitted as X-Ray

②  $E_{\gamma, n} > E_{\gamma, mono, k}$

Generated hole is filled with  $e^-$  from L-shell

$E_{\gamma} = E_{\gamma, mono, k} - E_{\gamma, mono, L}$

↳ Interaction of X-Ray with body

Goal: SNR ↑, enough X-Ray transmitted

CNR ↑, different absorption in different tissue

① Photoelectric Absorption

$\gamma_{in} \rightarrow \gamma_{out}$

Little energy. Will not make it out of body

Probability of occurrence:

$$P_{pe} \propto Z^4 \left( \frac{Z_{\text{shell}}}{E} \right)^3$$

② Compton Scattering

$$\Delta \lambda = \frac{h}{m_e c} (1 - \cos \varphi)$$

→ High energies also for good up

→ Scattered  $\gamma$  mistaken for primary radiation

→ Probability:  $P_c \propto \frac{Z}{E}$

↳ Linear ( $\mu$ ) and mass ( $\mu'$ ) attenuation coeff.

→ Transmitted X-Rays:  $N = N_0 \cdot \exp(-\mu(E) \cdot x)$

→  $\mu = \mu' \cdot \rho$

→ Better:  $C \propto (\mu_2 - \mu_1) \cdot d$

$C \propto \log(I_1) - \log(I_2)$

$C \propto \log(I_1) - \log(I_2)$

According to frequency

Characteristics "peaks"

Change  $E_{\gamma}$

↑ "Cut off" due to prefilter (collimator)

↑  $E_{\gamma}$  (Photon energy)

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Projection Imaging

Noise in X-Ray: Poisson distribution

$$P(N) = \frac{e^{-\mu} \mu^N}{N!}, \sigma = \sqrt{\mu}$$

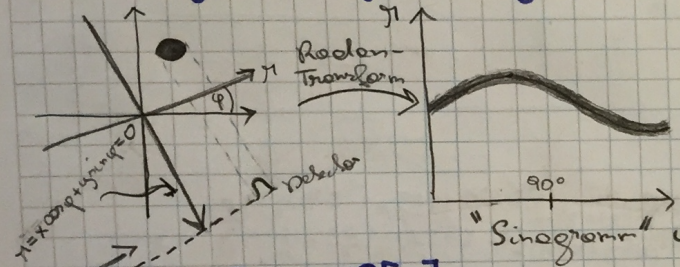
For large N:  $\sigma \sim \sqrt{N}$

## Quality of X-Ray Images:

- Signal-Noise-Ratio:  $SNR \propto \sqrt{N}$
- Spatial resolution: Use Line-Spread-F, LSF
- CNR: Drops for High energies (Scatter)

## Computed Tomography: $[HU] = 1000 \cdot \frac{\mu - \mu_{air}}{\mu_{air}}$

Idea: Project at different angles → RP

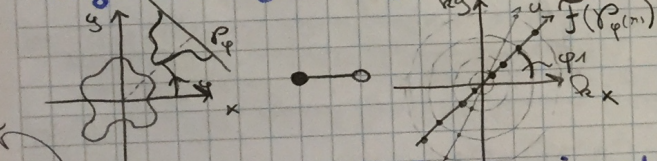


Projection:  $P_p(n) = -R[\mu] \cos(\phi)$   
 $= -\int \mu(x,y) \delta(x \cos \phi + y \sin \phi - n) dx dy$

Inverse Radon Transform numerically impractical

→ Use Fourier Transform!

→ Fourier Slice Theorem:  $F(P_p(n)) = S(F(\rho))$



→ Problem: Data in Fourier-Domain not equally spaced! (Bad for FFT)

## Filtrering:

- $3\sigma$  already in R-Space: Simply multiply
- Which filter?
  - Rectifying: Low Pass (from undersampled)
  - Ringing: Gaussian
  - Blurring: High Pass

→ Image = Filter \* Obj =  $F^{-1}(F(f) \cdot F(f))$

## Spiral CT: "Volumetric Image"

- Continuously move table while scanning
- Need to interpolate in z-direction
- Use "Multidetectors"

"There is no safe dose"

## X-Ray-Dose: (Womank Children → More Sensitive)

- Ionized Radiation → Damage!
  - Deterministic: Cell Death → High Threshold
  - Stochastic: Cancer → NO Threshold
- Absorbed Dose (Physically)  $[Gy] = [J/kg]$
- Equivalent Dose (Biological effects)  $[Sv]$ 
 $[ESv] = Q \cdot N [Gy]$ 
  - Energy per mass can be more harmful from different rad. in different tissue
- Quality Factor Q:
  - y, e<sup>-</sup>, muons ... 1
  - Neutrons ... 5-20 (=RBE)
  - Protons ... 5
  - α-Particles ... 20
- Radiation Factor N
  - Bone, colon, lung, stomach ... 0.12
  - Brain, Breast, Kidney, liver, muscles ... 0.05
  - Skin ... 0.01

## Nuclear Imaging: (Planar Scintigraphy / SPECT)

- Idea: Detect distribution of "Radioisotopes" (i.e. compound with unstable nucleus)
- Radioactivity:
  - α: Emitting:  $^4He$
  - β: Emitting:  $e^-$
  - γ: Emitting:  $\gamma$  (Photon)
- Half-life: Time:  $T_{1/2} = \frac{1}{\lambda} \cdot \log(2)$  [Q] =  $C_0 e^{-\lambda t}$
- Patient exposure depends on:
  - i) Physical  $T_{1/2}$  ("Mean Clearance")
  - ii) Biological  $T_{1/2}$  ("Mean Clearance")
- Ideal Radioisotope:
  - Reasonable  $T_{1/2}$  (99mTc: ~6h)
  - High: Need of good dose for suff.
  - Too low: No time to reach target
- Emission of monoenergetic photons
  - 3-Row: absorbed
  - Can distinct scattered γ
  - $E_{\gamma} \in [100, 200] ReV$  → Get out of body but don't penetrate collimator
  - High uptake in ROI
- Interaction γ & Tissue (Some or X-Ray)
  - i) Photo Effect → Ionization (Damage)
  - ii) Scatter:  $E_{\gamma}^{scat} = \frac{m_e \cdot c^2}{\frac{m_e \cdot c^2}{E_{\gamma}} + 1 - \cos \phi}$  (See X-RAY)

## Popular Radioisotopes for SPECT:

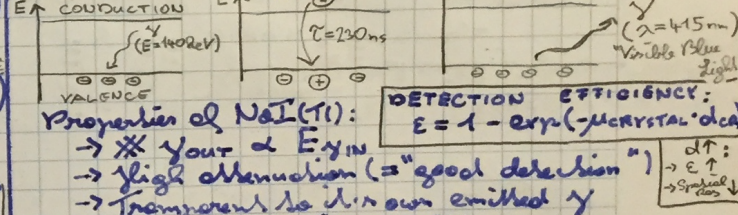
- $E_{\gamma} = 140 ReV$   $T_{1/2} \sim 6h$   $^{99m}Tc$
- Production:  $^{99}Mo \rightarrow ^{99m}Tc$
- $^{42}K \rightarrow ^{42}Ca$
- $^{111}In \rightarrow ^{111m}In$
- $^{201}Tl \rightarrow ^{201}Pb$

## The Gamma Camera:

- ① Collimator (v.l. "Anti-Scatter-Grid")
  - Goal: Increase Spatial Resol., but decrease sensitivity
- Parallel Hole:
  - PSF: Sensitivity reaching image plane
  - FWHM =  $\frac{L}{L_{eff}} (x + L_{eff})$
  - Where  $L_{eff} = L - \frac{2}{\mu_{scat}}$  due to finite transmission through Septa!
- DEPTH-DEPENDENCY

- Converging: FOV ↓, Spatial Res. ↑
- Diverging: FOV ↑, Spatial Res. ↓
- Fan-beam: like converging (for rectangular instead of circular geometry)
- Pinhole: for very small objects! (PSF depends on camera geometry)

## ② Scintillation Crystal (NaI(Tl))



DETECTION EFFICIENCY:  $\epsilon = 1 - \exp(-\mu_{crystal} \cdot d)$

Properties of NaI(Tl):

- \*  $\gamma$  out &  $E_{\gamma}$  in
- High attenuation (= "good detection")
- Transparent to its own emitted γ

## ③ Photomultiplier Tubes PMT

Problem: Energy of photon emitted by scint. are very low!

Goal: Magnify current!

## ④ Position Network (ANGER)

For each γ-Ray hitting the scintillation crystal multiple PMT's yield an output  $S_i$

The location of the incoming γ-Ray computes as:

$$\langle x \rangle = \frac{\sum S_i x_i}{\sum S_i}, \quad \langle y \rangle = \frac{\sum S_i y_i}{\sum S_i}$$

## ⑤ Pulse Height Analyser PHA

Goal: Discriminate between primary γ-Rays & scattered γ-Rays

$$\frac{\Delta E}{E_0} = \frac{1}{\sqrt{p \cdot N}}$$

→ Quality of Gamma-Cam:
 

- SNR =  $\sqrt{N}$  (See X-Ray)
- Spatial Res.:  $R_{tot} = \sqrt{R_{coll}^2 + R_{cam}^2}$



Position Emission Tomography Problem: Position travels a bit before annihilation

Really high SNR (no detector crystal collimator) → if detector crystal too expensive

# PET:

- Basic Principle: Radiotracer emits positrons! After annihilation forms 2  $\gamma$ -rays in 180° direction ( $E_\gamma = 511 \text{ keV}$ )
- Instrument for PET: Compton FDR
- Scintillation Crystal: (Mostly BGO)
  - Solent Crystal would have
  - High detection efficiency @ 511 keV
  - Short decay (to detect) time
  - Many problems per incident  $\gamma$ ! (Requires Ann. MT → cheaper)
- PSF for discrete pair:

$$FWHM = w \cdot \frac{\pi + x}{2\pi}$$

Overall Sensitivity:

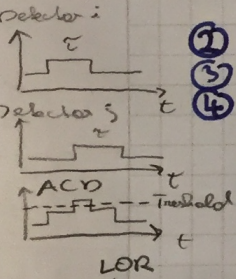
Detection efficiency:  $\epsilon = \Phi (1 - \exp(-\mu d))$

Geometric efficiency:  $\Omega = 4\pi \sin^2 \psi$

Radial coverage:  $\phi = x_{\gamma} / [x_{\gamma} + x_{\alpha}] \cdot (y_{\gamma} + y_{\alpha})$

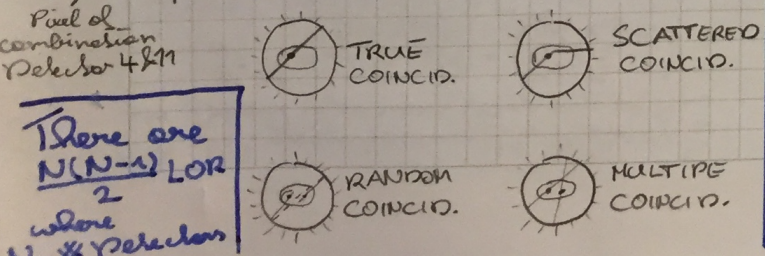
$\eta_{TOT} = \epsilon^2 \phi \cdot \frac{A}{4\pi}$  ( $< 1\%$ , typically)

- Again PMTs after Crystal
- Pulse flight analysis (ACD)
- Coincidence detection module:
  - Resolving time:  $2\tau$
  - If first  $\gamma$  is detected logic pulse is sent with duration  $\tau$ .
  - Second  $\gamma$  hits detector within this time → Pulses add above threshold → Accept LOR



From LOR to Sinogram: Every combination of two detectors corresponds to a point in the Sinogram → Inverse Radon!

Different Coincidence events:



There are  $\frac{N(N-1)}{2}$  LOR where  $N$  = # Detectors

- Beforehand: Correction for non-uniform detector efficiency → Use some reference image!
- Correction of PET-Data: (Both also for SPECT)
    - Attenuation Correction:  $\gamma$ -rays are attenuated in tissue → Less signal from deeper regions, even though same amount of FDR!
      - Typical procedure:
        - Geometric Sinogram of PET & CT
        - Correction map =  $\exp(0.234 \cdot CT \cdot \sin)$
        - PET Sinogram = PET Sinogram \* Correction map
      - For SPECT/PET: Scatter in of some amount
    - Scattered Coincidence Correction:
      - Energy window method: Tell PHA also to let pass signals from an energy window that only has contribution from scattered X-rays → Perform less times image reconstruction → weighting → Subtract!!!

- Correction of random coincidence:
  - Treat as noise! → Less in CNR
  - Start window long after true coinc. could have taken place → measure only randoms → Subtraction!
- After Correction: Usual Reconstruction
  - Either simple Backprojection / S-Radon
  - Filled BP (LOW-PASS)
  - Beware: 3D using Slice-Theory

- Radionuclides:
  - $^{14}\text{C}$ -compounds
  - Isotopic Substitution
    - Theoretically all organic compounds can be labeled
    - Receptor affinities unchanged
  - Short half life time (20 min)
  - $^{18}\text{F}$ -compounds: Inverse of  $^{14}\text{C}$  ( $t_{1/2} = 110 \text{ min}$ )

## Quantitative PET:

- $(\% \text{ID/g}) = \frac{C_t \cdot V_t}{C_p \cdot V_p} \cdot 100\%$ 
  - $C_t$ : tissue conc.
  - $V_t$ : tissue volume
  - $C_p$ : plasma conc.
  - $V_p$ : plasma volume
- SUV =  $(\% \text{ID/g}) \cdot \frac{100}{\text{Mass}}$ 
  - Obsv: Mass → Surface
- Distribution Volume:  $V_D = M/C_p$ 
  - "Volume of water required to dilute total amount of compound M to reach same concentration as it has in plasma"
- $V_D = V_D + V_T \cdot \frac{CT}{CN}$ , Part. Coeff.:  $\lambda = \frac{CT}{CN}$

- Receptor Binding:  $L + R \xrightleftharpoons[k_{off}]{k_{on}} RL$ 
  - In equilibrium:  $R_{tot} \cdot [L] \cdot [R] = R_{off} [RL]$ ,  $K_D = \frac{R_{off}}{R_{on}} = \frac{[L][R]}{[RL]}$
  - Mass conserv.:  $[R_T] = [R] + [RL]$
  - RT: Total amount of receptors
  - Equilibrium constant
  - Slope =  $-\frac{1}{K_D}$
- Saturation Equation:  $\frac{[RL]}{[L]} = -\frac{1}{K_D} [RL] + \frac{[R_T]}{K_D}$
- Compartment - Model:
  - PLASMA  $\xrightleftharpoons[k_2]{k_1}$  C1  $\xrightleftharpoons[k_4]{k_3}$  C2  $\xrightleftharpoons[k_6]{k_5}$  C3
  - MEASURE
  - FREE LIQUID
  - BOUND LIQUID
  - HEASURE

Kinetic Equations:

$$\frac{d}{dt} C_1 = k_1 C_p - (k_2 + k_3) C_1 + k_4 C_2$$

$$\frac{d}{dt} C_2 = k_3 C_1 - k_4 C_2$$

Flow Rate:  $E = 1 - \exp(-\frac{t}{\tau})$

Permeability:  $P = \frac{K_1 \cdot C_p}{K_2}$

Surface:  $S = \frac{R_{on} \cdot [L]}{R_{off}}$

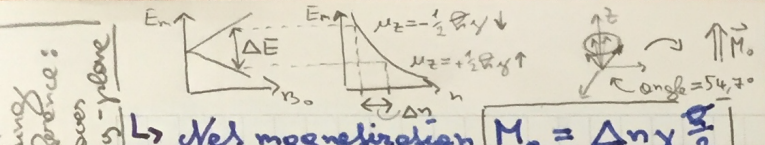
- If Impulse Response is known: Compute any output via convolution with input function (measured plasma-concentr.)
- Dependence of TAC's ("Time-activity curves") on reaction rates  $k_i$ :
  - INPUT-FUNC
  - WEIGHTING-FUNC
  - TAC of ROI

- Note: PET-measures everything:  $C_1 V_1 + C_2 V_2 + C_3 V_3$
- Example FDR:
  - It is also important to model the "cold" drug (the FDR competitor is glucose). Since the amount of Receptors is limited → Bounded Dose decreases!

## Magnetic Resonance Imaging (MRI)

- QM: Spin generates magnetic moment:  $\mu = \gamma S$ 
  - Spin comes in Quanta:  $S_z = \hbar m$ ,  $m = -s, -s+1, \dots, s$
  - For  $H^+$ :  $s = \frac{1}{2} \rightarrow S_z = \pm \frac{1}{2} \hbar \rightarrow \mu_z = \pm \frac{1}{2} \hbar \gamma$
- Potential Energy of dipole in magnetic field:  $E_m = -\mu \cdot B = -\frac{1}{2} \gamma \hbar B_0$
- More Energy → Less favourable → Population difference
- Boltzmann:  $\frac{n_{-1/2}}{n_{+1/2}} = \exp(-\frac{\Delta E}{k_B T})$ ,  $\Delta E = \hbar \gamma B_0 = \hbar \omega_L$





In Rotating Frame of reference:  $M_0$  only moves towards  $xy$ -plane

Net magnetization  $M_0 = \Delta n \gamma \frac{\hbar}{2}$

Black Equations (Dynamic of  $M_0$ ):

$$\frac{d\vec{M}}{dt} = \gamma (\vec{B} \times \vec{M}) - \left\{ \begin{matrix} M_x/T_2 \\ M_y/T_2 \\ (M_z - M_0)/T_1 \end{matrix} \right\}$$

Phenomenology of  $\vec{M}_0$ -Dynamics:

- Free Precession: ( $B_x = B_y = 0$ )
- On Resonance: ( $\omega_{RF} = \omega_L$ )
- Off Resonance: ( $\omega_{RF} \ll \omega_L$ )
- Spin-Spin-Relaxation: ( $T_2$ )  
Decrease  $M_{xy}$ -components! Due to Proton-Proton-Interaction!  
(Even faster decrease:  $T_2^* \rightarrow$  Due to field inhomogeneities  $\rightarrow$  Lose phase coherence)
- Spin-Lattice-Relaxation: ( $T_1$ )  
Realignment with  $z = -x$  in

Rotating frame of reference: ( $\omega_{RF} = \omega_L$ )  
Conversion of  $B_z$ :  $B_z = B_0 - \omega_{RF}/\gamma$

Steady-State transverse magnetiz. (Model: RF-Pulse results in instantaneous flip angle  $\alpha$ , until next pulse: fully relaxed)

$$M_{xy} = M_0 \sin \alpha \frac{1 - \exp(-T_R/T_2)}{1 - \cos \alpha \exp(-T_R/T_1)}$$

Optimum flip angle:  $\alpha_{opt} = \cos^{-1}(\exp(-T_R/T_1))$

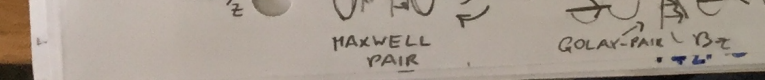
Signal Detection: Faraday Induction!

$$U_{IND} = i\omega \oint_{coil} \vec{B}(\vec{r}) d\vec{A} = M_{xy} \cdot V \cdot S(\vec{r})$$

Coil sensitivity:  $S(\vec{r}) = i\omega I_0^2 (\vec{r}_x - i\vec{r}_y)$

Choose coil:  $\rightarrow$  long distance: Bigger Coil  
 $U_{IND} \propto B_0^2 \rightarrow$  Small distance: Smaller Diameter

Image Acquisition: (Gradient Encoding)  
Key idea: Produce variable  $B_z$  field in all coord. directions! Phase frequency becomes a function of space!

$$G_x = \frac{\partial}{\partial x} B_z, G_y = \frac{\partial}{\partial y} B_z, G_z = \frac{\partial}{\partial z} B_z$$


Example: Axial Slice

$$\omega_L = \gamma B_0$$

- 3 Steps of Image Formation:
- Slice Selection:**  
Simultaneously switch on  $G_z$  and RF-Pulse! RF-Pulse has frequ.  $\omega$  and excitation BW of  $\Delta\omega$ ! Since Larmor-freq. is a function of  $z$ , only a slice will have magnetization rotated into  $xy$ -plane!  
(Achieve Rectangle in  $k$ -Domain  $\rightarrow$  would need infinitely long sinc!) Sinc:  $\text{Sinc}(x) = \text{Sinc}(x \cdot \text{bandwidth})$
  - Phase encoding:** (e.g. along  $y$ )  
Switch on the gradient  $G_y$  for a time  $t_e \rightarrow$  then switch off! Results in a phase shift of protons as a function of  $y$ :  $\varphi_{PH-EN} = \gamma G_y y t_e$
  - Frequency Encoding:** (Rare: along  $x$ )  
Apply gradient  $G_x$  during data acquisition  $\rightarrow$  Larmor frequency becomes function of  $x$

RESULT: Thin slice along  $z$ , with transverse magnetization, where  $\varphi = \varphi(x, y)$  and  $\omega = \varphi(x)$

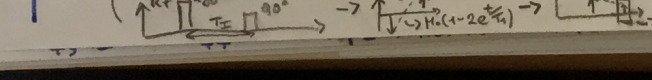
Image Reconstruction: Phase of signal from position  $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\varphi(\vec{r}, t) = \gamma B_0 t + \gamma \begin{pmatrix} G_x \cdot t \\ G_y \cdot t \\ G_z \cdot t \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$S(\vec{r}) = e^{i\varphi} = \int_{\text{obj}} S(\vec{r}) \cdot e^{i\vec{r} \cdot \vec{k}} d\vec{r}$$

Obtain Proton density via F-1 !!!  
Recall: Resolution  $\rightarrow$  Elements

- Travel through  $k$ -Space via Gradients!
- Imaging Sequences:
- Gradient Echo:**  
 $S \propto M_{xy} = S(x, y) \cdot \frac{(1 - e^{-T_R/T_1})}{1 - \cos \alpha \cdot e^{-T_R/T_1}} \cdot \sin \alpha \cdot e^{-T_E/T_2}$
  - Spin Echo:** (For fully relaxed  $M_0$ :  $T_R \gg T_1$ )  
 $S \propto M_{xy} = S(x, y) \cdot \sin \alpha \cdot \exp(-T_E/T_2)$
  - Inversion Recovery:**  
 $S \propto M_{xy} = S(x, y) \cdot \sin \alpha \cdot e^{-T_R/T_1} (1 - 2e^{-T_E/T_2})$



MRI-Quality:  $\sigma_{noise}^2 = 4R_n BW (R_{SAMP}^{EFF} T_{SAMP} + R_{coil} T_{coil} + R_{env} T_{env})$

$$SNR \propto \Delta V \cdot \sqrt{t_{scan}}$$

Ultrasound Imaging: ( $\nu = 1 - 50 \text{ MHz}$ )

Key Idea: Send "pressure beam" through tissue & measure reflected intensities at boundaries!

Basic Relations:

$$c = (\kappa \rho)^{-1/2}, \kappa \dots \text{compressibility}, \rho \dots \text{density}$$

$$p = \rho c u_z, u_z \dots \text{particle velocity}$$

$$Z = \rho u_z = \rho c = \sqrt{\rho/\kappa} \dots \text{Impedance}$$

$$I = \frac{1}{2} \rho u_z^2 \dots \text{Intensity}$$

Reflection & Transmission:

$$p_r = p_i \cdot \pi, \pi = \frac{Z_2 - Z_1}{Z_2 + Z_1}, p_t = p_i \cdot t, t = \frac{2Z_2}{Z_2 + Z_1}$$

Interaction in body:

- Rayleigh Scattering ( $\sigma_s \propto \lambda^{-4}$ )
- Speckles: Randomly in space, but "Distinctive Structure"
- Attenuation:  $p(z) = p_0 \cdot \exp(-\alpha z)$   
 $\alpha_0 = 20 \cdot \log(p_0/p(z)) \cdot \frac{1}{z} [\text{dB/m}]$   
Example (Kidney)  $\alpha = 1 \text{ dB/cm-MHz}$

Instrumentation:

- Piezoelectric Element (Pressure  $\Rightarrow$  Voltage)
- Energy: Match Impedances!
- Damping: Shorter RF-Pulse  $\rightarrow$  Better axial resolution (High BW)

Beam Geometry:

$$Z_{NFF} = \frac{\lambda^2}{\lambda_{Tissue}^2}$$

$$\lambda = \text{arcmin}(\frac{0.61\lambda}{n})$$

lateral resolution

NEAR FIELD FAR FIELD  $\rightarrow$  Wave Speed

Axial Resolution:  $\Delta z = \frac{1}{2} p_n \cdot c = \frac{1}{2} \lambda$

Linear Phase Array:

