

Diskrete Mathematik



Ch.1 Introduction

v1

Discrete mathematics is concerned with finite and countable infinite mathematical structures.

The main theme of this course: abstraction
Without it, writing good software becomes impossible.

A computer program is a discrete mathematical object, meaning that there is a mathematical proof that it is correct.

Mathematical statements are either true or false.

Important Signs:

- \forall for all
- \exists there exists
- \equiv_3 congruent to modulo 3

Ch. 2 Mathematical Reasoning & Proof

v2

A mathematical statement that is either true or false is called a **proposition**. When true, a proposition is often called a **theorem** or **lemma**, further if it's unclear it's called a **assumption**.

To connect mathematical statements we can use:

- | | |
|-----------------------|---|
| S and T | - both need to be true |
| S or T | - one needs to be true |
| S \Rightarrow T | - S implies T |
| S \Leftrightarrow T | - $S \Rightarrow T$ and $T \Rightarrow S$ |

A **proof** is used to show that a mathematical statement is true or false.

Informal proof - based on already proven axioms and known facts, formulated in common language

Formal proof - rigorous and formal, with the advantages of:

- preventing errors
- proof complexity and automatic verification
- precision and deeper understanding

Propositional Logic

The logical values (constants) "true" and "false" are usually denoted as 1 and 0

Logical operators:

In fact, \neg and \wedge are sufficient to express every logical function

- $\neg A$ NOT
- $A \wedge B$ AND
- $A \vee B$ OR
- $A \rightarrow B$ Implication
 "on both sides"
- $A \leftrightarrow B$

These operators are functions (e.g. \vee is a function of $\{0,1\} \times \{0,1\} \rightarrow \{0,1\}$, this can be shown as a tabel.

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A expression like this is called a formula

There are priority rules for logical operators:

1. \neg
2. \wedge / \vee
3. $\rightarrow / \leftrightarrow$

If two logical functions are equivalent,
we denote it with $F \equiv G$.

Basic equivalences:

↑ by using \equiv or \models it becomes
a statement.

Lemma 2.1.

- 1) $A \wedge A \equiv A$ and $A \vee A \equiv A$ (idempotence);
- 2) $A \wedge B \equiv B \wedge A$ and $A \vee B \equiv B \vee A$ (commutativity);
- 3) $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ and $(A \vee B) \vee C \equiv A \vee (B \vee C)$ (associativity);
- 4) $A \wedge (A \vee B) \equiv A$ and $A \vee (A \wedge B) \equiv A$ (absorption);
- 5) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (distributive law);
- 6) $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (distributive law);
- 7) $\neg\neg A \equiv A$ (double negation);
- 8) $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (de Morgan's rules).

If a formula is a logical consequence, it is denoted by $F \models G$. If we could interpret this as $F \leq G$, but we don't do that!

! When using \models or \equiv , the result is a statement. !

If a formula is always true, we call it a tautology (T) or valid. If it is true for at least one truth assignment it is satisfiable, if not, it is unsatisfiable (\perp).

The distinction between formula and statement is really important!

Predicate Logic

V3

Propositional logic is often not capable to express a statement (e.g. there are infinitely many prime numbers) in such cases we use predicate logic.

Predicate logic is based on **k-ary predicates** P in a universe U . P on U is a function $U^k \rightarrow \{0,1\}$. The predicate $P(x)$ assigns every x to a value 0=false or 1=true.

In predicate logic, we can also use functions on U , for example addition and multiplication.

Further we use the **Quantifiers** \forall and \exists .

$\forall x P(x)$ $P(x)$ is true for all x in U .
 $\exists x P(x)$ $P(x)$ is true for some x in U .

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Formulas are often leave room open for interpretation. They do this by having "free parts" that are not fixed. Such formulas can't be true or false until a interpretation for the "free parts" is applied.

If a formula has no interpretation that allow it to be true, it is called **unsatisfiable** if there is a interpretation that allows it to be true it is called **satisfiable** and if it is true for an interpretation, it is a **tautology**.

Logical Formulas and Mathematical Statements

Formulas and statements are different, because the symbols of a formula can be interpreted differently.

When a interpretation is fixed for a formula, it becomes a mathematical statement. If this is the case we can also tell if a formula is true or false.

While formulas are no statement, we can make statements about them:

- F is unsatisfiable
- F is valid ($\models F$)
- $F \models G$
- $\{F, G\} \models H$
- F is valid $\Rightarrow G$ is valid

Proof Patterns

Composition of Implications

If $S \Rightarrow T$ and $T \Rightarrow U$ are true, then $S \Rightarrow U$ is also true.

Proof. One writes down the truth tables for $(A \Rightarrow B) \wedge (B \Rightarrow C)$ and $A \Rightarrow C$. If the first evaluates to true, then the second one will as well.

Direct Proof of an Implication

The proof of an implication $S \Rightarrow T$, by assuming S and then proving T under this assumption.

Indirect Proof of an Implication

The proof of $S \Rightarrow T$ works by assuming T is false and proving S is false under this assumption.

Ex. Proof that if x is irrational, then \sqrt{x} has to be irrational too.

Modus Ponens

1. Find a statement R
2. Prove R
3. Prove $R \Rightarrow S$

Ex. $2^{3000} \equiv_{3001} 1$ (in Script.)

Case Distinction

1. Find a finite list of m . statements R_1, \dots, R_k (cases)
2. Prove that one R_i is true.
3. Prove $R_i \Rightarrow S$ for $i = 1, \dots, k$.

Ex. Checkerboard with one piece missing, can be covered by L-Shapes.

Proof by Contradiction

1. Find a statement T .
2. Prove that T is false.
3. Assume S is false and prove T has to be true for this. (a contradiction)

Existence Proofs

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Consider a set \mathcal{X} of parameters and for each $x \in \mathcal{X}$ a statement S_x . A existence proof is a proof that a statement is true for at least one $x \in \mathcal{X}$. It is **constructive** if it exhibits an a for which S_a is true, otherwise it is **non-constructive**.

Existence Proof via Pigeonhole Principle Schubfach Prinzip

If a set of n objects is partitioned into $k < n$ sets, then atleast one set must contain $\lceil \frac{n}{k} \rceil$ objects.

Ex. Among 100 people, atleast 9 are born in the same month.

Proofs by Counterexample

Considering a set \mathcal{X} of parameters and for each $x \in \mathcal{X}$ a statement S_x . A proof by counterexample is proof that S_x is not true for all $x \in \mathcal{X}$.

Proof by Induction

See notes on A&D.

Ch.3 Sets, Relations & Functions

A **set** is a collection of **objects**. For every object it is defined whether it is a **element** of a set or not (denoted by $x \in A$). Order and multiple occurrences of elements is irrelevant.

The number of elements in a finite set is the **cardinality**, denoted by $|A|$.

A set can also be defined by a property of the elements.

$\{x \in A \mid P(x)\}$ elements of A having property P

Sets are **equal** if they contain the same elements.

A set can be a element of another set

A set A is a **subset** of another set B if all elements of A are also part of B.

A ordered pair (a, b) is defined as $\{\{a\}, \{a, b\}\}$.

A set is **empty** if it does not contain any elements. This is often denoted as \emptyset .

\emptyset is a subset of every set

With the empty set, we could define the natural numbers.

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$$0 \stackrel{\text{def}}{=} \emptyset, \quad 1 \stackrel{\text{def}}{=} \{\emptyset\}, \quad 2 \stackrel{\text{def}}{=} \{\emptyset, \{\emptyset\}\}$$

Defining the function $s(n) = n \cup \{n\}$

One can then start to define operations like +

$$m + 0 = m \quad m + s(n) = s(m+n)$$

Power set

The power set of a set A , denoted $P(A)$, is the set of all subsets of A .

$$P(A) = \{S \mid S \subseteq A\}$$

e.x. $P(\emptyset) = \{\emptyset\}$ $P(\{a\}) = \{\emptyset, \{a\}\}$ \emptyset is part of every power set.

$P(A)$ has cardinality $2^{|A|}$.

Union & Intersection

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$
 union logical or

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$
 intersection logical and

This can be extended to a set of sets.

$$\bigcup A = \{x \mid x \in A \text{ for some } A \in A\}$$

$$\bigcap A = \{x \mid x \in A \text{ for all } A \in A\}$$

Typically sets in A are indexed by a index set I .
In this case one writes $\{A_i\}_{i \in I}$. The union becomes $\bigcup_{i \in I} A_i$ and the intersection $\bigcap_{i \in I} A_i$.

Complement

$$\bar{A} = \{x \in U \mid x \notin A\}$$
 logical negation

Difference

$$B \setminus A = \{x \in B \mid x \notin A\}$$

The Cartesian Product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

The set of all ordered pairs (a, b) (defined as $(a, b) = \{\{a\}, \{a, b\}\}$) where a is from set A and b is from set B .

Cardinality: $|A \times B| = |A| \cdot |B|$

Generally: $\times_{i=1}^k A_i = \{(a_1, \dots, a_k) \mid a_i \in A_i \text{ for } 1 \leq i \leq k\}$

Not associative!

Relations

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We can define relations between a set A and a set B . This (binary) relation ρ is a subset of $A \times B$. If $A=B$ then ρ is a relation on A .

$$(a, b) \in \rho \equiv a \rho b$$

Relations can be interpreted as graphs or as matrixes: $A = \{a, b, c\}$ $B = \{d, e, f\}$

$|A| \times |B|$ matrix

$$M^\rho = \begin{bmatrix} a & d & e & f \\ b & 1 & 0 & 1 \\ c & 1 & 0 & 0 \end{bmatrix}$$

$$\rho = \{(a, d), (b, d), (c, e), (a, f)\}$$

The concept of relations can be generalized from binary to k-ary for given sets A_1, \dots, A_k . This can for example be useful for database modeling. But in this course we only look at binary relations.

We can use set operations on relations.
(e.g. $\leq \cup \geq$ is the identity relation)

Further we can take the inverse of a relation ρ .

$$\hat{\rho} \stackrel{\text{def.}}{=} \{(a,b) \mid (b,a) \in \rho\}$$

We can also take the composition of relations.

$$\rho \circ \sigma \stackrel{\text{def.}}{=} \{(a,c) \mid \exists b \in B ((a,b) \in \rho \wedge (b,c) \in \sigma)\}$$

The composition $\rho \circ \rho = \rho^2$. Generalized ρ^n .

Special Properties

Name	Definition	Example
reflexive	$a \rho a$ is true for all $a \in A$, i.e. if $\text{id} \subseteq \rho$	\leq
irreflexive	$a \not\rho a$ is true for all $a \in A$, $\rho \cap \text{id} = \emptyset$	$<$
symmetric	$a \rho b \Leftrightarrow b \rho a$ is true, $\rho = \hat{\rho}$	$=$
antisymmetric	$(a \rho b \wedge b \rho a) \Rightarrow a = b$, $\rho \cap \hat{\rho} \subseteq \text{id}$	\leq
transitive	$(a \rho b \wedge b \rho c) \Rightarrow a \rho c$, $\rho^2 \subseteq \rho$	\leq

Transitive Closure $\rho^* = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \rho^n$

Equivalence Relations

Vg

A equivalence relation is a relation on a set A that is reflexive, symmetric and transitive (e.g. \equiv_m).

For $a \in A$, the set of elements of A that are equivalent to a is called the equivalence class $[a]_\theta$.

$$[a]_\theta \stackrel{\text{def}}{=} \{b \in A \mid b \theta a\}$$

The set A/Θ of equivalence classes of an equivalence relation Θ on A is a partition of A .

Partial Order Relations

A partial order on a set A is a relation that is reflexive, antisymmetric, and transitive. A set A with a partial order \leq on A is called a partially ordered set or poset and is denoted as $(A; \leq)$.

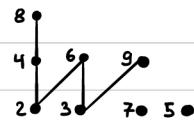
For a partial order we can define \prec similar to how $<$ is obtained from \leq .

$$a \prec b \stackrel{\text{def}}{\iff} a \leq b \wedge a \neq b$$

If any two elements of a poset are comparable, it is called totally ordered by \leq .

In a poset a element a covers b , if and only if $a \prec b$ with no $a \prec c$ and $c \prec b$.

A Hasse diagram of a poset is the graph whose vertices are elements of A and where there is an edge from a to b if and only if b covers a .



Hasse diagram of the poset $(\{2,3,4,5,6,7,8,9\}; \leq)$

For two given Posets a relation \leq can be defined on $A \times B$. V10

$$(a_1, b_1) \leq (a_2, b_2) \stackrel{\text{def}}{\iff} a_1 \leq a_2 \wedge b_1 \leq b_2$$

this is again a partial order relation. Further the relation \leq_{lex} is defined as

$$(a_1, b_1) \leq_{\text{lex}} (a_2, b_2) \stackrel{\text{def}}{\iff} a_1 < a_2 \vee (a_1 = a_2 \wedge b_1 \leq b_2)$$

Again this is a partial order relation. \leq_{lex} is useful if A and B are totally ordered.

Special Elements of Posets

1. minimal / maximal element ($a, b \in A$ exists no $b < a \wedge a < b$)
2. least / greatest element ($a, b \in A$ for all $a \leq b \wedge b \leq a$)
3. lower / upper bound of $S \subseteq A$ ($a \in A, b \in S$ for all $a \leq b \wedge b \leq a$)
4. greatest lower/least upper bound of $S \subseteq A$ (greatest/least element of all lower/upper bounds)

For a set $\{a, b\} \subseteq A$ a **meet** is $a \wedge b$ and a **join** is $a \vee b$. If every pair has a join and meet it is called a **lattice**.

Functions

Functions are a special form of relations. A function $f: A \rightarrow B$ from a domain A to a codomain B is a relation from A to B ($a f b$).

$$\forall a \in A \quad \exists b \in B \quad a f b \quad (\text{if is totally defined})$$
$$\forall a \in A \quad \forall b, b' \in B \quad (a f b \wedge a f b' \rightarrow b = b') \quad (\text{f is well defined})$$

The set of all functions $A \rightarrow B$ is B^A .

A partial function of $A \rightarrow B$ is a relation so that 2. holds true.

For a subset S of A , the image is:

$$f(S) \stackrel{\text{def}}{=} \{f(a) \mid a \in S\}$$

The subset $f(A)$ of B is the image / range of f , denoted as $\text{Im}(f)$.

For a subset T of B , the preimage of T ($f^{-1}(T)$) is

$$f^{-1}(T) \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in T\}$$

injective

for $a \neq b$ it is $f(a) \neq f(b)$

surjective

if $f(A) = B$ / for every $b \in B$, $b = f(a)$

bijective

both injective and bijective

Compositions of functions $g \circ f$ is defined as
 $(g \circ f)(a) = g(f(a))$. Different to normal relations!
It is associative.

Countable and Uncountable Sets

- i. two sets A and B are **equinumerous**, $A \sim B$, if there exist a bijection $A \leftrightarrow B$.
- ii. The set B **dominates** the set A , $A \leq B$, if $A \sim C$ for $C \subseteq B$ or if there exist a injective function $A \rightarrow B$.
- iii. A set A is **countable** if $A \leq B$ and **uncountable** otherwise.

$$A \leq B \wedge B \leq A \Rightarrow A \sim B$$

For finite sets we have $A \sim B$ if $|A| = |B|$.

All finite sets are countable.

Important Countable Sets

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- $\{0,1\}^*$, the set of finite binary sequences.
↳ $A \times B$ is countable if $A \in \mathbb{N} \wedge B \in \mathbb{N}$
 \mathbb{Q} is countable
- $\mathbb{N} \times \mathbb{N}$, the set of ordered pairs of natural numbers

It follows that:

- i. $n \in \mathbb{N}$, the set A^n of n -tuples over A is countable
- ii. The union $\bigcup_{i \in \mathbb{N}} A_i$ of a countable list A_1, \dots, A_i of countable sets is countable
- iii. The set A^* of finite sequences of A is countable.

Important Uncountable Sets

- $\{0,1\}^\omega$, the set of all infinite binary sequences
↳ can be proven with Cantor's diagonalization argument

Existence of Uncountable Functions

p. 67 script

←false, true

A function $f: \mathbb{N} \mapsto \{0, 1\}$ is called computable if there is a program that, for every $n \in \mathbb{N}$ has a output $f(n)$.

Ex. prime: $\mathbb{N} \rightarrow \{0, 1\}$

There are uncomputable functions $\mathbb{N} \rightarrow \{0, 1\}$

Ex. Halting problem important for CS

Ch.4 Number Theory

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Number Theory is the mathematical theory of the natural numbers \mathbb{N} or more generally of the integers \mathbb{Z} . While we focus on the numbers \mathbb{Z} , everything should also be applicable to any ring. (def. by ring axioms). We assume that we know what integers, operations and basic facts about integers are.

Divisors

For $a, b \in \mathbb{Z}$ we say a divides b , denoted $a|b$, if b is a multiple of a .

$$a|b \stackrel{\text{def.}}{\iff} \exists c \ b = a \cdot c \quad \begin{matrix} \text{quotient} \\ \text{divisor} \end{matrix}$$

If a is not a divisor of b , we get a rest. We can say that unique integers q, r exist for any $b, a \neq 0$.

$$b = q \cdot a + r \quad \begin{matrix} \text{remainder} \\ \text{where } 0 \leq r < |a| \end{matrix}$$

The remainder r is denoted as $R_a(b)$ or $b \bmod a$.

For two integers a, b (not both 0), an integer d is called the greatest common divisor, if

$$d|a \wedge d|b \wedge \forall c ((c|a \wedge c|b) \rightarrow c|d)$$

If $\gcd(a, b) = 1$, then a and b are relatively prime.

Further holds, $\gcd(m, n) = \gcd(m, n-q \cdot m) = \gcd(m, R_m(n))$.

For $a, b \in \mathbb{Z}$, the ideal generated by a and b , denoted (a, b) is the set $\{u \cdot a + v \cdot b \mid u, v \in \mathbb{Z}\}$. Similarly, the ideal generated by a single integer a is $(a) = \{u \cdot a \mid u \in \mathbb{Z}\}$.

For any (a, b) exists a (c) such that $(a, b) = (c)$. If $a \neq 0, b \neq 0$ then d is the gcd of a and b . It follows that $\gcd(a, b) = ua + vb$.

The least common multiple ℓ is defined as,

$$a|\ell \wedge b|\ell \wedge \forall c ((a|c \wedge b|c) \rightarrow \ell|c)$$

Primes

A positive integer $p > 1$ is called **prime** if the only positive divisor of p are 1 and p . Every other integer is a **composite**.

The fundamental theorem of arithmetic states that every positive integer can be written **uniquely** as the product of primes.

Congruences and Modular Arithmetic

We say a is congruent to b modulo m , if m divides $a-b$.

$$a \equiv_m b \stackrel{\text{def.}}{\iff} m | (a-b)$$

For any $m \geq 1$, \equiv_m is a equivalence relation on \mathbb{Z} .

If $a \equiv_m b$ and $c \equiv_m d$ then $a+c \equiv_m b+d$
and $ac \equiv_m bd$.

If $f(x_1, \dots, x_k)$ is a multi-variable polynomial with k variables. Then $a_i \equiv_m b_i$ for $1 \leq i \leq k$,

$$\Rightarrow f(a_1, \dots, a_k) \equiv_m f(b_1, \dots, b_k)$$

Modular arithmetic

Since \equiv_m is a equivalence relation on \mathbb{Z} we get the equivalence classes $[0], [1], \dots, [m-1]$. Each equivalence class $[a]$ has a natural representative $R_m(a) \in [a]$ in the set

$$\mathbb{Z}_m := \{0, \dots, m-1\} \text{ of remainders modulo } m.$$

For any $a, b, m \in \mathbb{Z}$ with $m \geq 1$:

$$\begin{aligned} a &\equiv_m R_m(a) \\ a \equiv_m b &\Leftrightarrow R_m(a) = R_m(b) \end{aligned}$$

It follows that

$$R_m(f(a_1, \dots, a_k)) \equiv_m R_m(f(R_m(a_1), \dots, R_m(a_k))).$$

Multiplicative Inverse

Considering $ax \equiv_m b$, obviously if x is a solution, so is $x + k \cdot m$ for any $k \in \mathbb{Z}$. Of special interest is the case where $\gcd(a, m) = 1$ and $b = 1$.

$$ax \equiv_m 1 \Leftrightarrow \gcd(a, m) = 1 \quad x \in \mathbb{Z}_m \text{ is unique}$$

x is called the multiplicative inverse of a mod m . Often denoted $x \equiv_m a^{-1}$ or $x \equiv_m \frac{1}{a}$.

To calculate the multiplicative inverse, the extended Euclidean algorithm is used.

Ex. $\gcd(99, 78) = \dots$

$$99 = 78 + 21 \Leftrightarrow 21 = 99 - 78$$

$$78 = 3 \cdot 21 + 15 \Leftrightarrow 15 = 78 - 3 \cdot 21 = 4 \cdot 78 - 3 \cdot 99$$

$$21 = 15 + 6 \Leftrightarrow 6 = 21 - 15 = 4 \cdot 99 - 5 \cdot 78$$

$$15 = 2 \cdot 6 + 3 \Leftrightarrow 3 = 15 - 2 \cdot 6 = \underline{-11 \cdot 99} + \underline{14 \cdot 78}$$

$$6 = 2 \cdot 3 + 0 \text{ ends when } r=0 \text{ or } r=1$$

\Rightarrow if $r=0$ there is no inverse

↑
inverse if
 $r=1$

Chinese Remainder Theorem

Let m_1, \dots, m_r be pairwise relatively prime and let $M = \prod_{i=1}^r m_i$. For every list a_1, \dots, a_r with $0 \leq a_i < m_i$ for $1 \leq i \leq r$, the system

$$\begin{aligned}x &\equiv_{m_1} a_1 \\x &\equiv_{m_2} a_2 \\&\vdots \\x &\equiv_{m_r} a_r\end{aligned}$$

has one unique solution $0 \leq x < M$.

Ex. solving CRT

① $x \equiv_3 2$
 $x \equiv_4 1$
 $x \equiv_5 4$

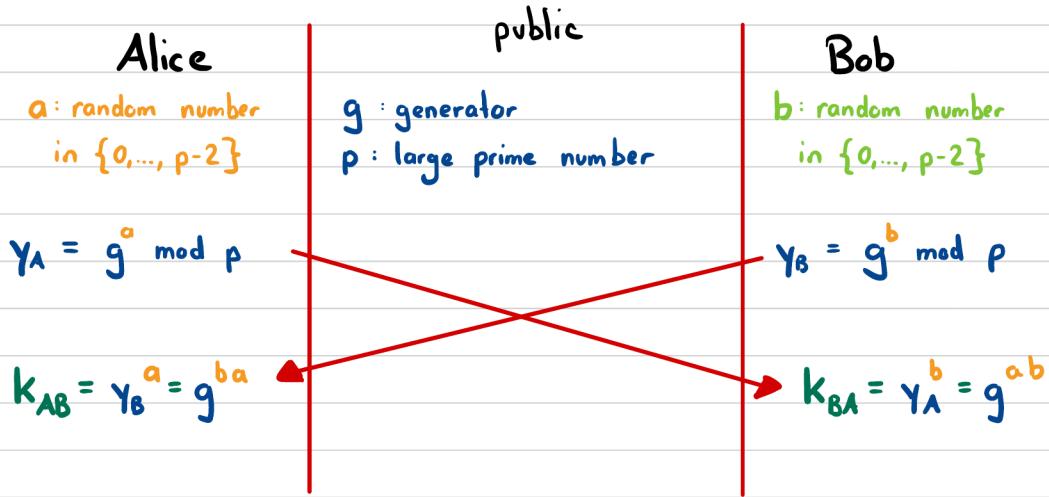
② $M_i = \frac{M}{m_i}$
• $M_1 = \frac{3 \cdot 4 \cdot 5}{3} = 20$
• $M_2 = \frac{3 \cdot 4 \cdot 5}{4} = 15$
• $M_3 = \frac{3 \cdot 4 \cdot 5}{5} = 12$

③ $N_i \cdot M_i \equiv_{m_i} 1$
• $N_1 \cdot 20 \equiv_3 1 \iff N_1 \cdot 2 \equiv_3 1 \Rightarrow N_1 = 2$
• $N_2 \cdot 15 \equiv_4 1 \iff N_2 \cdot 3 \equiv_4 1 \Rightarrow N_2 = 3$
• $N_3 \cdot 12 \equiv_5 1 \iff N_3 \cdot 2 \equiv_5 1 \Rightarrow N_3 = 3$

④ $\sum_{i=1}^r a_i \cdot M_i \cdot N_i = 2 \cdot 20 \cdot 2 + 1 \cdot 15 \cdot 3 + 4 \cdot 12 \cdot 3 = 80 + 45 + 144 = 269$
 $\equiv_{345} \underline{\underline{29}} = x$

Diffie Hellman

Diffie Hellman is one of the most used protocols for key-exchange. It uses the principle of a **one-way function**, a function that is easy to calculate but extremely hard to reverse.



It's easy to see that $k_{AB} = k_{BA}$. This is now the secret key, only known by Bob and Alice.

Ch.5 Algebra

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An operation on a set S is a function $S^n \rightarrow S$, where $n \geq 0$ is called the "arity" of an operation.

arity 1 = unary arity 2 = binary

An algebra is a pair $\langle S; \Omega \rangle$ where S is a set (carrier) and $\Omega = (\omega_1, \omega_2, \dots, \omega_n)$ is a list of operations on S .

$\langle \mathbb{Z}; +, -, 0, \cdot, 1 \rangle$ is an example of a algebra.

binary operators unary operators neutral elements

Monoids & Groups

A (left/right) neutral element of an algebra $\langle S; * \rangle$ is an element $e \in S$ such that $e * a = a$ or $a * e = a$ or both for all $a \in S$. There is at most one neutral element per operation.

A binary operation $*$ on a set S is associative if $a * (b * c) = (a * b) * c$.
one operation

A Monoid is an algebra $\langle M; *, e \rangle$ where $*$ is associative and e is the neutral element.

A (left/right) inverse b of an element a in an algebra $\langle S; *, e \rangle$ exists if $b * a = e$ or $a * b = e$ or both.

A group is an algebra $\langle G; *, ^\wedge, e \rangle$ satisfying the axioms:

G1: $*$ is associative

G2: e is a neutral element

G3: every element a has an inverse \hat{a} .

These axioms are not minimal and can be simplified.

A group or monoid is called commutative or abelian if $a * b = b * a$ for all $a, b \in G$.

Direct Products

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The direct product of n groups $\langle G_1; *_1 \rangle \dots \langle G_n; *_n \rangle$ is the algebra

$$\langle G_1 \times \dots \times G_n; \star \rangle$$

where the operation \star is component-wise (so are inverse and the NE)

$$(a_1, \dots, a_n) \star (b_1, \dots, b_n) = (a_1 *_1 b_1, \dots, a_n *_n b_n)$$

Group homomorphism

A function ψ from a group $\langle G; *, \wedge, e \rangle$ to a group $\langle H; \star, \sim, e' \rangle$ is a **group homomorphism** if

$$\psi(a * b) = \psi(a) \star \psi(b)$$

If ψ is bijective, then it is called a **isomorphism** and we write $G \cong H$.

$$\begin{array}{ccc} G & \xrightarrow{*} & G \\ \downarrow \psi & & \downarrow \psi \\ H & \xrightarrow{\star} & H \end{array}$$

Subgroups

A subset $H \subseteq G$ of a group $\langle G; *, \wedge, e \rangle$ is called a **subgroup** of G if $\langle H; \star, \sim, e \rangle$ is a group, i.e. H is closed with respect to all operations:

- 1) $a * b \in H$ for all $a, b \in H$
- 2) $e \in H$
- 3) $\hat{a} \in H$ for all $a \in H$

For any group $\langle G; \times, \wedge, e \rangle$, there exist two trivial subgroups: the subsets $\{e\}$ and G .

For now we will look at groups with the " \cdot " multiplicative operation, **multiplicative notation**, without loss of generality.

For $n \in \mathbb{Z}$, a^n is defined:

$$\cdot a^0 = e$$

$$\cdot a^n = a \cdot a^{n-1} \quad \text{for } n \geq 1$$

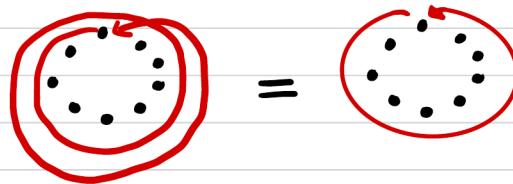
$$\cdot a^n = a^{-1}^{|\text{int}|} \quad \text{for } n \leq -1$$

Let G be a group and $a \in G$. The **order** of a , denoted $\text{ord}(a)$, is the least $m \geq 1$ such that $a^m = e$, if such exists. Otherwise $\text{ord}(a)$ is infinite.

In a finite group G , every element has a finite order.
 $|G|$ is called the order of G .
cardinality

Cyclic groups

If G is a group and $a \in G$ has finite order, then for any $m \in \mathbb{Z}$ we have $a^m = a^{\text{ord}(a)(m)}$



For group G and $a \in G$, the group generated by a , denoted $\langle a \rangle$, is defined

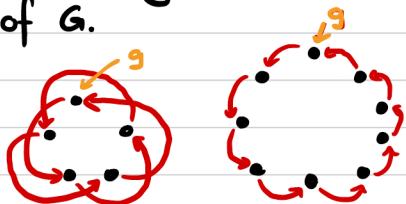
$$\langle a \rangle \stackrel{\text{def}}{=} \{a^n \mid n \in \mathbb{Z}\}$$

It is easy to see that $\langle a \rangle$ is the smallest subgroup of G containing the element a . For finite groups we have

$$\langle a \rangle \stackrel{\text{def}}{=} \{e, a, a^2, \dots, a^{\text{ord}(a)-1}\}$$

A group $G = \langle g \rangle$ generated by an element g is called **cyclic** and g is the **generator** of G .

If g is a generator, so is g^{-1} .



A cyclic group of order n is **isomorphic** to $\langle \mathbb{Z}_n; + \rangle$ and hence abelian.

The Order of Subgroups

! **Lagrange:** Let G be a finite group and $H \subseteq G$. Then the order of H divides the order of G , $|H| \mid |G|$

\Rightarrow For a finite group G , the order of every $a \in G$, divides $|G|$.

\Rightarrow Let G be a finite group. Then $a^{|G|} = e$ for every $a \in G$.

\Rightarrow Every group of prime order is cyclic and every element except the neutral element is a generator.

The group \mathbb{Z}_m^* and Euler's function

While $\mathbb{Z}_m = \{0, \dots, m-1\}$ is a group in respect to \oplus it is not for \circ . We need to exclude all elements without inverse.

$$\mathbb{Z}_m^* \stackrel{\text{def}}{=} \{a \in \mathbb{Z}_m \mid \gcd(a, m) = 1\} \quad (\text{excludes all with no inverse})$$

The Euler function $\varphi: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is defined as the cardinality of \mathbb{Z}_m^* :

$$\varphi(m) = |\mathbb{Z}_m^*|$$

If the prime factorization of m is:

$$m = \prod_{i=1}^r p_i^{e_i} \Rightarrow \varphi(m) = \prod_{i=1}^r (p_i - 1) p_i^{e_i - 1}$$

$\langle \mathbb{Z}_m^*, \circ, ^{-1}, 1 \rangle$ is a group.

If $m \geq 2$ and $\gcd(a, m) = 1$, then $a^{\varphi(m)} \equiv_m 1$. In particular for every prime p and a not divisible by p , $a^{p-1} \equiv_p 1$

\mathbb{Z}_m^* is cyclic if and only if $m = 2, m = 4, m = p^e$ or $m = 2p^e$, where p is a odd prime and $e \geq 1$.

RSA - Encryption

RSA public-key cryptosystem is another widely used protocol for secure communication.

To understand how it works, we need the following theorem, that follows from Lagrange's theorem.

Let G be some finite group (multiplicatively written) and let $e \in \mathbb{Z}$ be relatively prime to $|G|$. The function $f: x \rightarrow x^e$ is a bijection and the e -th root of $y \in G$, namely $x \in G$ satisfying $x^e = y$ is

$$x = y^d$$

where d is the multiplicative inverse of e modulo $|G|$,

$$ed \equiv_{|G|} 1$$

If $|G|$ is known, then d can be computed from $ed \equiv_{|G|} 1$ with the extended Euclidean algorithm. But there is no general method to compute it without knowing $|G|$.

With that we can look at how RSA works. In a group \mathbb{Z}_n^* , where $n=p \cdot q$ is the product of two large primes, the order $|\mathbb{Z}_n^*| = \varphi(n) = (p-1)(q-1)$ can only be computed knowing p and q .

Alice
Generate p, q

$$n = p \cdot q$$

$$f = (p-1)(q-1)$$

Select $e \in \mathbb{Z}_n^*$
 $d \equiv_f e^{-1}$ (inverse)

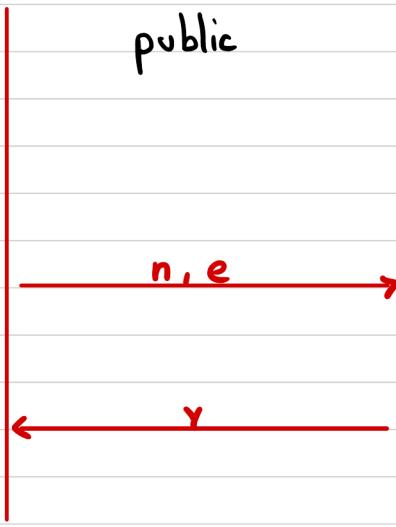
$$m = R_n(y^d) \stackrel{-1}{=} R_n(R_n(m^{ed}))$$

public

Bob

plaintext
 $m \in \{0, \dots, n-1\}$

ciphertext
 $y = R_n(m^e)$



Rings

Definition of a Ring: A ring $\langle R; +, -, 0, \cdot, 1 \rangle$ is an algebra for which

- i. $\langle R; +, -, 0 \rangle$ is a commutative group $(a+b=b+a)$
- ii. $\langle R; \cdot, 1 \rangle$ is a monoid
- iii. $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in R$
(distributive law)

Examples are $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C}

It holds that
 for all $a, b \in R$:

- i. $0a = 0$
- ii. $(-a)b = -(ab)$
- iii. $(-a)(-b) = ab$
- iv. if R is non-trivial
 then $1 \neq 0$

The characteristic of a ring is the order of 1 in the additive group if it is finite, otherwise it is 0. The characteristic gets denoted $\text{char}(R)$.

the number 1

Divisors (R denotes a commutative ring)

For $a, b \in R$, $a \neq 0$, if a divides b , $a|b$, there exists a $c \in R$ such that $b = a \cdot c$. b is a multiple and a is a divisor.

Note that every $a \neq 0$ divides 0 and 1/-1 divide every b .

- \Rightarrow i) $a|b$ and $b|c$ then $a|c$ (transitivity)
- ii) $a|b$ then $a|bc$ for all c
- iii) $a|b$ and $a|c$ then $a|(b+c)$

For $a, b \in R$ not both 0, there exists a gcd d such that $d|a \wedge d|b \wedge \forall c((c|a \wedge c|b) \rightarrow c|d)$

Units, Zerodivisors and Integral Domains

zerodivisor: element $a \neq 0$ such that $ab=0$ for some $b \in R, b \neq 0$.

unit: element in R that is invertible, i.e. $uv=vu=1$, we write $v=u^{-1}$. The set of all invertibles is denoted R^* and is a multiplicative group.

integral domain: a ring (nontrivial) without zero divisor. $\forall a \forall b (ab=0 \rightarrow a=0 \vee b=0)$

↪ In a integral domain if $a|b$ there exist a unique c such that $a \cdot c = b$, called **quotient**.

Polynomial Rings

A polynomial $a(x)$ over a ring R is of form:

$$a(x) = a_d \cdot x^d + a_{d-1} \cdot x^{d-1} + \dots + a_0 \cdot x^0 = \sum_{i=0}^d a_i \cdot x^i$$

for some non-negative integer d . The **degree** of $a(x)$, $\deg(a(x))$, is the greatest i for which $a_i \neq 0$. For the polynomial 0 , the degree is minus infinity. Let $R_{[x]}$ denote the set of polynomials (in x) over R .

Polynomials can be understood as finite lists (a_0, a_1, \dots, a_d) of elements in R .

⇒ For any ring R , $R_{[x]}$ is a ring. If D is a integral domain, so is $D_{[x]}$.

The units of $D_{[x]}$ are the constant polynomials that are units of D : $D_{[x]}^* = D^*$
 $\text{degree} = 0$

Fields

A field is a nontrivial commutative ring F in which every nonzero is a unit, i.e. $F^\times = F \setminus \{0\}$

\mathbb{Z}_p is a field if and only if p is prime. We can denote this by $GF(p)$, standing for Galois Field.

A field is an integral domain. (no zero divisors)

Polynomial over a field F are of special interest, since they have similar properties in common with the integers

A polynomial $a(x) \in F[x]$ is called monic if the leading coefficient is 1.

Example. $GF(3)[x] : 1 \cancel{x^3} + 2x^2 + x = (x+1)(x^2+x)$

A polynomial $a(x) \in F[x]$ with degree at least 1 is called irreducible if it is divisible only by constant polynomials and by constant multiples of $a(x)$. (corresponds to primality in \mathbb{Z})

Note that when multiplying polynomials the degree gets added.

The concept of gcd can also be carried over from \mathbb{Z} to $F[x]$.

The monic polynomial $g(x)$ of largest degree is the gcd of $a(x)$ and $b(x)$ if $g(x) | a(x)$ and $g(x) | b(x)$.

Also the concept of remainders can be carried over.
For any $a(x)$ and $b(x) \neq 0$ in $F[x]$ there exists a unique monic $q(x)$ quotient and a unique $r(x)$ remainder such that

$$a(x) = b(x) \cdot q(x) + r(x) \quad \text{and} \quad \deg(r(x)) < \deg(b(x))$$

Analogies between \mathbb{Z} and $F[x]$ *

These are some abstractions underlying both \mathbb{Z} and $F[x]$:

- a and b are associates if $a = u \cdot b$ for some unit u , denoted $a \sim b$.
- In an integral domain, a non-unit $p \in D \setminus \{0\}$ is irreducible if, whenever $p = ab$, then either a or b is a unit.
- $a \sim b \iff a|b \wedge b|a$
- A Euclidean domain is an integral domain D with a degree function $d: D \setminus \{0\} \rightarrow \mathbb{N}$ such that every a and $b \neq 0$ in D there exists q and r such that $a = bq + r$ and $d(r) < d(b)$ or $r = 0$.
- In a euclidean domain every element can be factored uniquely (up to taking associates) into irreducible elements.

Polynomials as functions

For a ring R , a polynomial $a(x) \in R[x]$ can be interpreted as a function $R \rightarrow R$ by evaluating $a(x)$ at $x \in R$. This gives us a function $R \rightarrow R: x \mapsto a(x)$.

A $\alpha \in R$ so that $a(\alpha) = 0$ is called a **root** of $a(x)$. For a field F , $\alpha \in F$ is a root of $a(x)$ iff $x - \alpha$ divides $a(x)$.

If a polynomial of degree ≥ 2 has no roots, it is irreducible.

If α is a root, then its **multiplicity** is the highest power of $x - \alpha$ that divides $a(x)$.

Example: GF(2): $x^5 + x$ has root 1 with multiplicity 4, since $(x+1)^4$ divides $x^5 + x$.

$$x-1 = x+1 \rightarrow x^5 + x : (x+1)^4 = 0$$

Polynom interpolation

A polynomial $a(x) \in F[x]$ of degree $\leq d$ can be uniquely determined by any $d+1$ values of $a(x)$, i.e. $a(\alpha_1), \dots, a(\alpha_{d+1})$ for any distinct $\alpha_1, \dots, \alpha_{d+1}$.

↳ same as in IR!

Example: let $\beta_i = a(\alpha_i)$

$$u_i = \frac{(x - \alpha_1) \cdots (x - \alpha_{i-1}) (x - \alpha_{i+1}) \cdots (x - \alpha_d)}{(x_i - \alpha_1) \cdots (x_i - \alpha_{i-1}) (x_i - \alpha_{i+1}) \cdots (x_i - \alpha_{d+1})}$$

$$a(x) = \sum_{i=1}^{d+1} \beta_i u_i(x)$$

	$a(x)$	$u_1(x)$	$u_2(x)$	\dots	$u_{d+1}(x)$
α_1	β_1	1	0	\dots	0
α_2	β_2	0	1	\dots	0
α_3	β_3	0	0	\dots	0
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
α_{d+1}	β_{d+1}	0	0	\dots	1

Finite Fields

We have already seen the finite field $GF(p)$, but what other finite fields are there.

$F[x]_{m(x)}$

The same way as in \mathbb{Z} we can compute $F[x]$ modulo a polynomial $m(x)$.

$$a(x), b(x) \in F[x] \quad a(x) \equiv_m b(x) \stackrel{\text{def}}{\iff} m(x) \mid (a(x) - b(x))$$

(Congruence modulo $m(x)$ is a equivalence relation on $F[x]$, and each equivalence class has a unique representation of degree $< \deg(m(x))$.

Let $m(x)$ be a polynomial of degree d over F . Then

$$F[x]_{m(x)} \stackrel{\text{def}}{\iff} \{a(x) \in F[x] \mid \deg(a(x)) < d\}$$

We can state a simple fact about the cardinality of $F[x]_{m(x)}$, when F is finite with q elements and $\deg(m(x)) = d \Rightarrow |F[x]_{m(x)}| = q^d$

$F[x]_{m(x)}$ is a ring with respect to addition and multiplication modulo $m(x)$.

$a(x) b(x) \equiv_{m(x)} 1$ only has a so $b(x)$ if $\gcd(a(x), m(x)) = 1$.
The solution is unique or in other words,

$$F[x]_{m(x)}^* = \{a(x) \in F[x]_{m(x)} \mid \gcd(a(x), m(x)) = 1\}$$

Constructing Extension Fields

The ring $F[x]_{m(x)}$ is a field if and only if $m(x)$ is irreducible (prime).

⇒ The ring $R[x]_{x^{n+1}}$ is isomorphic to C .

Error-Correcting Codes

Finite fields are important in CS, one of the applications is ECC.

To encode a message we use a injective function that maps a list $(a_0, \dots, a_k) \in A^k$ of $k+1$ symbols to a list $(c_0, \dots, c_n) \in A^n$ of $n > k+1$ encoded symbols, called a codeword.

A = alphabet

$$E: A^k \rightarrow A^n : (a_0, \dots, a_k) \mapsto E(a_0, \dots, a_k) = (c_0, \dots, c_n)$$

For such encoding functions E , one often considers the set

$$C = \text{Im}(E) = \{E(a_0, \dots, a_k) \mid a_0, \dots, a_k \in A\}$$

of codewords, which is called an **error-correction code**.

An (n, k) -ECC over an alphabet A with $|A| = q$ is a subset of A^n with cardinality q^k .

In CS one often uses $A = \{0, 1\}$ or $A = \{0, 1\}^3$.

The **Hamming distance** of two strings of equal length, is the number of positions in which they are different.

The **minimum distance** of an ECC is the minimum distance between any two codewords.

Decoding

A decoding function D for an (n, k) -ECC is a function
 $D: A^n \rightarrow A^k$.

A good decoding function takes an input $(r_0, \dots, r_n) \in A^n$ and decodes it to the most plausible information (a_0, \dots, a_k) .

A decoding function can be characterized in terms of **errors + that can be corrected**.

A decoding function D is t -error correcting for encoding function E if for any (a_0, \dots, a_k)

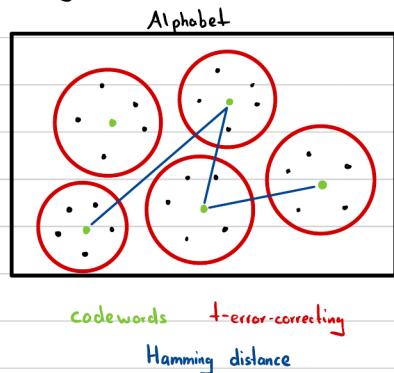
$$D((r_0, \dots, r_n)) = (a_0, \dots, a_k)$$

for any (r_0, \dots, r_n) with Hamming distance at most t from $E((a_0, \dots, a_k))$.

A code C is t -error correcting if there exists E and D such that $C = \text{Im}(E)$ where D is t -error correcting

A code C with minimum distance d is t -error correcting if and only if:

$$d \geq 2t + 1$$



Codes based on Polynomial Evaluation

A very powerful class of codes can be obtained by polynomial interpolation if A has a field structure, i.e. $A = \text{GF}(q)$.

Consider the encoding function:

$$E((a_0, \dots, a_k)) = (a(x_0), \dots, a(x_n))$$

where $a(x)$ is the polynomial $a_k x^k + \dots + a_1 x + a_0$. This code has a minimum distance of $n - k + 1$.

Ch.6 Logic

In this part we discuss the foundations of logic in a mathematically rigorous manner. On a fundamental level, the goal of logic is to express mathematical statements and express and verify proofs of such statements.

There are multiple different logics with different notations, but the concepts we discuss are general.

Proof Systems

For a formal treatment we need a well-defined syntax. Typically we express things as strings over some alphabet Σ .

In this section we study two types of mathematical objects: statements and proofs for these type of statements. By a type of statement we mean for example the class of statements of the form n is prime.

Let $S \subseteq \Sigma^*$ be the set of mathematical statements of a type and $P \subseteq \Sigma^*$ be the set of proof strings. Every statement $s \in S$ is either true or false, this can be expressed by a truth function:

$$T: S \rightarrow \{0, 1\}$$

This function defines the meaning (semantic) of objects in S .

A $p \in P$ is either a proof for $s \in S$ or not. This is expressed by a verification function:

$$\Phi: S \times P \rightarrow \{0, 1\}$$

Without a strong loss of generality we can say that $S = P = \{0, 1\}^*$.

A proof system is quadruple $\Pi = (S, P, \tau, \Phi)$.

A proof system $\Pi = (S, P, \tau, \Phi)$ is sound if no false statements have a proof. For all $s \in S$ for which there exists a $p \in P$ with $\Phi(s, p) = 1$, we have $\tau(s) = 1$.

It is complete if every true statement has a proof. For all $s \in S$ with $\tau(s) = 1$, there exists a $p \in P$ with $\Phi(s, p) = 1$.

In addition Φ should be efficient to compute.

Ex. see p.130

Important:

- while Φ must be efficient, proof generation generally is not.
- a proof system is restricted to one type of statement.
- Φ can proceed in many ways
- Statements and their negation are asymmetric, meaning they are of different difficulty to prove, if even possible.

Elementary General Concepts of Logic

The goal is to provide a proof system that can be applied to a very large class of mathematical statements.

However a proof system can never be applied to all possible mathematical statements.

A proof consists of a sequence of syntactic steps, called a derivation or a deduction. Each step consists of applying one of the allowed syntactic rules. The set of allowed syntactic rules is called a **calculus**.

Syntax, Semantics, Interpretation, Model

The **syntax** is the alphabet Λ of allowed symbols and specifies which strings in Λ^* are formulas (correct).

Ex. $A \vee B \checkmark$ $\underline{A} \vee \textcolor{red}{X}$ in each logic some symbols are understood as **variables** that can take a value in a domain.

The **semantics** define under which condition a formula is true or false.

A variable or predicate is called **free** if it needs to be assigned a fixed value before the formula has a truth value. This assignment is called a **interpretation**.

A semantic defines a function **free** so that $i \in \text{free}(F) \Rightarrow f_i$ is free.

A interpretation consists of a set $Z \subseteq \Lambda$ of symbols of Λ , a domain for each symbol in Z and a function that assigns to each symbol in Z a value in its associated domain.

Often such a domain is defined in terms of a so-called universe U .

A interpretation is called suited for a formula F if it assigns a value to all free symbols in F .

A semantic also defines a function $\sigma(F, A)$ assigning to each formula F and each interpretation A suitable for F , a truth value in $\{0, 1\}$.

One often simply writes $A(F)$.

A suitable interpretation A for which $A(F) = 1$ is called a model for F , $A \models F$, if M is a set of formulas we write $A \models M$. If A is not a model we write $A \not\models F$

Satisfiability, Tautology, Consequence, Equivalence

A formula F is called satisfiable if there exists a model for F and unsatisfiable otherwise. Unsatisfiable is denoted \perp .

A formula F is a tautology T if it is true for every suitable interpretation.

A formula G is a logical consequence of F , $F \models G$, if every interpretation suitable for F and G , which is a model for F is also a model for G .

If F is a tautology one only writes $\models F$.

Two formulas F and G are equivalent, $F \equiv G$, if $F \models G$ and $G \models F$.

The Logical Operators \vee, \wedge and \neg

If F and G are formulas, then $\neg F$, $(F \vee G)$ and $(F \wedge G)$ are formulas.

disjunction

conjunction

\leftrightarrow and \rightarrow are simply notational conventions. The semantics of these logical operators is defined as follows:

$$A(\neg F) = 1 \quad \text{iff} \quad A(F) = 0$$

$$A(F \wedge G) = 1 \quad \text{iff} \quad A(F) = 1 \quad \text{and} \quad A(G) = 1$$

$$A(F \vee G) = 1 \quad \text{iff} \quad A(F) = 1 \quad \text{or} \quad A(G) = 1$$

Some basic equivalences can now be stated:

Lemma 6.1. For any formulas F , G , and H we have

- 1) $F \wedge F \equiv F$ and $F \vee F \equiv F$ (idempotence);
- 2) $F \wedge G \equiv G \wedge F$ and $F \vee G \equiv G \vee F$ (commutativity);
- 3) $(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$ and $(F \vee G) \vee H \equiv F \vee (G \vee H)$ (associativity);
- 4) $F \wedge (F \vee G) \equiv F$ and $F \vee (F \wedge G) \equiv F$ (absorption);
- 5) $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$ (distributive law);
- 6) $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$ (distributive law);
- 7) $\neg\neg F \equiv F$ (double negation);
- 8) $\neg(F \wedge G) \equiv \neg F \vee \neg G$ and $\neg(F \vee G) \equiv \neg F \wedge \neg G$ (de Morgan's rules);
- 9) $F \vee \top \equiv \top$ and $F \wedge \top \equiv F$ (tautology rules);
- 10) $F \vee \perp \equiv F$ and $F \wedge \perp \equiv \perp$ (unsatisfiability rules).
- 11) $F \vee \neg F \equiv \top$ and $F \wedge \neg F \equiv \perp$.

Logical Consequences vs. Unsatisfiability

F is only a tautology if $\neg F$ is unsatisfiable.

The following statements are equivalent:

- $\{F_1, \dots, F_k\} \models G$
- $\{F_1, \dots, F_k\} \rightarrow G$ is a tautology
- $\{F_1, \dots, F_k, \neg G\}$ is unsatisfiable

There are four types of statements one may want to prove:

- Theorems
- Statements about F or M
- The statement $A \models F$ for given A, F
- Statements about the logic, for example that a calculus is sound.

Logical Calculi

The goal of logic is to provide a framework for expressing and verifying the proof. A proof should be a purely syntactic derivation consisting of simple, easy verifiable step. In each step, a new syntactical object is derived and at the end the desired theorem appears.

A well-defined set of rules to manipulate formulas is called a calculus (pl. calculi).

Hilbert-Style Calculi

In Hilbert-style calculus the syntactical objects that are manipulated are formulas.

A derivation rule is a rule for deriving a formula from a set of set of formulas (precondition). We write

$$\{F_1, F_2, \dots, F_n\} \vdash_R G \quad \text{or} \quad \frac{F_1 \ F_2 \ \dots \ F_n}{G} \quad (R)$$

if G can be derived from $\{F_1, F_2, \dots, F_n\}$ by R .

The application of a derivation rule R to set M of formulas means:

1. Select $N \subseteq M$
2. For the placeholder in R , select formulas $\in N$ such that $N \vdash_R G$
3. Add G to M

A **calculus** K is a finite set of derivation rules
 $K = \{R_1, R_2, \dots, R_n\}$.

A **derivation** of a formula G from a set M in a calculus K is a finite sequence of applications of rules in K .

- $M_0 := M$
- $M_i := M_{i-1} \cup \{G_i\}$ for $1 \leq i \leq n$ where $N \vdash_{R_j} G_i$ for some $N \subseteq M_{i-1}$ and some $R_j \in K$.
- $G_n = G$

We write $M \vdash_K G$.

A derivation rule R is **correct** if for every set M and every formula F ,

$$M \vdash_R F \Rightarrow M \models F$$

if this is the case for every rule in a calculus,
then the calculus is **sound**.

If for every M and F , $M \models F \Rightarrow M \vdash_K F$, then the calculus K is **complete**.

Derivations from Assumptions

A natural way to prove $F \rightarrow G$ is to assume F and to derive G .

If $F \vdash_K G$ holds for a sound calculus, then $\models (F \rightarrow G)$. More generally if $\{F_1, \dots, F_n\} \vdash G$, then $\models (\{F_1, \dots, F_n\} \rightarrow G)$.

For a given calculus, one can prove new derivation rules.

In general if $\{F_1, \dots, F_n\} \vdash_K G$, then one could extend K by the rule $\{F_1, \dots, F_n\} \vdash G$.

Propositional Logic