Linear Algebra I

Exercise Class

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1. Logic

Propositional logic (*Aussagenlogik*, in german), also known as *zeroth-order logic*, deals with propositions and relations between them while *first-order logic* (*Prädikatenlogik*, in german) introduces quantifiers and variables which allows to state involved propositions.

1.1 Basics

⚠ **Warning 1.** The *universal quantifier* $\forall$ and the *existential quantifier* $\exists$ should never be exchanged. To see why this is the case, observe that

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : n < m$$

and

$$\exists m \in \mathbb{N}\ \forall n \in \mathbb{N} : n < m$$

are two very different statements.

Claim 1. For arbitrary propositions A, B we have

$$(A \implies B) \iff (\neg A \vee B).$$

Proof. Consider the truth table:

A	B	$\neg A$	$\neg A \vee B$	$A \implies B$
t	t	f	t	t
t	f	f	f	f
f	t	t	t	t
f	f	t	t	t

Since the last two row match in every entry, this proves the claim. □

⚠ **Warning 2.** Let X be a set and $A(x)$ a proposition depending on $x \in X$. Then the statement

$$\forall x \in \emptyset : A(x)$$

is always true and the statement

$$\exists x \in \emptyset : A(x)$$

is always false.

1.2 Exercises

Exercise 1. Let A, B be arbitrary propositions. Which of the following are not equivalent to $A \vee B$?

(a) $\neg A \implies B$

(b) $\neg B \implies A$

(c) $\neg(\neg A \wedge \neg B)$

(d) $\neg(\neg A \vee \neg B)$

Solution. Note that by **Claim 1** we have

$$(\neg A \implies B) \iff \neg\neg A \vee B \overset{(1)}{\iff} A \vee B,$$

where at (1) we used **Theorem 1.2.17** (i) from the lecture. Similarly, (b) is equivalent to $A \vee B$. Now again by **Theorem 1.2.17** we have

$$\begin{aligned} \neg(\neg A \wedge \neg B) &\iff \neg\neg A \vee \neg\neg B \iff A \vee B, \\ \neg(\neg A \vee \neg B) &\iff \neg\neg A \wedge \neg\neg B \iff A \wedge B, \end{aligned}$$

so only (d) is not equivalent to $A \vee B$.

Exercise 2. Everybody loves Alex but Alex only loves me. Who is Alex?

(a) My mother

(b) Nobody

(c) My son

(d) I am Alex

Solution. Since everybody loves Alex, he/she also loves himself/herself. But since Alex only loves me, I have to be Alex.

Exercise 3. Let $X \subseteq \mathbb{R}$ be arbitrary. Under which assumptions on X can the statement

$$\forall \varepsilon > 0 \; \exists q \in \mathbb{Q} \; \forall x \in X : \quad |x - q| < \varepsilon$$

be proven?

(a) X is finite

(b) X is empty

(c) $X \subseteq \mathbb{Q}$

(d) $|X| = 1$

Solution. If we choose $X = \{1, 10\}$ then the statement is not true. (Why? Try to prove it.) Hence the assumptions from (a) or from (c) don't suffice to prove the statement. On the other hand, if X is empty then the statement is true since $\mathbb{Q} \neq \emptyset$ by **Warning 2**. If $|X| = 1$ then the statement can also be proven since $\mathbb{Q}$ is dense in $\mathbb{R}$ ($\longrightarrow$ see Analysis I). Hence (b) and (c) suffice.

2. Fields

2.1 Exercises

Exercise 4. Let k be a finite Field and let S be the sum of all its elements. Show that

$$S = 0 \iff |k| > 2$$

holds.

Proof. "$\Longrightarrow$". To show this direction, we want to use contraposition, so assume $|k| \leq 2$ holds. Since k is a field, we have $k = \{0, 1\}$ with $1 \neq 0$ and thus $S = 0 + 1 = 1 \neq 0$.

"$\Longleftarrow$". Now assume that $|k| > 2$ holds. Using this fact, we can choose an element

$$a \in k \setminus \{0, 1\}.$$

Since k is a field and $a \neq 0$, we know that a is invertible and thus the map

$$\varphi : k \to k, x \mapsto ax$$

is a bijection. Hence we have

$$S = \sum_{x \in k} x = \sum_{x \in k} \varphi(x) = \sum_{x \in k} ax = a \sum_{x \in k} x = aS$$

using distributivity. Hence we get the equation

$$S(1 - a) = 0$$

and now since $a \neq 1$ we can divide by the invertible element $1 - a \neq 0$ to get

$$S = 0$$

which concludes the proof. □

Exercise 5. Consider the field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.

(a) Compute the value of $\frac{3}{4} + \frac{1}{3}$ in $\mathbb{F}_5$.

(b) Solve $x^3 + x^2 = 6x$ for x in $\mathbb{F}_5$.

Solution.

(a) First observe that in $\mathbb{F}_5$ we have

$$4 \cdot 4 = 16 = 1 \implies \frac{1}{4} = 4$$

and similarly we find $\frac{1}{3} = 2$. Hence we get

$$\frac{3}{4} + \frac{1}{3} = 3 \cdot 4 + 2 = 14 = 4.$$

(b) By computing in $\mathbb{F}_5$ we simplify

$$\begin{aligned} x^3 + x^2 - 6x &= x(x^2 + x - 6) \\ &= x(x-2)(x+3) \\ &= x(x-2)^2. \end{aligned}$$

Hence the solutions in $\mathbb{F}_5$ are given by $x_1 := 0$ and $x_2 := 2$.