# Linear Algebra I
# Exercise Class

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## 1 The Dual Map

**Definition 1.1.** Let $V, W$ be $K$-vector spaces and $T : V \to W$ a linear map. Then we define the *dual map* of $T$ by

$$T^* : W^* \to V^*,\ f \mapsto f \circ T.$$

**Exercise 1.2.** Consider the linear map

$$D : \mathbb{R}[X] \to \mathbb{R}[X],\ p \mapsto p'$$

given by taking the derivative of a polynomial.

- Find an explicit formula for the dual map $D^*$.
- If we define $g : \mathbb{R}[X] \to \mathbb{R},\ p \mapsto p(3)$, what is $D^*(g)$?
- If we define $h : \mathbb{R}[X] \to \mathbb{R},\ p \mapsto \int_0^1 p(x)\, dx$, what is $D^*(h)$?

*Solution.*

- By definition, for any $f \in (\mathbb{R}[X])^*$ we have $D^*(f) = f \circ D$. Hence for any $p \in \mathbb{R}[X]$ we get the formula

$$(D^*(f))(p) = (f \circ D)(p) = f(p').$$

- Applying the formula we found, we get

$$(D^*(g))(p) = g(p') = p'(3)$$

and

$$(D^*(h))(p) = h(p') = \int_0^1 p'\,dx = p(1) - p(0)$$

for any $p \in \mathbb{R}[X]$.

# 2 Annihilator

Recall the following definition.

**Definition 2.1.** Let $V$ be a vector space. For any subspace $U \leq V$ we define the *annihilator* of $U$ by

$$\begin{aligned} U^\perp &:= \{f \in V^* \mid \forall u \in U : f(u) = 0\} \\ &= \{f \in V^* \mid f|_U = 0\} \leq V^*. \end{aligned}$$

*Recall* 2.2. Let $V, W$ be two $K$-vector spaces. Recall that for every basis $B$ of $V$ and every map $\tilde{f} : B \to W$ there exists a unique linear map $f : V \to W$ extending $\tilde{f}$, namely the map given by

$$f\Big(\sum_{k=1}^{n} \alpha_n b_n\Big) = \sum_{k=1}^{n} \alpha_k \tilde{f}(b_k)$$

for any $a_k \in K$, $b_k \in B$.

**Exercise 2.3.** Let $U, W \leq V$ be two subspaces of a finite dimensional $K$-vector space $V$.

(a) Show that $U \subseteq W$ if and only if $W^\perp \subseteq U^\perp$.

(b) Show that we have $(U \cap W)^\perp = U^\perp + W^\perp$.

*Solution.*

(a) *Proof.* "$\Longrightarrow$". Follows immediately by considering the definition.

"$\Longleftarrow$". Assume that $U \not\subseteq W$, so there exists a $u \in U \setminus W$. Now choose a basis $\{u, b_2, \dots, b_n\}$ of $V$ and define a linear map $f : V \to K$ by $f(u) = 1$ and $f(b_i) = 0$ for $i \in \{2, \dots, n\}$. But then we have $f \in W^\perp$ since $u \notin W$ and $f \notin U^\perp$ since $f(u) = 1 \neq 0$ in $K$. This proves $W^\perp \not\subseteq U^\perp$ and we can conclude by contraposition. □

(b) We are going to provide two different approaches to this problem. First, we present a proof which involves some explicit constructions.

*Proof 1.* "$\supseteq$". Let $f \in U^\perp + W^\perp$ be arbitrary, which means that it is of the form

$$f = f_1 + f_2$$

for $f_1 \in U^\perp$ and $f_2 \in W^\perp$. Now since $U \cap W$ is a subspace of $U$ and $W$, we get that for any $v \in U \cap W$ we have

$$f(v) = f_1(v) + f_2(v) = 0$$

by definition of $U^\perp$ and $W^\perp$, which shows $f \in (U \cap W)^\perp$.

"$\subseteq$". Let $f \in (U \cap W)^\perp$ be arbitrary. Choose a basis $\{b_1, \dots, b_n\}$ of $U \cap W$ and extend it such that $\{b_1, \dots, b_n, u_1, \dots, u_m\}$ is a basis of $U$ and $\{b_1, \dots, b_n, w_1, \dots, w_l\}$ is a basis of $W$. Note that then then we necessarily have

$$\{u_1, \dots, u_m\} \cap \{w_1, \dots, w_l\} = \emptyset.$$

Now chose vectors $v_1, \dots, v_r$ such that $\{b_1, \dots, b_n, u_1, \dots, u_m, w_1, \dots, w_l, v_1, \dots, v_r\}$ is a basis of $V$. Hence we can define two linear functions $f_1, f_2 : V \to K$ by

$$\begin{aligned} f_1(b_i) = f_1(u_j) = 0, \quad f_1(w_k) = f(w_k), \quad f_1(v_q) = f(v_q), \\ f_2(b_i) = f_2(w_k) = 0, \quad f_2(u_j) = f(u_j), \quad f_2(v_q) = 0 \end{aligned}$$

for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, $k \in \{1, \dots, l\}$ and $q \in \{1, \dots, r\}$. Then by definition we have $f_1 \in U^\perp$ and $f_2 \in W^\perp$. Furthermore, the assumption

$f \in (U \cap W)^\perp$ implies that $f(b_i) = 0$ holds for all $i \in \{1, \ldots, n\}$ and thus we have $f = f_1 + f_2$ by the uniqueness mentioned in Recall 2.2. This proves $(U \cap W)^\perp \subseteq U^\perp + W^\perp$. $\square$

Here is a simpler, non-constructive proof.

*Proof 2.* In a first step, show as in *proof 1* that $U^\perp + W^\perp \subseteq (U \cap W)^\perp$ holds and observe that we have the identity

$$(U + W)^\perp = U^\perp \cap W^\perp, \tag{1}$$

which is straight forward to verify. Now compute

$$\begin{aligned}
\dim((U \cap W)^\perp) &= \dim(V) - \dim(U \cap W) && (2)\\
&= \dim(V) + \dim(U + W) - \dim(U) - \dim(W) && (3)\\
&= \dim(V) + \dim(V) - \dim(V) \\
&\quad + \dim(U + W) - \dim(U) - \dim(W) \\
&= \dim(U^\perp) + \dim(W^\perp) - \dim((U + W)^\perp) && (4)\\
&= \dim(U^\perp) + \dim(W^\perp) - \dim(U^\perp \cap W^\perp) && (5)\\
&= \dim(U^\perp + W^\perp) && (6)
\end{aligned}$$

where at (2) and (4) we used theorem 6.4.5, at (3) and (6) we used theorem 3.4.4 and at (5) we used (1). Furthermore, since $V$ is finite dimensional, $V^*$ is also finite dimensional and thus since we have $U^\perp + W^\perp \subseteq (U \cap W)^\perp$, the equation above implies $U^\perp + W^\perp = (U \cap W)^\perp$. $\square$

# 3 Finding an Orthonormal Basis

**Definition 3.1.** For $n \geq 1$, we define the *canonical inner product* on $\mathbb{R}^n$ by

$$\begin{aligned}
\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}, \\
((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &\mapsto x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.
\end{aligned}$$

In this case, $\mathbb{R}^n$ (and any of its subspaces) is also called an *euclidean vector space*. Furthermore, a set $\mathcal{B} \subseteq \mathbb{R}^n$ is called an *orthonormal basis* of $\mathbb{R}^n$ if it is a basis and fulfills $\langle u, v\rangle = 0$ and $\langle u, u\rangle = 1$ for all $u, v \in \mathcal{B}$ with $u \neq v$.

*Remark* 3.2. The notion of an euclidean vector space can be generalized.

Later in this course, we are going to prove the following crucial theorem.

**Theorem 3.3.** *Every finite-dimensional euclidean vector space has an orthonormal basis.*

Based on this theorem, we can provide the following example.

**Example 3.4.** We want to find an orthonormal basis of the subspace

$$V := \langle v_1, v_2, v_3\rangle := \left\langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\rangle \leq \mathbb{R}^4.$$

We are going to do this in three steps.

(1) First set $w_1 := v_1$ and normalize it, meaning that we define

$$b_1 := \frac{1}{\sqrt{\langle w_1, w_1\rangle}} w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}.$$

Now $\langle b_1, b_1\rangle = 1$ is fulfilled as desired.

(2) Set

$$w_2 := v_2 - \langle v_2, b_1\rangle b_1 = \frac{1}{2} \begin{pmatrix}1\\-1\\2\\0\end{pmatrix}$$

and observe that with this choice we indeed have

$$\langle b_1, w_2\rangle = \langle b_1, v_2 - \langle v_2, b_1\rangle b_1\rangle = \underbrace{\langle b_1, v_2\rangle}_{=\langle v_2, b_1\rangle} - \langle v_2, b_1\rangle \cdot \underbrace{\langle b_1, b_1\rangle}_{=1} = 0.$$

Now again normalize $w_2$, so

$$b_2 := \frac{1}{\sqrt{\langle w_2, w_2\rangle}} w_2 = \frac{\sqrt{6}}{6}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

to achieve $\langle b_2, b_2\rangle = 1$ and note that we still have $\langle b_1, b_2\rangle = 0$.

(3) Similarly to step (2), set

$$w_3 := v_3 - \langle v_3, b_1\rangle b_1 - \langle v_3, b_2\rangle b_2 = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}$$

and

$$b_3 := \frac{1}{\sqrt{\langle w_3, w_3\rangle}} w_3 = \frac{\sqrt{3}}{6}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

Now $\{b_1, b_2, b_3\}$ indeed defines an orthonormal basis of $V$.

**Exercise 3.5.** Try to apply this algorithm to other subspaces and, while doing so, think about why it works.