Linear Algebra I
Exercise Class

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1 Linear Maps

Exercise 1.1. Let V be a finite dimensional vector space and $T : V \to V$ a linear map. Suppose that there exists a linear map $U : V \to V$ such that $T \circ U = \mathrm{id}_V$ holds.

(a) Prove that T is an isomorphism.

(b) Show that the conclusion from (a) does not hold if V is infinite dimensional.

Solution.

(a) *Proof.* We first want to show that T is surjective. Let $v \in V$ be arbitrary. Then

$$v = \mathrm{id}_V(v) = (T \circ U)(v) = T(U(v))$$

which proves $v \in \mathrm{im}(T)$. Hence T is surjective which implies

$$\dim_K(V) = \dim_K(\mathrm{im}(T)).$$

Now by theorem 4.2.9. from the lecture we also have

$$\begin{aligned}\dim_K(V) &= \dim_K(\ker(T)) + \dim_K(\mathrm{im}(T)) \\ &= \dim_K(\ker(T)) + \dim_K(V)\end{aligned}$$

and since $\dim_K(V) < \infty$ by assumption this implies $\dim_K(\ker(T)) = 0$. Hence $\ker(T) = \{0_V\}$ which by theorem 4.2.6. from the lecture implies that T is also injective. Thus T is bijective, so by lemma 4.2.19. T is an isomorphism. □

(b) We suggest two possible counterexamples.

- Let $V = \mathbb{R}^{\mathbb{N}}$ be the set of all real-valued sequences and define a *shift operator* by

$$T : V \to V,\ (a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, a_4, \dots)$$

which is indeed a linear map. Then the map

$$U : V \to V,\ (a_1, a_2, a_3, \dots) \mapsto (0, a_1, a_2, a_3, \dots)$$

is also linear and fulfills $T \circ U = \mathrm{id}_V$ but T is not bijective and thus cannot be an isomorphism.

- Let $V := \mathbb{R}[X]$ be the vector space of all polynomials with coefficients in $\mathbb{R}$ and define

$$D : V \to V,\ f \mapsto f'$$

to be the *derivative map*. Now for any $f \in V$ define a function I_f by

$$I_f(x) := \int_0^x f(t)\,dt$$

and set

$$U : V \to V,\ f \mapsto I_f$$

which is also linear. Now observe that by the *Fundamental Theorem of Calculus* we have $D \circ U = \mathrm{id}_V$ but D is not injective since all constant polynomials are mapped to 0_V under D.

Exercise 1.2. Let V be a vector space. A linear map $P : V \to V$ is called *idempotent* or a *projection* if $P^2 := P \circ P = P$ holds.

(a) Show that for every projection P we have

$$V = \ker(P) \oplus \mathrm{im}(P).$$

(b) Show that for any subspaces $U, W \leq V$ with $V = U \oplus W$ there exists a projection $P : V \to V$ with $\ker(P) = U$ and $\mathrm{im}(P) = W$.

Solution.

(a) *Proof.* Let P be any projection and set $U := \ker(P)$, $W := \mathrm{im}(P)$.

- Let $v \in V$ be arbitrary. Then

  $$P(v - P(v)) = P(v) - P^2(v) = P(v) - P(v) = 0$$

  and thus $v - P(v) \in \ker(P) = U$. Hence

  $$v = (v - P(v)) + P(v) \in U + W$$

  which proves $V = U + W$.

- Let $v \in U \cap W$ be arbitrary. Then there exists a $w \in W = \mathrm{im}(P)$ with $P(w) = v$. Hence we get

  $$v = P(w) = P^2(w) = P(v) \overset{(1)}{=} 0$$

  where at (1) we used $v \in U = \ker(P)$. This proves $U \cap W = \{0\}$.

Overall, this proves $V = U \oplus W = \ker(P) \oplus \mathrm{im}(P)$. □

(b) *Proof.* Let $U, W \leq V$ be two subspaces with $V = U \oplus W$. In the exercise class on 16. October 2023 we have shown that, in this case, for any vector $v \in V$ there exist *unique* vectors $u \in U$ and $w \in W$ with $v = u + w$. Hence the map

$$P : V \to V,\ v = u + w \mapsto w$$

is well-defined (*why can't we do this general?*), is a projection and fulfills the required conditions. □

2 Transformation Matrix

2.1 Recap and Explanation

Definition 2.1 (Transformation matrix)**.** Let $T : V \to W$ be a linear map between finite dimensional K-vector spaces V and W, let $\mathcal{B} = (v_1, \dots, v_n)$ be an ordered basis of V and let $\mathcal{C} = (w_1, \dots, w_m)$ an ordered basis of W. Then the *transformation matrix* of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is given by the unique matrix $[T]^{\mathcal{B}}_{\mathcal{C}} = (a_{ij})_{i,j} \in M_{m\times n}(K)$ with

$$\forall 1 \leq j \leq n : \quad T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \tag{1}$$

Remark 2.2. For any set of vectors $\mathcal{B} = (v_1, \dots, v_n)$ of V the map

$$\varphi_{\mathcal{B}} : K^n \to V, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum_{i=1}^{n} x_i v_i$$

is linear and it is an isomorphism if and only if $\mathcal{B}$ is a basis of V.

Proposition 2.3. *Let $T : V \to W$ and $\mathcal{B}, \mathcal{C}$ be as in Definition 2.1. Then $A := [T]^{\mathcal{B}}_{\mathcal{C}}$ is the unique matrix for which $T \circ \varphi_{\mathcal{B}} = \varphi_{\mathcal{C}} \circ L_A$ holds, meaning that the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\ T\ } & W \\ \varphi_{\mathcal{B}} \uparrow \wr & & \varphi_{\mathcal{C}} \uparrow \wr \\ K^n & \xrightarrow{\ L_A\ } & K^m \end{array}$$

Proof idea. The proof is based on the following computation. If we assume that the diagram commutes, then for every $1 \leq i \leq n$ we have

$$T(v_j) = T(\varphi_{\mathcal{B}}(e_j)) = \varphi_{\mathcal{C}}(L_A(e_j)) = \varphi_{\mathcal{C}} \left(\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} \right) = \sum_{i=1}^{m} a_{ij} w_i$$

which matches with formula (1). Furthermore, L_A is uniquely defined by the formula

$$L_A = \varphi_{\mathcal{C}}^{-1} \circ T \circ \varphi_{\mathcal{B}}$$

which uses that fact that in this case $\varphi_{\mathcal{C}}$ is an isomorphism. ∷

2.2 Exercises

Exercise 2.4. Let $V := \mathbb{R}[X]^{\leq 2} = \{a + bX + cX^2 \mid a, b, c \in \mathbb{R}\}$ be the vector space of polynomials with degree ≤ 2 with basis $\mathcal{B} := (1, X, X^2)$ and $W := \mathbb{R}^3$ with basis $\mathcal{C} := (e_1, e_2, e_3)$. Define a linear map by

$$T : V \to W,\ f \mapsto \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$$

(a) Compute $[T]^{\mathcal{B}}_{\mathcal{C}}$.

(b) Now consider the basis $\mathcal{B}' := \left(\frac{(X-1)(X-2)}{2}, X(2-X), \frac{X(X-1)}{2}\right)$ of V. Compute $[T]^{\mathcal{B}'}_{\mathcal{C}}$.

Solution.

(a) The matrix $A := [T]^{\mathcal{B}}_{\mathcal{C}}$ is the unique matrix that makes the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi_{\mathcal{B}} \uparrow \wr & & \varphi_{\mathcal{C}} \uparrow \wr \\ \mathbb{R}^3 & \xrightarrow{L_A} & \mathbb{R}^3 \end{array}$$

To determine L_A, we start by choosing an arbitrary vector $(a, b, c) \in \mathbb{R}^3$ and compute

$$(a, b, c) \xmapsto{\varphi_{\mathcal{B}}} a + bX + cX^2 \xmapsto{T} \begin{pmatrix} a \\ a + b + c \\ a + 2b + 4c \end{pmatrix}.$$

Observe that, in this case, we have $\varphi_{\mathcal{C}} = \mathrm{id}_{\mathbb{R}^3}$. Hence A fulfills the equation

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \varphi_{\mathcal{C}}(L_A(\begin{pmatrix} a \\ b \\ c \end{pmatrix})) = \begin{pmatrix} a \\ a + b + c \\ a + 2b + 4c \end{pmatrix}.$$

Using this equation, we see that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

holds.

(b) Set $B := [T]_{\mathcal{C}}^{\mathcal{B}'}$. Again, we compute

$$(a, b, c) \xmapsto{\varphi_{\mathcal{B}'}} a\tfrac{(X-1)(X-2)}{2} + bX(2 - X) + c\tfrac{X(X-1)}{2} \xmapsto{T} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This yields the equation

$$B \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

so $B = I_3$ holds.

Exercise 2.5. Let $V := \mathbb{R}[X]^{\leq 3}$ and $W := \mathbb{R}[X]^{\leq 2}$ with the bases $\mathcal{B} := (1, X, X^2, X^3)$ and $\mathcal{C} := (1, X, X^2)$ respectively and define the linear map

$$D : V \to W,\ f \mapsto f'.$$

Compute $[D]_{\mathcal{C}}^{\mathcal{B}}$.

Solution. The matrix $A := [D]_{\mathcal{C}}^{\mathcal{B}}$ is the unique matrix that makes the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{D} & W \\ {\scriptstyle\varphi_{\mathcal{B}}}\uparrow\wr & & {\scriptstyle\varphi_{\mathcal{C}}}\uparrow\wr \\ \mathbb{R}^4 & \xrightarrow{L_A} & \mathbb{R}^3 \end{array}$$

Choose an arbitrary vector $(a, b, c, d) \in \mathbb{R}^4$ and compute:

$$\begin{array}{ccc} a + bX + cX^2 + dX^3 & \xmapsto{D} & b + 2cX + 3dX^2 \\ {\scriptstyle\varphi_{\mathcal{B}}}\uparrow & & {\scriptstyle\varphi_{\mathcal{C}}}\uparrow \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} & \xmapsto{\qquad L_A \qquad} & (?) \end{array}$$

Now filling in the question mark (?) yields:

$$\begin{array}{ccc} a + bX + cX^2 + dX^3 & \xmapsto{D} & b + 2cX + 3dX^2 \\ \varphi_{\mathcal{B}} \uparrow & & \varphi_{\mathcal{C}} \uparrow \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} & \xmapsto{L_A} & \begin{pmatrix} b \\ 2c \\ 3d \end{pmatrix} \end{array}$$

Hence L_A is determined by the equation

$$A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 3d \end{pmatrix}$$

which implies that A is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$