Linear Algebra I
Exercise Class

Eric Ceglie

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1. Matrices

Let K be a field.

1.1 Recap

Definition. For matrices $A = (a_{ij})_{i,j} \in M_{m\times n}(K)$ and $B = (b_{jk})_{j,k} \in M_{n\times p}(K)$, we define their product by

$$AB := \Big(\sum_{j=1}^{n} a_{ij} b_{jk} \Big)_{\substack{1\le i\le m \\ 1\le k\le p}} \in M_{m\times p}(K).$$

Example. For

$$A := \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & 2 \end{pmatrix} \in M_{3\times 2}(\mathbb{F}_7),\ B := \begin{pmatrix} 0 & 3 \\ 2 & 6 \end{pmatrix} \in M_{2\times 2}(\mathbb{F}_7),$$

we have

$$AB = \begin{pmatrix} 1 & 6 \\ 6 & 3 \\ 4 & 5 \end{pmatrix} \in M_{3\times 2}(\mathbb{F}_7).$$

1.2 Exercises

Exercise 1. Let $A, B \in M_{n\times n}(K)$ be arbitrary. Does

$$(A+B)^2 = A + 2AB + B^2$$

hold in general?

Solution. No. For $n = 2$ and $K = \mathbb{R}$ consider the matrices

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0. \end{pmatrix}.$$

Then we have

$$(A+B)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$A^2 + 2AB + B^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

Hence the formula does not hold in this case since $1 \neq 2$ in $\mathbb{R}$.

Exercise 2. Prove that for $A := \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ the formula

$$A^n = \begin{pmatrix} 1 & 3(2^n - 1) \\ 0 & 2^n \end{pmatrix} \tag{1}$$

holds for $n \geq 1$.

Proof. We prove this by induction.

- For $n = 1$ one directly verifies that the formula holds.
- Now let $n > 1$ and assume that (1) holds for $n - 1$. Then we have

$$\begin{aligned} A^n = A^{n-1}A &= \begin{pmatrix} 1 & 3(2^{n-1} - 1) \\ 0 & 2^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 + 3(2^n - 2) \\ 0 & 2^n \end{pmatrix} = \begin{pmatrix} 1 & 3(2^n - 1) \\ 0 & 2^n \end{pmatrix} \end{aligned}$$

  which concludes the proof. □

1.3 Row-Reduced Echelon Form

We know the following theorem from lecture:

Theorem. Every matrix A is row-equivalent to a matrix in row-reduced echelon form.

$\longrightarrow$ QUESTION. How do we find an appropriate matrix W such that WA is in row-reduced echelon form?

We are going to introduce a general algorithm with the concrete example

$$A := \begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 2 & 18 \\ -11 & 1 & -8 & -86 \end{pmatrix} \in M_{3\times 4}(\mathbb{R}).$$

Now our goal is to find a *permutation matrix* $P \in M_{3\times 3}(\mathbb{R})$ and an *lower triangular matrix* $L \in M_{3\times 3}(\mathbb{R})$ such that $R := PLA$ is in row-reduced echelon form. To do so, start by writing:

$$
\begin{array}{ccc|cccc}
I_3 & \longrightarrow & L & A & \longrightarrow & \multicolumn{2}{c}{\text{"row-reduced"}} \\
\hline
1 & & & 1 & 0 & 0 & 4 \\
& 1 & & 2 & 0 & 2 & 18 \\
& & 1 & -11 & 1 & -8 & -86
\end{array}
\tag{2}
$$

In a first step, we are going to apply transformations of the form $M(r, \lambda)$ and $S(r, s, \lambda)$ to bring the right side in (2) into row-reduced form. While doing so, apply the very same operations simultaneously to the left side, to receive the matrix L. This looks as follows:

$$
\xrightarrow[]{\substack{S(1,2,-2),\\ S(1,3,11)}}
\begin{array}{ccc|cccc}
1 & & & 1 & 0 & 0 & 4 \\
-2 & 1 & & 0 & 0 & 2 & 10 \\
11 & & 1 & 0 & 1 & -8 & -42
\end{array}
$$

$$
\xrightarrow{S(2,3,4)}
\begin{array}{ccc|cccc}
1 & & & 1 & 0 & 0 & 4 \\
-2 & 1 & & 0 & 0 & 2 & 10 \\
3 & 4 & 1 & 0 & 1 & 0 & -2
\end{array}
$$

$$
\xrightarrow{M(2,\frac{1}{2})}
\begin{array}{ccc|cccc}
1 & & & 1 & 0 & 0 & 4 \\
-1 & \frac{1}{2} & & 0 & 0 & 1 & 5 \\
3 & 4 & 1 & 0 & 1 & 0 & -2
\end{array}
$$

Now set

$$
L := \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 3 & 4 & 1 \end{pmatrix}
$$

and observe that with this choice LA is in row-reduced form. In a last step, do the same procedure with operations of the form $P(r, s)$ as follows:

$$
\begin{array}{ccc|cccc}
I_3 & \longrightarrow & P & LA & \longrightarrow & R & \\
\hline
1 & & & 1 & 0 & 0 & 4 \\
& 1 & & 0 & 0 & 1 & 5 \\
& & 1 & 0 & 1 & 0 & -2
\end{array}
$$

$$
\xrightarrow{P(2,3)}
\begin{array}{ccc|cccc}
1 & & & 1 & 0 & 0 & 4 \\
& & 1 & 0 & 1 & 0 & -2 \\
& 1 & & 0 & 0 & 1 & 5
\end{array}
$$

By setting

$$
P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

we finally get

$$PLA = R = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Exercise 3. Try to apply this algorithm to other matrices and, while doing so, think about *why* it works.

1.4 Inverting a Matrix

Here let $A \in M_{n\times n}(K)$ be an invertible matrix.

$\longrightarrow$ QUESTION. How can we algorithmically find A^{-1}?

As above, we illustrate a general algorithm by applying it to the example

$$A = \begin{pmatrix} 2 & 3 & \frac{1}{2} \\ 2 & 2 & 0 \\ 2 & -3 & -4 \end{pmatrix} \in \mathrm{GL}_3(K).$$

Again, start by writing:

$$\begin{array}{ccc|ccc} I_3 & \longrightarrow & A^{-1} & A & \longrightarrow & I_3 \\ \hline 1 & & & 2 & 3 & \frac{1}{2} \\ & 1 & & 2 & 2 & 0 \\ & & 1 & 2 & -3 & -4 \end{array} \tag{3}$$

Now we are going to apply elementary row-transformations until the right side of (3) becomes I_3 as follows:

$$\xrightarrow{\substack{S(1,2,-1),\\ S(1,3,-1)}} \begin{array}{ccc|ccc} 1 & & & 2 & 3 & \frac{1}{2} \\ -1 & 1 & & 0 & -1 & -\frac{1}{2} \\ -1 & & 1 & 0 & -6 & -\frac{9}{2} \end{array}$$

$$\xrightarrow{\substack{M(1,\frac{1}{2})\\ M(2,-1)}} \begin{array}{ccc|ccc} \frac{1}{2} & & & 1 & \frac{3}{2} & \frac{1}{4} \\ 1 & -1 & & 0 & 1 & \frac{1}{2} \\ -1 & & 1 & 0 & -6 & -\frac{9}{2} \end{array}$$

$$\xrightarrow{\substack{S(2,1,-\frac{3}{2})\\ S(2,3,6)}} \begin{array}{ccc|ccc} -1 & \frac{3}{2} & & 1 & 0 & -\frac{1}{2} \\ 1 & -1 & & 0 & 1 & \frac{1}{2} \\ 5 & -6 & 1 & 0 & 0 & -\frac{3}{2} \end{array}$$

$$\xrightarrow{M(3,-\frac{2}{3})} \begin{array}{ccc|ccc} -1 & \frac{3}{2} & & 1 & 0 & -\frac{1}{2} \\ 1 & -1 & & 0 & 1 & \frac{1}{2} \\ -\frac{10}{3} & 4 & -\frac{2}{3} & 0 & 0 & 1 \end{array}$$

$$\xrightarrow[]{\substack{S(3,1,\frac{1}{2})\\ S(3,2,-\frac{1}{2})}} \left.\begin{array}{ccc|ccc} -\frac{8}{3} & \frac{7}{2} & -\frac{1}{3} & 1 & 0 & 0 \\ \frac{8}{3} & -3 & \frac{1}{3} & 0 & 1 & 0 \\ -\frac{10}{3} & 4 & -\frac{2}{3} & 0 & 0 & 1 \end{array}\right.$$

Hence we conclude that

$$A^{-1} = \begin{pmatrix} -\frac{8}{3} & \frac{7}{2} & -\frac{1}{3} \\ \frac{8}{3} & -3 & \frac{1}{3} \\ -\frac{10}{3} & 4 & -\frac{2}{3} \end{pmatrix}$$

holds.

Exercise 4. Again, try to apply this algorithm to other matrices and, while doing so, think about *why* it works.