Linear Algebra I
Exercise Class

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1 The Quotient Space

Exercise 1.1. Let $V := \mathbb{R}^{[0,1]}$ be the $\mathbb{R}$-vector space of all functions $f : [0,1] \to \mathbb{R}$ and consider

$$U := \{f \in V \mid f(0) = f(1) = 0\}.$$

(a) Show that U is a subspace of V.

(b) Identify V/U.

*(c) Find a subspace $W \subseteq V$ such that $V/W \cong \mathbb{R}^{\mathbb{N}}$.

Solution.

(a) *Proof.* For any $f, g \in U$ and $\lambda \in \mathbb{R}$ we have

$$(\lambda f + g)(0) = \lambda f(0) + g(0) = 0$$

and similarly $(\lambda f + g)(1) = 0$, so $\lambda f + g \in U$. Furthermore, we have $0_V \in U$ and thus U is indeed a subspace. $\square$

(b) **Claim.** $V/U \cong \mathbb{R}^2$.

Proof. For any $(x,y) \in \mathbb{R}^2$ define a function

$$f_{(x,y)} : [0,1] \to \mathbb{R},\ t \mapsto \begin{cases} x & \text{if } t = 0 \\ 0 & \text{if } 0 < t < 1 \\ y & \text{if } t = 1 \end{cases}$$

and observe that $f_{(x,y)} \in V$. Using this, define

$$\varphi : \mathbb{R}^2 \to V/U,\ (x,y) \mapsto f_{(x,y)} + U.$$

Then φ is well-defined and for any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \varphi(\lambda(x_1,y_1)+(x_2,y_2)) &= f_{(\lambda(x_1,y_1)+(x_2,y_2))} + U \\ &= \lambda f_{(x_1,y_1)} + f_{(x_2,y_2)} + U \\ &= \lambda(f_{(x_1,y_1)} + U) + f_{(x_2,y_2)} + U \\ &= \lambda\varphi(x_1,y_2) + \varphi(x_2,y_2) \end{aligned}$$

which proves that φ is a linear map.

Injectivity. Let $(x,y) \in \mathbb{R}^2$ be arbitrary with $\varphi(x,y) = f_{(x,y)} + U = 0_{V/U}$. Equivalently, we then have $f_{(x,y)} \in U$, so by definition of U we get

$$f_{(x,y)}(0) = f_{(x,y)}(1) = 0.$$

But by definition of $f_{(x,y)}$ we also have

$$f_{(x,y)}(0) = x, \quad f_{(x,y)}(1) = y$$

and thus $(x,y) = (0,0)$. This proves $\ker(\varphi) = \{0\}$ and thus φ is injective.

Surjectivity. Let $f + U \in V/U$ be arbitrary and set

$$x := f(0), \quad y := f(1).$$

We want to prove that now $\varphi(x,y) = f$ holds. Observe that we have

$$\begin{aligned} (f - f_{(x,y)})(0) &= f(0) - f_{(x,y)}(0) \\ &= f(0) - x \\ &= f(0) - f(0) = 0 \end{aligned}$$

and by the same reasoning $(f - f_{(x,y)})(1) = 0$ follows. But this means that $f - f_{(x,y)} \in U$ or equivalently

$$f + U = f_{(x,y)} + U = \varphi(x, y),$$

which proves surjectivity. □

*(c) Take $W := \{f \in V \mid \forall n \in \mathbb{N} : f(\frac{1}{n}) = 0\}$. To see that this indeed fulfills $V/W \cong \mathbb{R}^{\mathbb{N}}$, define a map

$$\psi : \mathbb{R}^{\mathbb{N}} \to V/W,\ (x_n)_{n\in\mathbb{N}} \mapsto f_{(x_n)_{n\in\mathbb{N}}} + W,$$

where

$$f_{(x_n)_{n\in\mathbb{N}}} : [0, 1] \to \mathbb{R},\ t \mapsto \begin{cases} x_n & \text{if } \exists n \in \mathbb{N} :\ t = \frac{1}{n} \\ 0 & \text{else} \end{cases}$$

and proceed as in (b).

Exercise 1.2. For $n \in \mathbb{N}$, set $V := M_{n\times n}(\mathbb{R})$ and

$$U := \{A \in V \mid A^T + A = 0\}.$$

(a) Show that U is a subspace if V and compute $\dim(U)$.

(b) Identify V/U.

Hint. Use the first isomorphism theorem.

(c) Using (b), find a complement of U.

Solution.

(a) Direct computation shows that U is indeed a subspace and we have $\dim(U) = \frac{n(n-1)}{2}$. Here we omit a proof (the idea is to explicitly write down a basis of U).

(b) **Claim.** $V/U \cong \{A \in V \mid A^T = A\} = \{A \in V \mid A \text{ is symmetric}\} =: W$.

> Key idea. Find an appropriate linear map $T : V \to V$ with $\ker(T) = U$ and identify $\operatorname{im}(T)$. By the first isomorphism theorem, we then have $V/U = V/\ker(T) \cong \operatorname{im}(T)$.

Proof. Define

$$T : V \to V,\ A \mapsto A + A^T$$

and observe that T is linear and $\ker(T) = U$ follows immediately.

Claim. We have $\operatorname{im}(T) = W$.

"$\subseteq$". Let $B \in \operatorname{im}(T)$, so there exists a matrix $A \in V$ with

$$B = A + A^T.$$

Now compute

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$$

and thus B is symmetric, so $B \in W$.

"$\supseteq$". Let $B \in W$ be any symmetric matrix and write $B = (b_{ij})_{i,j=1,\dots,n}$. Then since $B^T = B$ holds, we have

$$\forall i, j \in \{1, \dots, n\} : \quad b_{ij} = b_{ji}.$$

Now set

$$a_{ij} := \begin{cases} b_{ij} & \text{if } i < j \\ \frac{b_{ij}}{2} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

for $i, j \in \{1, \dots, n\}$ and $A := (a_{ij})_{i,j=1,\dots,n} \in V$, so

$$A = \begin{pmatrix} \frac{b_{11}}{2} & b_{12} & b_{13} & \dots & & b_{1n} \\ 0 & \frac{b_{22}}{2} & b_{23} & \dots & & b_{2n} \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ & \vdots & & & & \\ 0 & 0 & \dots & & 0 & \frac{b_{nn}}{2} \end{pmatrix}.$$

Then we have

$$T(A) = A + A^T = B$$

and thus $B \in \mathrm{im}(T)$.

Hence we have shown that $\mathrm{im}(T) = W$ holds. Now using the first isomorphism theorem, we get

$$V/U = V/\ker(T) \cong \mathrm{im}(T) = W$$

which concludes the proof. $\square$

(c) By theorem 7.1.6 from the lecture notes, any complement $W' \leq V$ of U must fulfill $W' \cong V/U$. By (b), this might suggest to take $W' := W$ to find a complement of U. One directly verifies that $W \leq V$ and that we have $U \cap W = \{0\}$. Combining (a) and (b) we get

$$\dim(W) = \dim(V/U) = \dim(V) - \dim(U) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Hence $\dim(U + W) = n^2 = \dim(V)$ and thus $V = U \oplus W$.

Note that this means that we can uniquely decompose any matrix into a sum of a symmetric and an antisymmetric matrix.