Linear Algebra I

Exercise Class

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16. October 2023

1. Vector Spaces and Subspaces

1.1 Recap

We first recall the important definitions.

Definition. (*Vector space*) A *vector space* over a field K is a tuple $(V, +, \cdot, 0_V)$ consisting of a set V with two maps

$$\begin{aligned} + :\ & V \times V \to V,\ (v_1, v_2) \mapsto v_1 + v_2, \\ \cdot :\ & K \times V \to V,\ (\lambda, v) \mapsto \lambda \cdot v \end{aligned}$$

and a selected element $0_V \in V$ such that
(*VR1*) $\forall v_1, v_2 \in V :\ v_1 + v_2 = v_2 + v_1$,
(*VR2*) $\forall v_1, v_2, v_3 \in V :\ v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$,
(*VR3*) $\forall v \in V :\ 0_V + v = v$,
(*VR4*) $\forall v \in V\ \exists w \in V :\ v + w = 0_V$,
(*VR5*) $\forall v_1, v_2 \in V\ \forall \lambda \in K :\ \lambda \cdot (v_1 + v_2) = \lambda \cdot v_1 + \lambda \cdot v_2$,
(*VR6*) $\forall v \in V\ \forall \lambda_1, \lambda_2 \in K :\ (\lambda_1 + \lambda_2) \cdot v = \lambda_1 \cdot v + \lambda_2 \cdot v$,
(*VR7*) $\forall v \in V\ \forall \lambda_1, \lambda_2 \in K :\ \lambda_1 \cdot (\lambda_2 \cdot v) = (\lambda_1 \cdot \lambda_2) \cdot v$,
(*VR8*) $\forall v \in V :\ 1_K \cdot v = v$.

Definition. (*Subspace*) Let V be a K-vector space. A subset $U \subseteq V$ is called a *subspace* if
(*UR0*) $U \neq \emptyset$,
(*UR1*) $\forall u_1, u_2 \in U :\ u_1 + u_2 \in U$,
(*UR2*) $\forall u \in U\ \forall \lambda \in K :\ \lambda \cdot u \in U$.
In this case, we also write $U \leq V$.

1.2 Vector spaces

Exercise. Let $V := \mathbb{R}_{>0}$ and define two operations

$$\begin{aligned} \oplus : V \times V \to V, &\quad (x, y) \mapsto xy \\ \odot : \mathbb{Q} \times V \to V, &\quad (q, x) \mapsto x^q. \end{aligned}$$

Is there an element $0_V \in V$ such that $(V, \oplus, \odot, 0_V)$ is a $\mathbb{Q}$-vector space?

Solution. Yes, set $0_V := 1 \in \mathbb{R}_{>0}$ and verify all vector space axioms (*VR1*)-(*VR8*).

Exercise. Let $V := \mathbb{R}^2$ and define two operations

$$\begin{aligned} +_V &: V \times V \to V, \\ \cdot_V &: \mathbb{R} \times V \to V \end{aligned}$$

by

$$\begin{aligned} (x, y) +_V (x', y') &:= (x + x', y + y'), \\ \lambda \cdot_V (x, y) &:= (\lambda x, y). \end{aligned}$$

Is there an element $0_V \in V$ such that $(V, +_V, \cdot_V, 0_V)$ is a $\mathbb{R}$-vector space?

Solution. No, the scalar multiplication $\cdot_V$ does not fulfil the axiom (*VR6*).

Exercise. Let $V := \mathbb{R}^n$ for $n \in \mathbb{N}$, fix a vector $v_0 \in V$ and define two operations

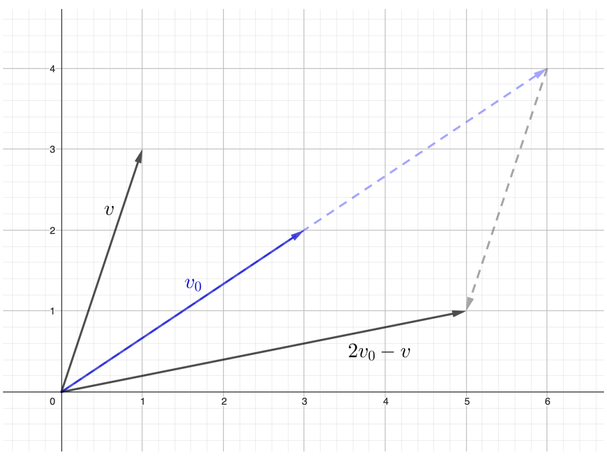
$$\begin{aligned} \vec{+} &: V \times V \to V, \quad (v, w) \mapsto v + w - v_0 \\ \vec{\cdot} &: \mathbb{R} \times V \to V, \quad (\lambda, v) \mapsto \lambda(v - v_0) + v_0. \end{aligned}$$

Is there an element $0_V \in V$ such that $(V, \vec{+}, \vec{\cdot}, 0_V)$ is an $\mathbb{R}$-vector space?

Solution. Yes, set $0_V := v_0$ and verify all vector space axioms (*VR1*)-(*VR8*). Notice how this is just the "usual" vector space structure on $\mathbb{R}^n$ but translated by v_0. In V the additive in inverse of a vector $v \in V$ is given by $2v_0 - v$ because

$$v \vec{+} 2v_0 - v = v_0 = 0_V$$

holds. This situation might be illustrated by the following picture.



Try to similarly visualize the scalar multiplication $\vec{\cdot}$ in V.

1.3 Subspaces

Exercise. Which of the following subsets of $\mathbb{R}^3$ are subspaces?

(a) $V_1 := \{(x, 1, z) \mid x, z \in \mathbb{R}\}$

(b) $V_2 := \{(y + z, 2x, -y) \mid x, y, z \in \mathbb{R}\}$

(c) $V_3 := \{(x^2, 0, 0) \mid x \in \mathbb{R}\}$

(d) $V_4 := \{(e^x \sin(y), 0, 0) \mid x, y \in \mathbb{R}\}$

Solution.

(a) Since $(0, 0, 0) \notin V_1$, it is not a subspace.

(b) V_2 is indeed a subspace. Since $(0, 0, 0) \in V_2$ holds, we get $V_2 \neq \emptyset$. Furthermore, for $x, \tilde{x}, y, \tilde{y}, z, \tilde{z} \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ we have

$$(y + z, 2x, -y) + (\tilde{y} + \tilde{z}, 2\tilde{x}, -\tilde{y}) = ((y + \tilde{y}) + (z + \tilde{z}), 2(x + \tilde{x}), -(y + \tilde{y})) \in V_2$$

and

$$\lambda(y + z, 2x, -y) = (\lambda y + \lambda z, 2\lambda x, -\lambda y) \in V_2.$$

(c) Observe that $(1, 0, 0) \in V_3$ holds but

$$-1 \cdot (1, 0, 0) = (-1, 0, 0) \notin V_3$$

and thus V_3 is not a subspace.

(d) Since the function $(x, y) \mapsto e^x \sin(y)$ is surjective, we have

$$V_4 = \{(t, 0, 0) \mid t \in \mathbb{R}\}$$

which clearly is a subspace of $\mathbb{R}^3$.

Exercise. Let V be a vector space and $U_1, U_2 \leq V$ two subspaces such that $U_1 \cup U_2$ is also a subspace. Prove that then one of the spaces is contained in the other.

Proof. Assume that $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$ holds. Then there exist vectors

$$u_1 \in U_1 \setminus U_2, \quad u_2 \in U_2 \setminus U_1$$

which fulfil $u_1, u_2 \in U_1 \cup U_2$. But since $U_1 \cup U_2$ is a subspace, we then get

$$u_1 + u_2 \in U_1 \cup U_2,$$

which implies WLOG $u_1 + u_2 \in U_1$. Now since $-u_1 \in U_1$ holds, we get $u_2 \in U_1$ which is a contradiction. □

Exercise. Let $U_1, U_2 \leq V$ be two subspaces of a vector space V such that $V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$. Prove that for all $v \in V$ there are unique vectors $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = u_1 + u_2$ holds.

Proof. Let $v \in V$ be arbitrary. The existence of such a representation directly follows from $V = U_1 + U_2$. Now let $u_1, \tilde{u}_1 \in U_1$ and $u_2, \tilde{u}_2 \in U_2$ be vectors with

$$v = u_1 + u_2 = \tilde{u}_1 + \tilde{u}_2.$$

Then we get

$$\underbrace{u_1 - \tilde{u}_1}_{\in U_1} = \underbrace{\tilde{u}_2 - u_2}_{\in U_2} \in U_1 \cap U_2$$

and thus by using $U_1 \cap U_2 = \{0\}$

$$u_1 = \tilde{u}_1, \quad u_2 = \tilde{u}_2$$

follows which proves uniqueness. □