Linear Algebra I
Exercise Class

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23. October 2023

1. Vector Spaces

1.1 The Vector Space $\mathbb{R}^I$

Here we introduce an important object that often comes up very naturally, although it might seem somewhat abstract at first sight.

For any set I, we define $\mathbb{R}^I$ to be the set of all functions $f : I \to \mathbb{R}, i \mapsto a_i$. Then using pointwise addition and multiplication $\mathbb{R}^I$ naturally becomes an $\mathbb{R}$-vector space.

How can we think of $\mathbb{R}^I$*?* Note that we can uniquely identify any function $f \in \mathbb{R}^I$ by its values $(a_i)_{i \in I}$. Hence if I is finite with $n := |I| < \infty$, we have

$$\mathbb{R}^I \text{ ``}=\text{''} \mathbb{R}^n.$$

Now what if $I = \mathbb{N}$ *holds?* Then any vector $f \in \mathbb{R}^{\mathbb{N}}$ can be identified with its values $(a_i)_{i \in \mathbb{N}}$ and thus $\mathbb{R}^{\mathbb{N}}$ is just the vector space of all real-values sequences. The picture

$$f \text{ ``}=\text{''} (a_i)_{i \in \mathbb{N}} = (a_1, a_2, a_3, \ldots)$$

might help to draw connections. Similarly, any $f \in \mathbb{R}^{\mathbb{Z}}$ might be visualized by

$$f \text{ ``}=\text{''} (\ldots, a_{-1}, a_0, a_1, \ldots).$$

What if $I = \mathbb{R}$ *holds?* In this situation, the picture

$$f \text{ ``}=\text{''} (\ldots, a_0, \ldots, a_{0.001}, \ldots, a_\pi, \ldots)$$

is just *very misleading* because (as Cantor's diagonal argument shows) any attempt to *actually list* all real numbers fails.

We see that our intuition does not cover all possibilities and thus to *really* understand $\mathbb{R}^I$, one needs to rigorously proof mathematical statements. This process is often a *delicate interplay between intuition and logic.*

Claim. The set of all functions with *finite support*

$$\mathbb{R}^{(I)} := \{(a_i)_{i \in I} \in \mathbb{R}^I \mid a_i = 0 \text{ for all but finitely many } i \in I\}$$

is a subspace of the vector space $\mathbb{R}^I$.

Exercise. Try to prove this.

1.2 Exercise Sheet 4

Exercise 3. Let X be a set and V a K-vector space. Then $\mathcal{F} := \mathcal{F}(X,V) = V^X$ denotes the K-vector space of all function $f : X \to V$ equipped with pointwise addition and scalar multiplication.

Now fix $y \in X$ and $v \in V$ set

$$\mathcal{F}_{y,v} := \{f \in \mathcal{F} \mid f(y) = v\}.$$

(a) Prove that $\mathcal{F}_{y,v}$ is a subspace of $\mathcal{F}$ if and only if $v = 0$ holds.

(b) Find a subspace $W \leq \mathcal{F}$ such that

$$\mathcal{F} = \mathcal{F}_{y,v} + W \quad \text{and} \quad \mathcal{F}_{y,v} \cap W = \{0_{\mathcal{F}}\}.$$

Solution. Can be found on Lineare Algebra I Herbst 2023.

2. Span and Basis

2.1 How to Find a Basis

In exercise 6 on exercise sheet 5 you are going to prove the following result, which is very useful to determine whether given vectors are independent or not.

Proposition 1. Let K be a field, $n, m \in \mathbb{N}$ and let $A \in \mathrm{M}_{n\times n}(K)$ be an invertible matrix. Then $v_1, \ldots, v_m \in K^n$ are linearly independent if and only if $Av_1, \ldots, Av_m$ are linearly independent.

Example. Let

$$v_1 := \begin{pmatrix} 1 \\ -2 \\ 3 \\ 4 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ -3 \\ 2 \\ 1 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 2 \\ 5 \\ 0 \\ 5 \end{pmatrix}.$$

Which of the vectors above form a basis of $\langle v_1, v_2, v_3 \rangle \leq \mathbb{R}^4$?

For this, define the matrix

$$A := (v_1, v_2, v_3) = \begin{pmatrix} 1 & 0 & 2 \\ -2 & -3 & 5 \\ 3 & 2 & 0 \\ 4 & 1 & 5 \end{pmatrix} \in M_{4\times 3}(\mathbb{R}).$$

Now using elementary row-operations, we can find an invertible matrix $W \in M_{4\times 4}(\mathbb{R})$ such that

$$WA = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we can immediately see that

$$2v_1 - 3v_2 - v_3 = 0$$

holds which means that $\{v_1, v_2, v_3\}$ are not linearly independent. Thus we need to remove at least one vector to obtain a basis. Note that the equation above also implies

$$\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle.$$

But we can also observe that

$$Av_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Av_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

are linearly independent and thus by **proposition 1** this implies that v_1 and v_2 are indeed linearly independent. Putting everything together, this proves that $\{v_1, v_2\}$ is a basis of $\langle v_1, v_2, v_3 \rangle$.

2.2 Vector Space of Polynomials

In **example 3.1.3.** from the lecture notes you have seen that the set of polynomials with real coefficients and degree ≤ 3, denoted by $\mathbb{R}[X]^{\leq 3}$, is an $\mathbb{R}$-vector space.

Exercise. Ist the set $B := \{X + X^3, X^2 + 2X, X^3 - X^2\}$ a basis of $\mathbb{R}[X]^{\leq 3}$.

Solution. We first check if the vectors of B are independent. For this, let $a, b, c \in \mathbb{R}$ be arbitrary with

$$a(X + X^3) + b(X^2 + 2X) + c(X^3 - X^2) = 0.$$

Rearranging the equation yields

$$(a + 2b)X + (b - c)X^2 + (a + c)X^3 = 0$$

and by comparing the coefficients we obtain the following system of equations:

$$\begin{aligned} a + 2b &= 0 \\ b - c &= 0 \\ a + c &= 0 \end{aligned}$$

One can directly check that $a = b = c = 0$ is the unique solution of this system of linear equations. This proves that the vectors of B are indeed linearly independent.

But again by comparing coefficients, one can show that the polynomial $1 \in \mathbb{R}[X]^{\leq 3}$ cannot be represented as a linear combination of vectors in B. Hence B is not a basis of $\mathbb{R}[X]^{\leq 3}$.