Linear Algebra I
Exercise Class

Eric Ceglie

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1. Fibonacci Sequences

Theorem 1.1.6. Let V be the set of all Fibonacci Sequences over $\mathbb{R}$, i.e.

$$V = \{\mathcal{F}_{a,b} \mid a, b \in \mathbb{R}\}.$$

Then

(a) for all $\mathcal{F}, \mathcal{G} \in V$ we have $\mathcal{F} + \mathcal{G} \in V$.

(b) for all $\mathcal{F} \in V$ and $\alpha \in \mathbb{R}$ we have $\alpha\mathcal{F} \in V$.

Proof. Let $\mathcal{F}, \mathcal{G} \in V$ and $\alpha \in \mathbb{R}$ be arbitrary and write

$$\mathcal{F} = (a_n)_{n\geq 0}, \quad \mathcal{G} = (b_n)_{n\geq 0}$$

for $a_n, b_n \in \mathbb{R}$.

(a) We define

$$c_n := a_n + b_n$$

for all $n \geq 0$ and observe that

$$\begin{aligned} c_n &= a_n + b_n = a_{n-1} + a_{n-2} + b_{n-1} + b_{n-2} \\ &= (a_{n-1} + b_{n-1}) + (a_{n-2} + b_{n-2}) \\ &= c_{n-1} + c_{n-2}. \end{aligned}$$

holds for $n \geq 2$. Hence we get

$$\mathcal{F} + \mathcal{G} = (c_n)_{n\geq 0} \in V.$$

(b) Similarly, define

$$d_n := \alpha a_n$$

for $n \geq 0$ and observe that

$$\begin{aligned} d_n &= \alpha a_n = \alpha(a_{n-1} + a_{n-2}) \\ &= \alpha a_{n-1} + \alpha a_{n-2} \\ &= d_{n-1} + d_{n-2} \end{aligned}$$

holds for $n \geq 2$. Hence we get

$$\alpha \mathcal{F} \in V$$

which conclude the proof. $\square$

Exercise 1.1.8. Let $a, b \in \mathbb{R}$ be arbitrary such that $(a, b) \neq (0, 0)$ and set

$$W := \{\mathcal{F}_{\alpha a, \alpha b} \mid \alpha \in \mathbb{R}\} \subseteq V,$$

where V defined as in **Theorem 1.1.8.** Show that W contains infinitely many elements but is not equal to V.

Proof. Since $(a, b) \neq (0, 0)$, we may assume WLOG ("Without loss of generality") that $a \neq 0$ holds.

- Since $a \neq 0$ holds, we have

  $$|\{\alpha a \mid \alpha \in \mathbb{R}\}| = \infty$$

  and thus $|W| = \infty$ follows immediately by definition of $\mathcal{F}_{\alpha a, \alpha b}$.

- Here we distinguish two cases.
  - CASE $b = 0$. Then $\mathcal{F}_{a,1} \in V \setminus W$ and thus W is not equal to V.
  - CASE $b \neq 0$. Assume that

    $$\mathcal{F}_{a,2b} \in W$$

    holds, so there exists a $\alpha \in \mathbb{R}$ such that

    $$\mathcal{F}_{a,2b} = \mathcal{F}_{\alpha a, \alpha b}$$

    holds. But this implies

    $$a = \alpha a \text{ and } 2b = \alpha b$$

    and since we have $a, b \neq 0$ we can divide both equations by a and b respectively to get

    $$1 = \alpha \text{ and } 2 = \alpha$$

    which is a contradiction since $1 \neq 2$ holds in $\mathbb{R}$. Hence we have $\mathcal{F}_{a,2b} \notin W$ and since $\mathcal{F}_{a,2b} \in V$ holds this concludes the proof. $\square$

Exercise 1.1.10. Let $\mathcal{F} \in V$. Show that there exist elements $c, d \in \mathbb{R}$ such that

$$\mathcal{F} = c\mathcal{F}_{1,1} + d\mathcal{F}_{1,-1}$$

holds.

Proof. Let $\mathcal{F} \in V$ be arbitrary and write

$$\mathcal{F} = (a_n)_{n\geq 0}$$

for $a_n \in \mathbb{R}$. To find c and d we have to solve the system of equations given by

$$\begin{aligned} a_0 &= c \cdot 1 + d \cdot 1 = c + d \\ a_1 &= c \cdot 1 + d \cdot (-1) = c - d, \end{aligned}$$

where a_0, a_1 are both known. A solution is given by

$$c = \frac{a_0 + a_1}{2}, \quad d = \frac{a_0 - a_1}{2}$$

which concludes the proof. □

Where do φ and ψ come from?

Recall the *translation map* S given by

$$S : V \to V, (a_0, a_1, \ldots) \mapsto (a_1 a_2, \ldots).$$

Since this is an interesting map, we want to find eigensequences of S. For this, assume that $\mathcal{A} = (a_n)_{n\geq 0} \in V$ is an eigensequence of S, meaning that there exists a $\alpha \in \mathbb{R}$ such that

$$S(\mathcal{A}) = \alpha\mathcal{A}$$

holds. But this implies

$$(a_1, a_2, a_3, \ldots) = (\alpha a_0, \alpha a_1, \alpha a_2, \ldots)$$

and by recursively plugging in the given equations we get

$$\begin{aligned} a_1 &= \alpha a_0 \\ a_2 &= \alpha a_1 = \alpha^2 a_0 \\ a_3 &= \ldots = \alpha^3 a_0 \\ &\vdots \\ a_n &= \alpha^n a_0 \end{aligned}$$

for $n \geq 0$. Hence

$$\mathcal{A} = a_0(1, \alpha, \alpha^2, \ldots)$$

holds and since $\mathcal{A}$ was chosen to be a Fibonacci sequence we get

$$\alpha^2 = \alpha + 1 \iff \alpha^2 - \alpha - 1 = 0.$$

Now using the well-known quadratic formula we get

$$\alpha \in \left\{ \frac{1 \pm \sqrt{5}}{2} \right\},$$

so the eigenvalue α corresponding to $\mathcal{A}$ is given by

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad \psi = \frac{1-\sqrt{5}}{2}.$$

Exercise 1.1.19. Find numbers $c, d \in \mathbb{R}$ such that

$$c\mathcal{F}_{1,\varphi} + d\mathcal{F}_{1,\psi} = \mathcal{F}_{0,1} \tag{1}$$

holds.

Solution. Again, as in **exercise 1.1.10**, the condition (1) gives us the equations

$$\begin{aligned} 0 &= c + d \\ 1 &= c\varphi + d\psi. \end{aligned}$$

A simple computation now shows that a solution is given by

$$c = \frac{1}{\varphi - \psi}, \quad d = \frac{1}{\psi - \varphi}$$

which concludes the exercise.

Corollary 1.1.20. Let $\mathcal{F}_{0,1} = (a_n)_{n \geq 0}$. Then an explicit formula for a_n is given by

$$a_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

Proof. Combining **exercise 1.1.19** with the observation that

$$\mathcal{F}_{1,\varphi} = (1, \varphi, \varphi^2, \ldots), \quad \mathcal{F}_{1,\psi} = (1, \psi, \psi^2, \ldots)$$

holds we get

$$a_n = \frac{1}{\varphi - \psi}\varphi^n + \frac{1}{\psi - \varphi}\psi^n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

which concludes the proof. □