Linear Algebra I
Exercise Class

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1 Complements of Subspaces

1.1 Recap

Definition 1.1. Let V be a K-vector space and $U \leq V$ a subspace. A subspace $W \leq V$ is called a *complement* of U if

- $V = U + W$,
- $U \cap W = \{0\}$

hold. In this case, we also write $V = U \oplus W$.

Theorem 1.2. *Every subspace of a vector space has a complement.*

Theorem 1.3. *Let V be a finite-dimensional vector space and $U, W \leq V$ two subspaces. Then we have*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

1.2 Finding a Complement

We are going to introduce an algorithm that generally works with an example.

Example 1.4. Let $V := \mathbb{R}^4$ and consider the subspace $U := \langle v_1, v_2, v_3 \rangle$ for

$$v_1 := \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}.$$

Find a complement $W \leq V$ of U.

Solution. Proceed in in three steps as follows.

(1) Consider the matrix

$$A := \begin{pmatrix} | & | & | & \\ v_1 & v_2 & v_3 & I_4 \\ | & | & | & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that we added the canonical basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ to our vectors v_1, v_2, v_3 to make sure that we obtain a basis of $\mathbb{R}^4$. Now we need to determine a basis that extends $\{v_1, v_2, v_3\}$ to a basis of $\mathbb{R}^4$.

(2) By only using elementary row operations, transform A into a matrix in row-reduced echelon form

$$\tilde{A} := \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

(3) In a third step, we are now going to closely investigate the matrix $\tilde{A}$ to conclude. First observe that the first three columns of $\tilde{A}$ are linearly dependent and thus v_1, v_2, v_3 are also linearly dependent. But by looking at $\tilde{A}$ we also see that

$$v_3 = 2v_1 - v_2$$

holds and that v_1 and v_2 are linearly independent, so

$$U = \langle v_1, v_2 \rangle.$$

Again by looking at $\tilde{A}$, we see that that the vectors v_1, v_2, e_1, e_2 are linearly independent and since $\dim(\mathbb{R}^4) = 4$ holds they form a basis of $\mathbb{R}^4$. Now using Theorem 1.3 we can conclude that we have

$$\mathbb{R}^4 = \langle v_1, v_2 \rangle \oplus \langle e_1, e_2 \rangle.$$

2 Linear Maps

2.1 Recall

Examples 2.1.

- If we consider $\mathbb{C}$ as an $\mathbb{R}$-vector space, then $\{1, i\}$ is a basis of $\mathbb{C}$.
- If we consider $\mathbb{C}$ as a $\mathbb{C}$-vector space, then $\{1\}$ is a basis of $\mathbb{C}$.

Definition 2.2. Let V, W be two K-vector spaces. A map $T : V \to W$ is called a *linear map* if the following conditions hold:

(i) $\forall v_1, v_2 \in V : T(v_1 + v_2) = T(v_1) + T(v_2)$,

(ii) $\forall v \in V \, \forall \alpha \in K : T(\alpha v) = \alpha T(v)$.

In this case, we also say that T is *K-linear*. Furthermore, a linear map $T : V \to V$ is called an *endomorphism*.

Definition 2.3. Let $m, n \geq 1$ and $A \in M_{m \times n}(K)$ for a field K. Then we define

$$T_v : K^n \to K^m, \; v \mapsto Av.$$

Lemma 2.4. *The map T_v is linear.*

2.2 Exercises

Exercise 2.5. Recall that we can view $V := \mathbb{C}$ as an $\mathbb{R}$- or $\mathbb{C}$-vector space. Suppose that a map $T : V \to V$ is $\mathbb{R}$-linear. Is it then necessarily also $\mathbb{C}$-linear? Prove it or find a counter example.

Solution. The claim is false. Indeed, define

$$T : V \to V,\ z \mapsto \mathrm{Re}(z),$$

where for any $a, b \in \mathbb{R}$ we set $\mathrm{Re}(a + bi) := a$. One directly verifies that T is $\mathbb{R}$-linear but we also have

$$T(i \cdot 1) = T(i) = 0 \neq i = i \cdot T(1)$$

which shows that T is not $\mathbb{C}$-linear.

3 Equivalence Relations

3.1 Recap

Definition 3.1. Let X be a set. A *relation* on X is a subset $R \subseteq X \times X$ and for $x, y \in X$ we write

$$x \sim y \;:\Longleftrightarrow\; (x, y) \in R.$$

The relation $\sim$ is called an *equivalence relation* if

(i) $\forall x \in X : x \sim x \quad$ (*reflexive*),

(ii) $x, y \in X : x \sim y \implies y \sim x \quad$ (*symmetric*),

(iii) $\forall x, y, z \in X : x \sim y$ and $y \sim z \implies x \sim z \quad$ (*transitive*).

In this case, we define

$$[x] := \{y \in X \mid x \sim y\}$$

for any $x \in X$ and

$$X/{\sim} := \{[x] \mid x \in X\}.$$

Exercise 3.2. Define an relation on $\mathbb{R}$ by

$$x \sim y \;:\Longleftrightarrow\; x - y \in \mathbb{Z}$$

for $x, y \in \mathbb{R}$.

(a) Show that $\sim$ is an equivalence relation.

Now define the map

$$f : \mathbb{R}/{\sim} \to \mathbb{R},\ [x] \mapsto x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leq x\}$.

(b) Show that f is well-defined.

(c) If we instead considered the relation

$$x \sim y \;:\Longleftrightarrow\; x - y \in \mathbb{Q},$$

would f still be well-defined?

Solution.

(a) This can be proven by a straightforward computation (*try to write it out*).

(b) *Proof.* Let $x, y \in \mathbb{R}$ be arbitrary with $x \sim y$. Then by definition of $\sim$ there exists an integer $n \in \mathbb{Z}$ such that $x - y = n$. Hence we have

$$\begin{aligned} x - \lfloor x \rfloor &= y + n - \lfloor y + n \rfloor \\ &= y + n - (\lfloor y \rfloor + n) \qquad (1) \\ &= y - \lfloor y \rfloor \end{aligned}$$

which shows that f is well-defined. Note that at (1) we used a basic property of $\lfloor \cdot \rfloor$ which only holds if n is an integer. (*Can you prove this property*?) □

(c) No, in this case f is not well-defined since for example

$$1.1 - \lfloor 1.1 \rfloor = 0.1 \neq 0.2 = 1.2 - \lfloor 1.2 \rfloor$$

but $1.1 \sim 1.2$.