Non-Life Insurance Mathematics

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Reference

[1] M. Wuthrich, *Non-life insurance: mathematics & statistics*, Available at SSRN 2319328, (2023).

Model Selection

χ^2-Goodness-of-Fit Analysis

For data $N = (N_1, \dots, N_T)$ consider the one-sided test

$$H_0 : N_t \overset{ind.}{\sim} \mathrm{Poi}(\lambda v_t) \text{ vs. } H_1 : \text{we have over-dispersion.}$$

Under H_0. By the aggregation theorem we have

$$N_t \overset{(d)}{=} \sum_{k=1}^{v_t} X_k$$

for $X_k \overset{i.i.d.}{\sim} \mathrm{Poi}(\lambda)$. Hence by the CLT

$$\frac{N_t/v_t - \lambda}{\sqrt{\lambda/v_t}} \overset{(d)}{=} \frac{\sum_{k=1}^n X_k - v_t\lambda}{\sqrt{v_t\lambda}} \xrightarrow{(d)} \mathcal{N}(0,1).$$

Thus for v_t large enough

$$\chi^*(N) := \sum_{t=1}^T \frac{(N_t/v_t - \lambda)^2}{\lambda/v_t} \overset{(d)}{\approx} \chi^2_T$$

and

$$\widehat{\chi}^*(N) := \sum_{t=1}^T \frac{(N_t/v_t - \widehat{\lambda}^{\mathrm{MLE}})^2}{\widehat{\lambda}^{\mathrm{MLE}}/v_t} \overset{(d)}{\approx} \chi^2_{T-1}.$$

Kolmogorov-Smirnov Test

Theorem (Glivenko-Cantelli)**.** *If $Y_i \overset{i.i.d.}{\sim} G_0$ for a unknown distribution G_0 then $\|\hat{G}_n - G_0\|_\infty \xrightarrow{a.s.} 0$ as $n \to \infty$.*

Assume $Y_i \overset{i.i.d.}{\sim} G$ for a unknown *continuous* distribution G and consider the null hypothesis $H_0 : G = G_0$. Then

$$D_n := \sqrt{n}\|\hat{G}_n - G_0\|_\infty \xrightarrow{(d)} \text{Kolmogorov Distribution } K$$

as $n \to \infty$.
Hence we reject H_0 on significance level $q \in (0,1)$ if $D_n > K^{\leftarrow}(1-q)$.

Anderson-Darling Test

The KS test is modified by introducing a *weight function* $\psi : [0,1] \to \mathbb{R}_+$ (for example $\psi(t) = (t(1-t))^{-1}$). The new test statistics is given by

$$A_n := \sqrt{n} \sup_y |\hat{G}_n(y) - G_0(y)| \sqrt{\psi(G_0(y))}.$$

Its limit is often computed numerically.

Akaike's Information Criterion

Assume two densities g_1 and g_2 were MLE fitted to some data $Y = (Y_1, \dots, Y_n)$. Then we define the AIC values by

$$\mathrm{AIC}^{(i)} := -2\ell_Y^{(i)}(\hat{\theta}_i^{\mathrm{MLE}}) + 2d^{(i)}$$

for $i = 1, 2$, where $d^{(i)}$ are the number of estimated parameters. Then

$$\text{select model 1} \iff \mathrm{AIC}^{(1)} \le \mathrm{AIC}^{(2)}.$$

Bayesian Information Criterion

Assume two densities g_1 and g_2 were MLE fitted to some data $Y = (Y_1, \dots, Y_n)$. Then we define the BIC values by

$$\mathrm{BIC}^{(i)} := -2\ell_Y^{(i)}(\hat{\theta}_i^{\mathrm{MLE}}) + \log(n)d^{(i)}$$

for $i = 1, 2$, where $d^{(i)}$ are the number of estimated parameters. Then

$$\text{select model 1} \iff \mathrm{BIC}^{(1)} \le \mathrm{BIC}^{(2)}.$$

Approximations of Compound Distributions

Assume $S = \sum_{k=1}^N Y_k$ has a compound distribution. Then explicitly

$$\mathbb{P}(S \le t) = \sum_{k=0}^\infty \mathbb{P}(N = k) \overbrace{\mathbb{P}(Y_1 + \ldots + Y_k \le t)}^{=G^{*k}(t)},$$

where $G^{*k}(t)$ is difficult to compute for large k.
Hence for modeling insurance data, we independently decompose $S = S_{\mathrm{sc}} + S_{\mathrm{lc}}$ such that S_{lc} can be computed explicitly and we approximate S_{sc} by assuming large expected numbers of small claims.

Normal Approximation (no skewness)

Theorem. *Assume $S \sim CompPoi(\lambda v, G)$ with G having finite second moment. Then*

$$\frac{S - \lambda v\mathbb{E}[Y]}{\sqrt{\lambda v \mathbb{E}[Y^2]}} \xrightarrow{(d)} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

Hence in this case $\mathbb{P}(S \le t) \approx \Phi\left(\frac{t - \lambda v\mathbb{E}[Y]}{\sqrt{\lambda v\mathbb{E}[Y^2]}}\right)$.

Translated Approximation (positive skewness)

For $k \in \mathbb{R}$ let

$$X = k + Z, \text{ where } Z \sim \begin{cases} \Gamma(\gamma, c) & \text{or} \\ \mathrm{LN}(\mu, \sigma^2) \end{cases}$$

and fit (k, γ, c) or (k, μ, σ^2) by solving

$$\mathbb{E}[X] = \mathbb{E}[S], \quad \mathrm{Var}(X) = \mathrm{Var}(S), \quad \varsigma_X = \varsigma_S.$$

Panjer Algorithm

Assume that N has a *Panjer distribution*, i.e. for constants $a, b \in \mathbb{R}$ we have $p_k = (a + b/k)p_{k-1}$ where $p_k := \mathbb{P}(N = k)$, $g_m := \mathbb{P}(Y = m)$ and $f_r := \mathbb{P}(S = r)$. Then

$$f_r = \mathbb{P}(S = r) = \begin{cases} \sum_{k=0}^\infty p_k g_0^k & \text{if } r = 0, \\ \frac{1}{1 - ag_0} \sum_{k=1}^r (a + bk/r) g_k f_{r-k} & \text{if } r \ge 1. \end{cases}$$

Fast Fourier Transform

Let $\mathcal{A} := \{0, 1, \dots, n-1\}$ and $f : \mathcal{A} \to \mathbb{R}$. For $z \in \mathcal{A}$ set

$$\hat{f}(z) := \sum_{l=0}^{n-1} f(l) e^{-\frac{2\pi i}{n} lz} \rightsquigarrow f(l) = \frac{1}{n} \sum_{z=0}^{n-1} \hat{f}(z) e^{\frac{2\pi i}{n} zl}.$$

If $S \sim (f(l))_{l \in \mathcal{A}}$ is a distribution then $\hat{f}(z) = M_S(-\frac{2\pi i}{n} z)$.
Hence if $S \sim f$ is a compound sum with $Y \sim g$, we have

$$\hat{f}(z) \approx M_n(\log \hat{g}(z)) \rightsquigarrow f(l) \approx \frac{1}{n} \sum_{z=0}^{n-1} M_n(\log \hat{g}(z)) e^{\frac{2\pi i}{n} zl}.$$

Premium Calculation Principles

Net profit condition (NPC): $\pi > \mathbb{E}[S]$

Utility Theory

Idea. Characterize risk via a strictly increasing *utility function* $u : I \to \mathbb{R}$ and u is called *risk-averse* if it is strictly concave. Then

$$X \succcurlyeq Y \;:\Longleftrightarrow\; \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

and u is *more risk-averse* than v if

$$\forall X \in \mathcal{X}: \quad u^{-1}(\mathbb{E}[u(X)]) \leq v^{-1}(\mathbb{E}[v(X)]).$$

Define the *absolute/relative risk aversion* by

$$\rho^u_{\mathrm{ARA}}(x) := -\frac{u''(x)}{u'(x)}, \quad \rho^u_{\mathrm{RRA}}(x) := -x\frac{u''(x)}{u'(x)}.$$

Then the *utility indifference price* $\pi^u_S(c_0)$ for initial capital c_0 is defined by the solution of

$$u(c_0) = \mathbb{E}[u(c_0 + \pi^u_S(c_0) - S)]$$

and for risk-averse u we have $\pi^u_S(c_0) > \mathbb{E}[S]$.

Examples. • *CARA:* $u(x) = 1 - \alpha^{-1}e^{-\alpha x}$ for $\alpha > 0$.

• *CRRA:* $u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1, \\ \log x & \text{for } \gamma = 1. \end{cases}$

Results. • $\pi^u_S(c_0)$ does not depend on $c_0 \iff u(x) = a - be^{-cx}$ for $a \in \mathbb{R}$, $b, c > 0$.

• u is more risk-averse than $v \iff \rho^u_{\mathrm{ARA}} \geq \rho^v_{\mathrm{ARA}}$ $\implies \pi^u_S(c_0) \geq \pi^v_S(c_0)$

• $\pi^u_S(c_0)$ is decreasing in $c_0 \iff \rho^u_{\mathrm{ARA}}(x)$ is decreasing.

Esscher Premium

Let $S \sim F$ and $\alpha > 0$ and assume that M_S exists on $(-r_0, r_0)$. The *Esscher distribution* is defined by

$$F_\alpha(s) := \frac{1}{M_s(\alpha)} \int_{-\infty}^{s} e^{\alpha x}\, dF(x)$$

and the *Esscher premium* is given by

$$\pi_\alpha := \mathbb{E}_\alpha[S] := \int_{\mathbb{R}} s\, dF_\alpha(s) = \frac{M'_S(\alpha)}{M_S(\alpha)} = b'_S(\alpha) > b'_S(0) = \mathbb{E}[S].$$

Probability Distortion

Let $h : [0,1] \to [0,1]$ be a *distortion function*, so it is continuous, increasing, concave and $h(0) = 0$, $h(1) = 1$ and assume $h(p) > p$.
The *probability distorted price* is defined by

$$\pi_h := \mathbb{E}_h[S] := \int_0^\infty h(\mathbb{P}(S > x))\, dx > \mathbb{E}[S].$$

Cost-of-Capital Principles

Let $\varrho : \mathcal{X} \to \overline{\mathbb{R}}$ be a *risk measure* and set a *cost-of-capital* rate $r_{\mathrm{CoC}} > r_0$, where r_0 denotes the risk-free rate an investor receives on a risk-free bank account.
The *cost-of-capital premium* is defined by

$$\pi_{\mathrm{CoC}} := \mathbb{E}[S] + r_{\mathrm{CoC}}\, \varrho(S - \mathbb{E}[S]).$$

A risk-measure is called *coherent* if it is defined on a convex cone $\mathcal{X}$ containing $\mathbb{R}$ and fulfills
(1) normalization,
(2) monotonicity,
(3) translation invariance,
(4) positive homogeneity,
(5) subadditivity.

Examples. Let $S \sim F$ be a distribution.

• The *standard deviation risk measure*

$$\varrho(S) = \alpha \mathrm{Var}(S)^{1/2}$$

is not subadditive and thus not coherent.

• The *Value-at-Risk (VaR)*

$$\varrho(S) = \mathrm{VaR}_{1-q} = F^{\leftarrow}(1-q)$$

is not subadditive and thus not coherent.

• The *expected shortfall*

$$\mathrm{ES}_{1-q}(S) = \frac{1}{q}\int_{1-q}^{1} \mathrm{VaR}_\alpha(S)\, d\alpha$$

is a coherent risk-measure. Furthermore, if F is continuous

$$\mathrm{ES}_{1-q}(S) = \mathbb{E}[S \mid S \geq \mathrm{VaR}_{1-q}(S)] = \mathrm{TVaR}_{1-q}(S).$$