Linear Algebra II
Exercise Class

Eric Ceglie

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1 The Rule of Sarrus

Let K be a field.

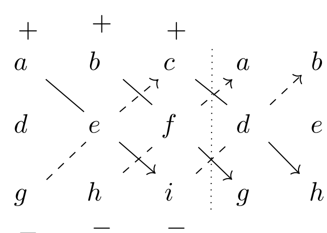
Theorem 1.1 (Sarrus' formula)**.** *For any matrix*

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in M_{3\times 3}(K)$$

we have

$$\det A = aei + bfg + cdh - gec - hfa - idb.$$

Remark 1.2. Note that this formula is particularly easy to remember using the following pattern:



Proof. Applying the Laplace expansion formula to the first row yields

$$\begin{aligned} \det A &= a(ei - fh) - b(di - gf) + c(dh - eg) \\ &= aei - afh - bdi + bgf + cdh - ceg, \end{aligned}$$

concluding the proof. ☐

2 Eigenvectors and Eigenvalues

Let V be a finite-dimensional K-vector space.

2.1 Recap

Definition 2.1. Let $T : V \to V$ be a linear map. A $\lambda \in K$ is called an *eigenvalue* if there exists a vector $v \in V \setminus \{0\}$ with $Tv = \lambda v$. Such a vector is called an *eigenvector*.

Remark 2.2. This definition can easily be extended to infinite-dimensional vector spaces.

Definition 2.3. For a matrix $A \in M_{n\times n}(K)$ we define its *characteristic polynomial* by

$$\chi_A(x) := \det(A - x \cdot I_n) \in K[x].$$

Similarly, for a linear map $T : V \to V$ we define

$$\chi_T(x) := \chi_{[T]^{\mathcal{B}}_{\mathcal{B}}}(x) = \det([T]^{\mathcal{B}}_{\mathcal{B}} - x \cdot I_n),$$

where $\mathcal{B}$ is any basis of V.

Exercise 2.4. Let

$$A := \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}.$$

(a) Compute $\chi_A(x)$.

(b) Find all eigenvalues and eigenvectors of A.

Solution.

(a) By applying Sarrus' formula we get

$$\begin{aligned}\chi_A(x) = \det(A - x \cdot I_n) = \det \begin{pmatrix} 2-x & 2 & 3 \\ 1 & 2-x & 1 \\ 2 & -2 & 1-x \end{pmatrix} \\ = (2-x)^2(1-x) + 4 - 6 - 6(2-x) + 2(2-x) - 2(1-x) \\ = -x^3 + 5x^2 - 2x - 8 = -(x+1)(x-2)(x-4).\end{aligned}$$

(b) Since the set of eigenvalues is equal to the set of zeros of χ_A, by (a) we see that the eigenvalues are precisely $\{-1, 2, 4\}$. The corresponding eigenvectors are given by

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}.$$

2.2 Diagonalization

Why do eigenvectors diagonalize?

Let $T : V \to V$ be a linear map and $\mathcal{B} = (v_1, \dots, v_n)$ a basis of V consisting of eigenvectors of T with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. To see why this basis has nice properties, we are going to compute the transformation matrix $A := [T]_{\mathcal{B}}^{\mathcal{B}}$. Recall that A is the unique matrix making the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \varphi_{\mathcal{B}} \uparrow \wr & & \varphi_{\mathcal{B}} \uparrow \wr \\ K^n & \xrightarrow{L_A} & K^n \end{array}$$

where

$$\varphi_{\mathcal{B}} : K^n \xrightarrow{\sim} V, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum_{i=1}^n x_i v_i.$$

Observe that for any vector $(a_1, \dots, a_n) \in K^n$ we have

$$T(\varphi_{\mathcal{B}}(a_1, \dots, a_n)) = T\Big(\sum_{i=1}^n a_i v_i\Big) = \sum_{i=1}^n \lambda_i a_i v_i$$

and thus

$$A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda_1 a_1 \\ \vdots \\ \lambda_n a_n \end{pmatrix}.$$

One now checks that this implies

$$[T]_{\mathcal{B}}^{\mathcal{B}} = A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Hence we see that in this case, given an appropriate basis, the map T only stretches the coordinate axes.

Exercise 2.5. A square matrix $A \in M_{n\times n}(K)$ is called *idempotent* if $A^2 = A$ holds.

(a) Show that every idempotent matrix is diagonalizable.

(b) Show that two idempotent matrices are similar if and only if they have the same rank.

Solution. Let $A, B \in M_{n\times n}(K)$ be idempotent.

(a) *Proof.* To diagonalize A, we will find a basis of K^n consisting of eigenvalues of A. Let $v \in K^n$ be arbitrary and observe that we have

$$v = Av + (I_n - A)v$$

with

$$A \cdot Av = A^2 v = Av,$$
$$A \cdot (I_n - A)v = Av - A^2 v = Av - Av = 0$$

since A is idempotent. Hence Av is an eigenvector of A to the eigenvalue 1 and $(I_n - A)v$ is an eigenvector to the eigenvalue 0, so we have written v as a linear combination of eigenvectors. This proves that the eigenvectors to the eigenvalues $\{0, 1\}$ span K^n and thus A is diagonalizable with values $\{0, 1\}$ on the diagonal. □

(b) *Proof.* Recall from the lecture that similar matrices have the same rank. Hence it only remains to show that if $\mathrm{rank}(A) = \mathrm{rank}(B)$ then A and B are similar. By (a) there exists an invertible $U \in \mathrm{GL}_n(K)$ such that

$$UAU^{-1} = \mathrm{diag}(\underbrace{1, \dots, 1}_{r \text{ times}}, \underbrace{0, \dots, 0}_{n-r \text{ times}}) =: D$$

with $r := \mathrm{rank}(D) = \mathrm{rank}(A)$. Now since $r = \mathrm{rank}(A) = \mathrm{rank}(B)$, by similar reasoning we get that there exists a $W \in \mathrm{GL}_n(K)$ with

$$WBW^{-1} = D.$$

Hence

$$A = U^{-1}DU = U^{-1}WBW^{-1}U,$$

concluding the proof since $U^{-1}W \in \mathrm{GL}_n(K)$. □