Linear Algebra II
Exercise Class

Eric Ceglie

6. May 2024

1 The Adjoint and the Dual Map

In this section, let V, W be two inner product spaces over $\mathbb{C}$ and $T : V \to W$ a linear map.

1.1 Recap

Definition 1.1. The *dual map* of T is defined by

$$T^*_{\text{dual}} : W^* \to V^*,\ f \mapsto f \circ T.$$

Theorem 1.2 (Riesz-representation theorem)**.** *Assume that V is finite-dimensional and for any $v \in V$ define the map*

$$\varphi_v : V \to K,\ u \mapsto \langle u, v \rangle.$$

Then the map

$$\Phi_V : V \xrightarrow{\sim} V^*,\ v \mapsto \varphi_v$$

is a bijection.

Assuming V and W are both finite-dimensional, we can construct the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi_V"'] & W \arrow[d, "\Phi_W"] \\ V^* & W^* \arrow[l, "T^*_{\text{dual}}"] \end{tikzcd}$$

1.2 The Adjoint

Proposition 1.3. *There exists at most one linear map $T^* : W \to V$ such that*

$$\forall v \in V \ \forall w \in W : \quad \langle Tv, w\rangle_W = \langle v, T^*w\rangle_V \tag{1}$$

holds.

Definition 1.4. If there exists a linear map $T^* : W \to V$ satisfying (1), we call T^* the *adjoint* of T.

We can now use the diagram from above to explicitly construct the adjoint of T, if V and W are finite-dimensional.

Theorem 1.5. *Assume that V and W are finite-dimensional. Then the adjoint of T is given by*

$$T^* = \Phi_V^{-1} \circ T^*_{\text{dual}} \circ \Phi_W.$$

Proof. It suffices to verify (1). To this end, let $v \in V$ and $w \in W$ be arbitrary. Observe that then

$$\begin{aligned}
\langle v, (\Phi_V^{-1} \circ T^*_{\text{dual}} \circ \Phi_W)(w)\rangle_V &= \Phi_V(\Phi_V^{-1} \circ T^*_{\text{dual}} \circ \Phi_W(w))(v) \\
&= (T^*_{\text{dual}} \circ \Phi_W)(w)(v) \\
&= T^*_{\text{dual}}(\varphi_w)(v) \\
&= \varphi_w(Tv) = \langle Tv, w\rangle_W.
\end{aligned}$$

This concludes the proof. □

1.3 Exercises

Exercise 1.6. Let $n \geq 1$ and consider the map

$$\langle \cdot, \cdot \rangle : M_{n\times n}(\mathbb{C}) \times M_{n\times n}(\mathbb{C}) \to \mathbb{C}, \ (A, B) \mapsto \mathrm{Tr}(AB^*),$$

where $B^* = \overline{B^T}$.

(a) Prove that $\langle \cdot, \cdot \rangle$ defines an inner product on $M_{n\times n}(\mathbb{C})$.

(b) Consider a matrix $M \in M_{n\times n}(\mathbb{C})$ with the property $M = M^*$. Define a linear map

$$S : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C}),\ A \mapsto AM.$$

Show that S is self-adjoint with respect to the above inner product.

Solution. See exam collection linear algebra II FS 23 (`https://exams.vmp.ethz.ch/exams/pipzz0mu.pdf`) exercise 7.

2 Basis of the Sequence Space

Let $\ell(\mathbb{N})$ be the $\mathbb{R}$-vector space of all $\mathbb{R}$-valued sequences, or equivalently the space of all functions $f : \mathbb{N} \to \mathbb{R}$.

Exercise 2.1. Prove that $\ell(\mathbb{N})$ has no countable basis.

We present two different approaches.

Proof 1. The first proof uses the classical approach of Cantor's diagonal argument. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \ell(\mathbb{N})$ be any countable subset. We aim to show that such a set cannot span the whole of $\ell(\mathbb{N})$. To this end, we will define a vector $g \in \ell(\mathbb{N})$ which is linearly independent of $\{f_n\}_{n\in\mathbb{N}}$.

Start by defining the first two values $g(1), g(2) \in \mathbb{R}$ such that the vector $(g(1), g(2))$ is linearly independent from

$$(f_1(1), f_1(2)).$$

Define the values $g(3), g(4), g(5) \in \mathbb{R}$ such that the vector $(g(3), g(4), g(5))$ is linearly independent from

$$(f_1(3), f_1(4), f_1(5)),$$
$$(f_2(3), f_2(4), f_2(5)).$$

Define the values $g(6), g(7), g(8), g(9) \in \mathbb{R}$ such that the vector $(g(6), g(7), g(8), g(9))$ is linearly independent from

$$(f_1(6), f_1(7), f_1(8), f_1(9)),$$
$$(f_2(6), f_2(7), f_2(8), f_2(9)),$$
$$(f_3(6), f_3(7), f_3(8), f_3(9)).$$

Note that by continuing this construction we obtain a $g \in \ell(\mathbb{N})$ with the property

$$\forall n \in \mathbb{N}: \quad g \notin \operatorname{span}\{f_1, \dots, f_n\}.$$

But this implies $g \notin \operatorname{span}\{f_n\}_{n\in\mathbb{N}}$, concluding the proof. □

Proof 2. The following proof may be considered to be more simple, but it uses more theory. Define a linear map by

$$S : \ell(\mathbb{N}) \to \ell(\mathbb{N})$$

by

$$Sf(n) := f(n+1)$$

for $f \in \ell(\mathbb{N})$ and $n \in \mathbb{N}$. Note that then for every $\lambda \in \mathbb{R}$ the vector

$$(1, \lambda, \lambda^2, \dots)$$

is an eigenvector of S to the eigenvalue λ. Moreover, recall that eigenvectors to different eigenvalues are linearly independent. Hence the set

$$\{(1, \lambda, \lambda^2, \dots) \mid \lambda \in \mathbb{R}\} \subseteq \ell(\mathbb{N})$$

is an uncountable linearly independent subset. This proves that $\ell(\mathbb{N})$ has no countable basis. □

This exercise shows that there is no hope in finding a countable basis. But we know that there exists a basis, so one might still hope to be able to construct a basis.

However, here is a general rule of thumb: If something has to be uncountable, it might very well be dependant or even equivalent to the *axiom of choice*.

Axiom 2.2 (Axiom of choice). *For any set X consisting of non-empty sets, there exists a choice function $f : X \to \bigcup_{A\in X} A$ that is defined on X and maps each set of X to an element of that set. Formally, the axiom is given by*

$$\forall X \left(\emptyset \notin X \to \exists f \Big(f \in \Big(\bigcup_{A \in X} A \Big)^X \wedge \, \forall A \in X (f(A) \in A) \Big) \right).$$

Recall that when proving that every vector space has a basis, one applies the axiom of choice. It turns out that one cannot get rid of this. Indeed, the axiom of choice is equivalent to the fact that every vector space has a basis.[1] This means that, on a formal level, one could replace Axiom 2.2 with the statement "every vector space has a basis" to obtain the very same theory.

With that being said, how does the concrete example of $\ell(\mathbb{N})$ relate to the axiom of choice? If we do not assume the axiom of choice, then the existence of a basis for $\ell(\mathbb{N})$ is not provable. However, assuming that $\ell(\mathbb{N})$ has a basis does not imply the axiom of choice.[2] Hence the statement "the vector space $\ell(\mathbb{N})$ has a basis" is strictly weaker than the axiom of choice but it requires the axiom of choice to prove it. This tells us that there is no constructive way to find a basis of $\ell(\mathbb{N})$.[3]

We end this discussion with a joke due to the mathematician Jerry L. Bona:

> "The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?"

[1] See `https://en.wikipedia.org/wiki/Axiom_of_choice#Equivalents`.

[2] See `https://math.stackexchange.com/questions/3924289/basis-for-mathbbr-mathbbn-implies-axiom-of-choice`.

[3] See `https://math.stackexchange.com/questions/122571/is-there-a-constructive-way-to-exhibit-a-basis-for-mathbbr-mathbbn/`.