Linear Algebra II
Exercise Class

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1 The Jordan Normal Form

1.1 Exercises

Exercise 1.1. Let $A \in M_{7\times 7}(\mathbb{R})$ and assume that

$$\chi_A(x) = (x-2)^2(x-3)^5$$

and

$$(A - 3I_7)^3 + (A - 3I_7)^4 = 0.$$

What are all the possible Jordan normal forms for A, up to permutation of the Jordan blocks?

Solution. Observe that

$$(A - 3I_7)^3 + (A - 3I_7)^4 = (A - 3I_7)^3(A - 2I_7)$$

and thus we have

$$m_A(x) \mid (x-3)^3(x-2)$$

for the minimal polynomial of A. Hence the possible Jordan normal forms are

$$\begin{pmatrix} 2 & & & & & & \\ & 2 & & & & & \\ & & 3 & 1 & & & \\ & & & 3 & 1 & & \\ & & & & 3 & & \\ & & & & & 3 & \\ & & & & & & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & & & & & & \\ & 2 & & & & & \\ & & 3 & 1 & & & \\ & & & 3 & 1 & & \\ & & & & 3 & & \\ & & & & & 3 & 1 \\ & & & & & & 3 \end{pmatrix}$$

if $m_A(x) = (x-3)^3(x-2)$ and

$$\begin{pmatrix} 2 &&&&&& \\ & 2 &&&&& \\ && 3 & 1 &&& \\ &&& 3 &&& \\ &&&& 3 && \\ &&&&& 3 & \\ &&&&&& 3 \end{pmatrix}, \quad \begin{pmatrix} 2 &&&&&& \\ & 2 &&&&& \\ && 3 & 1 &&& \\ &&& 3 &&& \\ &&&& 3 & 1 & \\ &&&&& 3 & \\ &&&&&& 3 \end{pmatrix}$$

if $m_A(x) = (x-3)^2(x-2)$ and

$$\begin{pmatrix} 2 &&&&&& \\ & 2 &&&&& \\ && 3 &&&& \\ &&& 3 &&& \\ &&&& 3 && \\ &&&&& 3 & \\ &&&&&& 3 \end{pmatrix}$$

if $m_A(x) = (x-3)(x-2)$.

Exercise 1.2. For $n \in \mathbb{N}_0$ consider

$$V_n := \{p \in \mathbb{C}[x] \mid \deg(p) \leq n\}$$

and let $\mathcal{B}_n := (1, x, \dots, x^n)$ be the standard basis of V_n. Define a linear map

$$T_n : V_n \to V_n$$

by

$$T_n p(x) := p(x+1) - p(x)$$

for $p \in V_n$. Now assume $n \geq 1$.

(a) Show that $\operatorname{im}(T_n) \subseteq V_{n-1}$ holds.

(b) Compute $\ker(T_n)$ and conclude that $\operatorname{im}(T_n) = V_{n-1}$ holds.

(c) Show that 0 is the only eigenvalue of T_n and find the Jordan normal form of T_n.

(d) Compute $[T_4]_{\mathcal{B}_4}^{\mathcal{B}_4}$.

(e) Find a polynomial $p \in V_4$ with $T_4 p(x) = x^3$.

(f) Using (e), find an explicit formula for $\sum_{k=1}^{n} k^3$ in terms of $n \in \mathbb{N}_0$.

Solution.

(a) *Proof.* Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in V_n$ be arbitrary and observe that

$$\begin{aligned} T_n p(x) = p(x+1) - p(x) &= \sum_{k=0}^{n} (a_k (x+1)^k - a_k x^k) \\ &= \sum_{k=0}^{n} \left(a_k \sum_{j=0}^{k} \binom{k}{j} x^j - a_k x^k \right) \\ &= \sum_{k=0}^{n} \left(\sum_{j=0}^{k-1} \binom{k}{j} a_k x^j + a_k x^k - a_k x^k \right) \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k-1} \binom{k}{j} a_k x^j, \end{aligned}$$

so $\deg(T_n p) \leq n-1$ which proves $\operatorname{im}(T_n) \subseteq V_{n-1}$. □

(b) *Proof.* Continuing the computation from (a), we obtain

$$\begin{aligned} T_n p(x) &= \sum_{k=0}^{n} \sum_{j=0}^{k-1} \binom{k}{j} a_k x^j \\ &= \sum_{0 \leq j < k \leq n} \binom{k}{j} a_k x^j \\ &= \sum_{j=0}^{n-1} \left(\sum_{k=j+1}^{n} \binom{k}{j} a_k \right) x^j. \end{aligned}$$

Now assume that $T_n p = 0$ which by the above implies

$$\forall\, 0 \leq j < n : \quad \sum_{k=j+1}^{n} \binom{k}{j} a_k = 0.$$

Hence for $j = n-1$ we get

$$\binom{n}{n-1} a_n = 0 \implies a_n = 0.$$

Now for $j = n-2$ we get

$$\binom{n-1}{n-2} a_{n-1} + \binom{n}{n-2} a_n = 0 \implies a_{n-1} = 0.$$

Similarly, this reasoning continuous until we achieve

$$a_n = a_{n-1} = \ldots = a_1 = 0,$$

which proves $\ker(T_n) \subseteq V_0$. On the other hand, it is apparent that T_n sends all constant polynomials to 0, which proves $\ker(T_n) = V_0$, which is a one-dimensional space. Hence

$$\dim(\mathrm{im} T_n) = \dim(V_n) - \dim(\ker T_n) = n$$

and thus by (a) we have $\mathrm{im}(T_n) = V_{n-1}$. $\square$

(c) *Proof.* We have already established that for any $p \in V_n$ we have

$$\deg(T_n p) < \deg(p).$$

Now assume that $p \neq 0$ and that there exists some $\lambda \in \mathbb{C}$ with $T_n p = \lambda p$. Then the formula above implies

$$\deg(\lambda p) < \deg(p),$$

so by comparing coefficients we obtain $\lambda = 0$. Moreover, by (b) we know that

$$g_0 = \dim(\ker T_n) = 1.$$

This tells us that the Jordan normal form of T_n has only one Jordan block corresponding to the only eigenvalue $\lambda = 0$ and is thus equal to $J_{n+1}(0)$. $\square$

(d) Recall that $A := [T]_{\mathcal{B}_4}^{\mathcal{B}_4} \in M_{5\times 5}(\mathbb{C})$ is the unique matrix making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{T_4} & W \\ {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow \wr & & {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow \wr \\ \mathbb{C}^5 & \xrightarrow{L_A} & \mathbb{C}^5 \end{array}$$

Hence we compute:

$$\begin{array}{ccc} 1 & \xmapsto{T_4} & 0 \\ {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow & & \downarrow {\scriptstyle \varphi_{\mathcal{B}_4}^{-1}} \\ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} & & \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \end{array} \quad \begin{array}{ccc} x & \xmapsto{T_4} & 1 \\ {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow & & \downarrow {\scriptstyle \varphi_{\mathcal{B}_4}^{-1}} \\ \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} & & \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \end{array} \quad \begin{array}{ccc} x^2 & \xmapsto{T_4} & 1+2x \\ {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow & & \downarrow {\scriptstyle \varphi_{\mathcal{B}_4}^{-1}} \\ \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} & & \begin{pmatrix}1\\2\\0\\0\\0\end{pmatrix} \end{array}$$

$$\begin{array}{ccc} x^3 & \xmapsto{T_4} & 1+3x+3x^2 \\ {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow & & \downarrow {\scriptstyle \varphi_{\mathcal{B}_4}^{-1}} \\ \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} & & \begin{pmatrix}1\\3\\3\\0\\0\end{pmatrix} \end{array} \quad \begin{array}{ccc} x^4 & \xmapsto{T_4} & 1+4x+6x^2+4x^3 \\ {\scriptstyle \varphi_{\mathcal{B}_4}} \uparrow & & \downarrow {\scriptstyle \varphi_{\mathcal{B}_4}^{-1}} \\ \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} & & \begin{pmatrix}1\\4\\6\\4\\0\end{pmatrix} \end{array}$$

This now yields

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(e) Using (d), finding a polynomial $p \in V_4$ with $T_4 p(x) = x^3$ is reduced to solving

$$Av = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

for $v \in \mathbb{C}^5$. A possible solution is given by $v = (0, 0, \frac{1}{4}, -\frac{1}{2}, \frac{1}{4})$ and thus the polynomial

$$p(x) = \frac{1}{4}x^2 - \frac{1}{x}x^3 + \frac{1}{4}x^4$$

fulfills $T_4 p(x) = x^3$.

(f) Using the polynomial p found in (e), we may now write

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} T_4 p(k) = \sum_{k=1}^{n} (p(k+1) - p(k)) \\ &= p(n+1) - p(1) = \frac{1}{4}(n+1)^2 - \frac{1}{2}(n+1)^3 + \frac{1}{4}(n+1)^4 \\ &= \frac{1}{4}(n+1)^2[1 - 2(n+1) + (n+1)^2] = \frac{1}{4}(n+1)^2 n^2 \\ &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2. \end{aligned}$$

Observe that this formula also holds for $n = 0$.