Linear Algebra II
Exercise Class

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1 Exercise Sheet 15

Exercise 6 A complex number $\zeta \in \mathbb{C}$ is said to be a *n-th root of unity* if $\zeta^n = 1$ and ζ is said to be a *primitive* n-th root of unity if additionally

$$\forall 1 \leq m < n: \quad \zeta^m \neq 1.$$

The *n-th cyclotomic polynomial* (*Kreisteilungspolynom*) $\Phi_n(z) \in \mathbb{Z}[z]$ is the unique polynomial of maximal degree with leading coefficient 1 that is a divisor of $z^n - 1$ and coprime to all $z^d - 1$ for $1 \leq d < n$.
Solution. See `https://metaphor.ethz.ch/x/2024/fs/401-1152-02L/ex/l15.pdf`.

2 Eigenspaces

Let V be a K-vector space.

Definition 2.1. Let $T : V \to V$ be a linear map and λ an eigenvalue of T. The *eigenspace* of λ is given by

$$E_\lambda := \{v \in V \mid Tv = \lambda v\} = \ker(T - \lambda \cdot \mathrm{id}_V).$$

Exercise 2.2. Let V be a $\mathbb{R}$-vector space and $T : V \to V$ a linear map with $T^2 = \mathrm{id}_V$. Prove that

$$V = E_1 \oplus E_{-1}$$

holds.

Proof. Let $v \in V$ be arbitrary. Then

$$v = \frac{v + Tv}{2} + \frac{v - Tv}{2}$$

holds and

$$T\left(\frac{v+Tv}{2}\right)=\frac{Tv+T^2v}{2}=\frac{Tv+v}{2}$$

which implies $\frac{v+Tv}{2}\in E_1$ and similarly $\frac{v-Tv}{2}\in E_{-1}$. This proves $V=E_1+E_{-1}$. Since for any two eigenspaces we have $E_1\cap E_{-1}=\{0\}$, this concludes the proof. □

Remark 2.3. Observe that Exercise 2.2 is a generalization of the fact that any function $f:\mathbb{R}\to\mathbb{R}$ can be uniquely written as the sum of an even and an odd function.
Indeed, let V be the vector space of all functions $f:\mathbb{R}\to\mathbb{R}$ and apply the previous exercise to the map

$$T:V\to V,\ f\mapsto\tilde{f},$$

where we set $\tilde{f}:\mathbb{R}\to\mathbb{R},\ x\mapsto f(-x)$. Clearly $T^2=\mathrm{id}_V$ and the eigenspaces E_1 and E_{-1} precisely correspond to the even and odd functions.

3 Algebraic and Geometric Multiplicity

Let V be finite-dimensional.

3.1 Recap

Definition 3.1. Let $T:V\to V$ be a linear map and λ an eigenvalue of T.

- The *geometric multiplicity* of λ is defined by $g_\lambda:=\dim E_\lambda$.
- The *algebraic multiplicity* of λ is defined by $a_\lambda:=\max\{k\in\mathbb{N}\mid(\lambda-x)^k \text{ divides } \chi_T(x)\}$.

Recall 3.2. We always have $g_\lambda\leq a_\lambda$. Moreover, if $\lambda_1,\dots,\lambda_k$ are all eigenvalues of T then T is diagonalizable if and only if $g_{\lambda_i}=a_{\lambda_i}$ holds for all $1\leq i\leq k$. In this case, we also have $V=\bigoplus_{i=1}^k E_{\lambda_i}$.

Exercise 3.3. Compute the determinant of the matrix

$$M:=\begin{pmatrix}2&1&\dots&1\\1&2&\dots&\vdots\\\vdots&\vdots&\ddots&1\\1&\dots&1&2\end{pmatrix}\in M_{n\times n}(\mathbb{Q}).$$

Solution. Define

$$A:=\begin{pmatrix}1&\dots&1\\\vdots&\ddots&\vdots\\1&\dots&1\end{pmatrix}$$

and observe that $M = A + I_n$, so

$$\det(M) = \det(A + I_n) = \chi_A(-1).$$

Hence it suffices to determine χ_A. To this end, observe that the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots , \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

are $n-1$ linearly independent eigenvectors of A to the eigenvalue 0. Hence we have

$$a_0 \geq g_0 = \dim E_0 \geq n - 1$$

which by definition of the algebraic multiplicity implies

$$(-x)^{n-1} \mid \chi_A(x).$$

Moreover, the vector $(1, \dots, 1)^T$ is an eigenvector of A to the eigenvalue n, so by similar reasoning we have

$$n - x \mid \chi_A(x).$$

Now since $\deg(\chi_A) = n$ implies

$$\chi_A(x) = (-x)^{n-1}(n - x)$$

and thus we arrive at

$$\det(M) = \chi_A(-1) = n + 1.$$