Linear Algebra II
Exercise Class

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1 Inner Products and Adjoints

Definition 1.1. Let V be an arbitrary inner product space and $T : V \to V$ a linear map. T is called *unitary* if $T^*T = TT^* = \mathrm{id}_V$ holds.

1.1 Exercises

Exercise 1.2. Let V be an arbitrary inner product space and $T : V \to V$ linear. Prove that T is unitary if and only if T is a surjective isometry.[1]

Proof. "$\Longrightarrow$". Assume that T is unitary, so by definition we have $T^*T = TT^* = \mathrm{id}_V$. Then T is surjective because $TT^*V = \mathrm{id}_V(V) = V$. Moreover, for any $v \in V$ we have

$$\|v\|^2 = \langle v, v\rangle = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2,$$

so T is also an isometry.

"$\Longleftarrow$". Assume that T is a surjective isometry. Since T is an isometry, we have

$$\forall v, w \in V : \quad \langle Tv, Tw\rangle = \langle v, w\rangle$$

by lemma 18.1.4. from lecture. Hence for any $v, w \in V$ we

$$\langle v, w\rangle = \langle T^*Tv, w\rangle$$

[1] In lecture, you saw that the situation simplifies if $\dim(V) < \infty$.

and taking $w = v - T^*Tv$ implies

$$\|v - T^*Tv\| = 0.$$

This proves $T^*T = \mathrm{id}_V$.

Now $v \in V$ be arbitrary and by surjectivity of T choose some $w \in V$ with $Tw = v$. Then we have

$$w = T^*Tw = T^*v$$

and thus multiplying by T yields,

$$v = Tw = TT^*v$$

which proves $TT^* = \mathrm{id}_V$, so T is unitary. $\square$

Exercise 1.3. Fix an integer $n \geq 1$ and define

$$P_n := \{f : \mathbb{Z} \to \mathbb{C} \mid \forall k \in \mathbb{Z} : \ f(k+n) = f(k)\}.$$

(a) Show that P_n is a subspace of the $\mathbb{C}$-vector space of functions $f : \mathbb{Z} \to \mathbb{C}$.

(b) Show that

$$\langle f, g \rangle := \frac{1}{n} \sum_{k=0}^{n-1} f(k)\overline{g(k)}$$

defines an inner product on P_n.

(c) For every integer $0 \leq j < n$ define

$$e_j(k) := e^{\frac{2\pi i j}{n} k}$$

for $k \in \mathbb{Z}$. Show that $e_1, \ldots, e_n$ is an orthonormal basis of P_n.

(d) For any $f \in P_n$, define

$$\widehat{f} : \mathbb{Z} \to \mathbb{C}$$

by

$$\widehat{f}(l) := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(k) e^{-\frac{2\pi i k}{n} l}.$$

Show that $\widehat{f} \in P_n$ and that the map

$$\mathcal{F} : P_n \to P_n,\ f \mapsto \widehat{f}$$

is linear.

(e) Compute the adjoint of $\mathcal{F}$. If $\mathcal{F}$ self-adjoint, normal or unitary?

Solution.

(a) This is just a formal verification.

(b) *Proof.* Linearity in the first component and anti-symmetry are immediate. For the last axiom, observe that for any $f \in P_n$ we have

$$\langle f, f \rangle = \frac{1}{n} \sum_{k=0}^{n-1} |f(k)|^2 \geq 0,$$

where equality holds if and only if there exists some $0 \leq k < n$ with $f(k) \neq 0$. But since f is n-periodic, this is equivalent to $f = 0$. □

(c) Observe that for any $0 \leq j < n$ and $k \in \mathbb{Z}$ we have

$$e_j(k+n) = e^{\frac{2\pi i j}{n}(k+n)} = e^{\frac{2\pi i j}{n}k} e^{2\pi i j} = e^{\frac{2\pi i j}{n}k} = e_j(k),$$

so we have $e_j \in P_n$. Moreover, for $0 \leq j, l < n$ we have

$$\begin{aligned} \langle e_j, e_l \rangle &= \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{2\pi i j}{n}k} e^{-\frac{2\pi i l}{n}k} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{2\pi i k}{n}(j-l)} \\ &= \begin{cases} 1 & \text{if } j = l, \\ \frac{1}{n} \frac{1-(e^{\frac{2\pi i k}{n}(j-l)})^n}{1-e^{\frac{2\pi i k}{n}(j-l)}} = 0 & \text{if } j \neq l, \end{cases} \end{aligned}$$

which proves that $e_1, \ldots, e_n$ forms an orthonormal system, in particular these n vectors are linearly independent. Now since $\dim(P_n) = n$ holds,[2] this proves that $e_1, \ldots, e_n$ indeed forms an orthonormal basis of P_n.

[2] One may establish this for example by explicitly writing down an isomorphism $P_n \cong \mathbb{C}^n$.

(d) This is again a formal verification.

(e) Observe that for any two $f, g \in P_n$ we have

$$\begin{aligned}
\langle \mathcal{F}(f), g \rangle &= \frac{1}{n} \sum_{k=0}^{n-1} \widehat{f}(k)\overline{g(k)} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} f(l) e^{-\frac{2\pi i l}{n} k} \right) \overline{g(k)} \\
&= \frac{1}{n} \sum_{l=0}^{n-1} f(l) \overline{\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{\frac{2\pi i k}{n} l} g(k)}.
\end{aligned}$$

Hence if we define

$$\widetilde{g}(l) := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(k) e^{\frac{2\pi i k}{n} l}$$

then the adjoint of $\mathcal{F}$ is given by

$$\mathcal{F}^* g = \widetilde{g}.$$

Note that this proves that $\mathcal{F}$ is unitary. Indeed, we have

$$\begin{aligned}
\mathcal{F}^* \mathcal{F} f(l) &= \mathcal{F}^* \widehat{f}(l) \\
&= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \widehat{f}(k) e^{\frac{2\pi i k}{n} l} \\
&= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f(j) e^{-\frac{2\pi i j}{n} k} e^{\frac{2\pi i k}{n} l} \\
&= \frac{1}{n} \sum_{j=0}^{n-1} f(j) \underbrace{\sum_{k=0}^{n-1} e^{\frac{2\pi i k}{n} (l-j)}}_{= n\, \delta_{l,j}} \\
&= \sum_{j=0}^{n-1} f(j) \delta_{l,j} = f(l)
\end{aligned}$$

by the same computation as in (c). This proves $\mathcal{F}^* \mathcal{F} = \mathrm{id}_{P_n}$ and thus $\mathcal{F}$ is unitary since P_n is finite-dimensional.