Linear Algebra II
Exercise Class

Eric Ceglie

18. March 2024

1 The Minimal Polynomial

1.1 Recap and Some Theory

Let V be a K-vector space and $T : V \to V$ a linear map.

Theorem 1.1. *For every monic polynomial $\varphi \in K[x]$ the following are equivalent:*

- $\varphi(T) = 0$ *and for all* $\psi \in K[x]$ *with* $\psi(T) = 0$ *we have* $\varphi \mid \psi$,
- $\varphi(T) = 0$ *and for all* $\psi \in K[x] \setminus \{0\}$ *with* $\psi(T) = 0$ *we have* $\deg\varphi \leq \deg\psi$.

Moreover, such a polynomial is uniquely determined, called the minimal polynomial *of T denoted by m_T.*

Theorem 1.2. *If $\dim_K(V) < \infty$ then the minimal polynomial m_T always exists.*

Proof idea. Recall that if $n := \dim_K(V) < \infty$ then

$$\mathrm{End}_K(V) \cong M_{n\times n}(K),$$

so $\dim_K(\mathrm{End}_K(V)) = n^2 < \infty$. Hence $T^0, T, T^2, \dots, T^{n^2}$ are linearly independent, which provides us with a polynomial $\varphi \in K[x]$ satisfying $\varphi(T) = 0$. Taking the minimal monic polynomial with this property yields the minimal polynomial. ∷

Example 1.3. Note that in general the minimal polynomial might not exist. The following will provide an explanation for this fact.

Proposition 1.4. *Let $\varphi \in K[x]$ be arbitrary with $\varphi(T) = 0$. Then for every eigenvalue $\lambda \in K$ of T we have $\varphi(\lambda) = 0$.*

Proof. Let $\lambda \in K$ be an eigenvalue of T and $v \in V \setminus \{0\}$ a corresponding eigenvector. Now recall that $\varphi(T) = 0 = 0_{\mathrm{End}_K(V)}$ is a map, namely the map given by $v \mapsto 0$. Now write $\varphi(x) = \sum_{k=0}^{m} a_k x^k$ for $a_k \in K$. Then

$$\begin{aligned} 0 = 0_{\mathrm{End}_K(V)}(v) &= \varphi(T)(v) \\ &= \left(\sum_{k=0}^{m} a_k T^k\right)(v) = \sum_{k=0}^{m} a_k T^k(v) \\ &= \sum_{k=0}^{m} a_k \lambda^k v = \varphi(\lambda) v. \end{aligned}$$

Since $v \neq 0$ this implies $\varphi(\lambda) = 0$. □

Corollary 1.5. *An endomorphism of an infinite-dimensional vector space with infinitely many different eigenvalues has no minimal polynomial.*

Proof. As by definition we require $m_T(T) = 0$ for the minimal polynomial and every polynomial has only a finite number of roots, the claim follows from Proposition 1.4. □

Example 1.6. With Corollary 1.5 we now know how to find an endomorphism for which its minimal polynomial does not exist.
Consider the vector space of real-valued sequences $V := \mathbb{R}^{\mathbb{N}}$ and the linear map

$$T : V \to V,\ (x_n)_n \mapsto (n x_n)_n.$$

Then every $n \in \mathbb{N}$ is an eigenvalue and thus the minimal polynomial of T does not exist by Corollary 1.5.
Another interesting example is the derivative operator

$$D : \mathbb{R}[x] \to \mathbb{R}[x],\ f \mapsto f'.$$

If there was a polynomial $\varphi \in \mathbb{R}[x]$ with $\varphi(D) = 0$ then all polynomials would be solutions to the same linear differential equation with constant coefficients given by

$$\varphi(D)(y) = 0,$$

which is clearly not possible.

Exercise 1.7. Let

$$A := \begin{pmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

.

(a) Find the characteristic polynomial χ_A.

(b) Find the minimal polynomial m_A.

(c) Compute $2A^5 - 3A^3 + A^2 - 7I_3$.

Solution.

(a) We compute

$$\begin{aligned}\chi_A(x) = \det(xI_3 - A) &= \det \begin{pmatrix} x & 2 & 2 \\ -1 & x-3 & -1 \\ 0 & 0 & x-2 \end{pmatrix} \\ &= (x^2 - 3x + 2)(x-2) = (x-1)(x-2)^2.\end{aligned}$$

(b) Observe that since $m_A \mid \chi_A$ part (a) implies that we only have two options for the minimal polynomial, namely

$$(x-1)(x-2) \quad \text{or} \quad (x-1)(x-2)^2.$$

Since

$$(A - I_3)(A - 2I_3) = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get $m_A(x) = (x-1)(x-2)$.

(c) Note that by part (b) we know that

$$0 = (A-1)(A-2) = A^2 - 3A + 2I_3,$$

which implies

$$A^2 = 3A - 2I_3.$$

This allows compute

$$\begin{aligned}
2A^5 - 3A^3 + A^2 - 7I_3 &= 2AA^2A^2 - 3AA^2 + A^2 - 7I_3 \\
&= 2A(3A - 2I_3)^2 - 3A(3A - 2I_3) + (3A - 2I_3) - 7I_3 \\
&= 2A(9A^2 - 12A + 4I_3) - 3(3A^2 - 2A) + 3A - 9I_3 \\
&= 2A(9(3A - 2I_3) - 12A + 4I_3) - 3(3(3A - 2I_3) - 2A) + 3A - 9I_3 \\
&= 2A(15A - 14I_3) - 21A + 18I_3 + 3A - 9I_3 \\
&= 30A^2 - 28A - 18A + 9I_3 \\
&= 30(3A - 2I_3) - 46A + 9I_3 \\
&= 90A - 60I_3 - 46A + 9I_3 \\
&= 44A - 51I_3 \\
&= \begin{pmatrix} 0 & -88 & -88 \\ 44 & 132 & 44 \\ 0 & 0 & 88 \end{pmatrix} - \begin{pmatrix} 51 & 0 & 0 \\ 0 & 51 & 0 \\ 0 & 0 & 51 \end{pmatrix} = \begin{pmatrix} -51 & -88 & -88 \\ 44 & 81 & 44 \\ 0 & 0 & 37 \end{pmatrix}.
\end{aligned}$$

Exercise 1.8. Show that every matrix $A \in M_{n\times n}(K)$ with minimal polynomial $\chi_A(x) = x^n$ is nilpotent, i.e. $\exists m \in \mathbb{N} : A^m = 0$.

Proof. The theorem of Cayley-Hamilton immediately implies $A^n = \chi_A(A) = 0$. □

2 Jordan Normal Form

Let V be a K-vector space over an algebraically closed field K.

2.1 Recap

Theorem 2.1. *Let $T : V \to V$ be a linear map. Then there exists a basis $\mathcal{B}$ of V such that*

$$[T]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} J_{n_1}(\alpha_1) & & & \\ & J_{n_2}(\alpha_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\alpha_k) \end{pmatrix}$$

for $n_1, \dots, n_k \geq 1$ with $n_1 + \ldots + n_k = n$ and $\alpha_1, \dots, \alpha_k \in K$. If $\lambda \in K$ is an eigenvalue of T then

$$g_\lambda = |\{1 \leq j \leq k \mid \alpha_k = \lambda\}|$$

holds for the geometric multiplicity of λ. Moreover, the minimal polynomial of T is given by

$$m_T(x) = \prod_{\substack{\lambda \in K \\ eigenvalue}} (x - \lambda)^{s(\lambda)}, \tag{1}$$

where $s(\lambda) := \max\{n_j \mid 1 \leq j \leq k : \alpha_j = k\}$ *is the size of the biggest Jordanblock corresponding to* λ*.*

2.2 Exercises

Exercise 2.2. Recall that a matrix $A \in M_{n\times n}(K)$ is called idempotent if $A^2 = A$ holds. Show that every idempotent matrix is diagonalizable without directly using the concept of eigenvectors.

Proof. By assumption we have $A^2 - A = 0$ and thus $m_A \mid x(x-1)$ for the minimal polynomial. Hence the minimal polynomial factors into linear factors with multiplicity 1 and roots in $\{0, 1\}$, so using (1) and Theorem 2.1 A is diagonalizable with values in $\{0, 1\}$ on the diagonal. □