Linear Algebra II
Exercise Class

Eric Ceglie

22. April 2024

1 Normed Vector Spaces

1.1 Recap

Let $K \in \{\mathbb{R}, \mathbb{C}\}$ and let V be a K-vector space.

Definition 1.1. A *norm* on V is a map

$$\|\cdot\| : V \to \mathbb{R}_{\geq 0},\ v \mapsto \|v\|$$

such that for all $v, w \in V$ and $\alpha \in \mathbb{R}$ we have

(1) $\|v + w\| \leq \|v\| + \|w\|$,

(2) $\|\alpha v\| = |\alpha| \|v\|$,

(3) if $\|v\| = 0$ then $v = 0$.

1.2 Exercises

Exercise 1.2. Let V be a finite-dimensional $\mathbb{R}$-vector space. For any subset $E \subseteq V^*$ define

$$\|v\|_{1,E} := \sum_{\ell \in E} |\ell(v)|,$$
$$\|v\|_{\infty,E} := \max_{\ell \in E} |\ell(v)|$$

for $v \in V$.

(a) For an arbitrary subset $E \subseteq V^*$, which of the three axioms from Definition 1.1 are satisfied by $\|\cdot\|_{1,E}$ and $\|\cdot\|_{\infty,E}$. For the ones you are not able to prove, find a specific example for which it fails.

(b) Find a condition for $E \subseteq V^*$ which is equivalent to $\|\cdot\|_{1,E}$ and $\|\cdot\|_{\infty,E}$ both being a norm and prove it.

(c) Construct a norm $\|\cdot\|$ on the $\mathbb{R}$-vector space $\mathbb{R}^2$, for which the unit ball

$$\{v \in \mathbb{R}^2 \mid \|v\| \leq 1\}$$

is a regular hexagon.

Solution.

(a) We claim that axioms (1) and (2) always hold. The this end, let $v, w \in V$ and $\alpha \in K$ be arbitrary and observe that

$$\begin{aligned} \|v+w\|_{1,E} &= \sum_{\ell \in E} |\ell(v+w)| \\ &= \sum_{\ell \in E} |\ell(v) + \ell(w)| \\ &\leq \sum_{\ell \in E} (|\ell(v)| + |\ell(w)|) \\ &= \|v\|_{1,E} + \|w\|_{1,E} \end{aligned}$$

and similarly

$$\begin{aligned} \|v+w\|_{\infty,E} &= \max_{\ell \in E} |\ell(v+w)| \\ &\leq \max_{\ell \in E} (|\ell(v)| + |\ell(w)|) \\ &\leq \max_{\ell \in E} |\ell(v)| + \max_{\ell \in E} |\ell(w)| \\ &= \|v\|_{\infty,E} + \|w\|_{\infty,E}, \end{aligned}$$

where we also used that for a arbitrary finite sets $A, B \subseteq \mathbb{R}$ we have $\max_{a \in A, b \in B}(a+b) \leq \max_{a \in A} a + \max_{b \in B} b$. This proves axiom (1) in both cases. Axiom (2) is just a formal verification.

Now we claim that (3) does not hold in general. To this end, consider $V := \mathbb{R}^2$ as an $\mathbb{R}$-vector space and the standard basis b_1, b_2 of V. Now take $E := \{b_1^*\} \subseteq V^*$. Then we have $b_1^*(b_2) = 0$ and thus

$$\|b_2\|_{1,E} = \|b_2\|_{\infty,E} = |b_1^*(b_2)| = 0$$

but $b_2 \neq 0$, so in this case axiom (3) does not holds.

(b) *Claim.* For a subset $E \subseteq V^*$ the functions $\|\cdot\|_{1,E}$ and $\|\cdot\|_{\infty,E}$ are both norms if and only if we have $\langle E \rangle = V^*$.

Proof. We present a detailed proof for $\|\cdot\|_{1,E}$ as for $\|\cdot\|_{\infty,E}$ it works in the very same manner. Also note that by (a) it only remains to verify the (3) axiom of Definition 1.1.

Let $b_1, \dots, b_n$ be a basis of V and let $E \subseteq V^*$ be arbitrary.

"$\Longleftarrow$". Assume that $\langle E \rangle = V^*$ holds and let $v \in V$ be arbitrary with $\|v\|_{1,E} = 0$. Our goal now is to conclude that then $v = 0$ holds. To this end, write $v = \sum_{k=1}^n \alpha_k b_k$ for $\alpha_k \in \mathbb{R}$ and let $1 \leq k \leq n$ be arbitrary. Observe that by definition of the dual basis we now have $\alpha_k = b_k^*(v)$. Since $\langle E \rangle = V^*$, we can find $\beta_1, \dots, \beta_m \in \mathbb{R}$ and $\ell_1, \dots, \ell_m \in E$ with

$$b_k^* = \sum_{j=1}^m \beta_j \ell_j.$$

But now we may estimate

$$\begin{aligned} |\alpha_k| &= |b_k^*(v)| \\ &= \left| \sum_{j=1}^m \beta_j \ell_j(v) \right| \\ &\leq \sum_{j=1}^m |\beta_j| |\ell_j(v)| \\ &\leq \max_{1 \leq j \leq m} |\beta_j| \sum_{\ell \in E} |\ell(v)| \\ &= \max_{1 \leq j \leq m} |\beta_j| \|v\|_{1,E} = 0. \end{aligned}$$

This proves $\alpha_1 = \ldots = \alpha_n = 0$ and thus $v = 0$.

"$\Longrightarrow$". Assume that $\langle E \rangle \subsetneq V^*$ holds. We wish to show that in this case $\| \cdot \|_{1,E}$ is not a norm. To this end, observe that the assumption implies that there exists some $1 \leq k \leq n$ such that $b_k^* \in V^* \setminus \langle E \rangle$. Now assume that there exists a $\ell \in E$ with $\ell(b_k) \neq 0$. Observe that

$$\dim(\ker(\ell)) = \dim(V) - \dim(\operatorname{im}(\ell)) \geq n - 1$$

since $\dim(\operatorname{im}(\ell)) \leq \dim(\mathbb{R}) = 1$. This implies

$$\forall j \neq k : \quad \ell(b_j) = 0.$$

But then we would have

$$b_k^* = \ell(b_k)^{-1} \ell,$$

contradicting $b_k^* \notin \langle E \rangle$. Hence we established that

$$\forall \ell \in E : \quad \ell(b_k) = 0$$

and thus

$$\|b_k\|_{1,E} = \sum_{\ell \in E} |\ell(b_k)| = 0$$

which proves that $\| \cdot \|_{1,E}$ is not a norm since $b_k \neq 0$. □

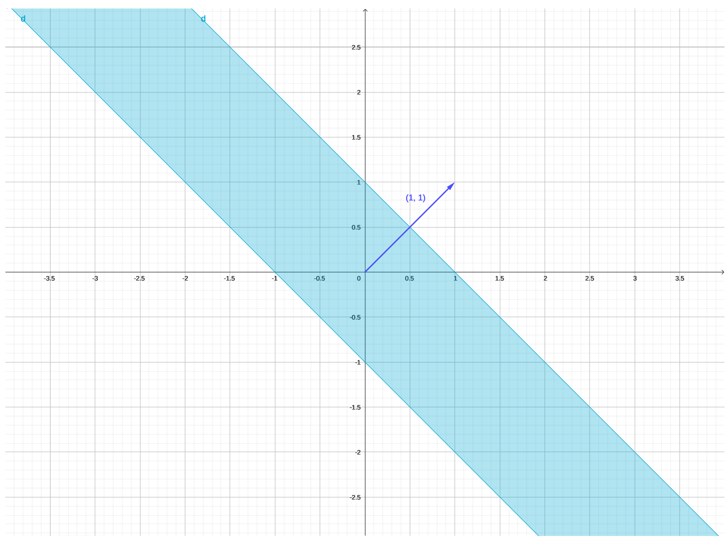
(c) First observe that for any $(a, b) \in \mathbb{R}$ the map

$$\ell_{a,b} : \mathbb{R}^2 \to \mathbb{R}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto ax + by = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

is linear, so it is an element of the dual $(\mathbb{R}^2)^*$. Now note that for the function $\|\cdot\|_{\infty,\{\ell_{1,1}\}}$ all vectors on the line $\{(x, y) \in \mathbb{R}^2 \mid x = -y\}$ yields zero. Hence the set

$$\{v \in \mathbb{R}^2 \mid \|v\|_{\infty,\{\ell_{1,1}\}} = |\ell_{1,1}(x, y)| \leq 1\}$$

looks like a tube of diameter 2 with direction $\binom{1}{-1}$.



Using this intuition, we may use three of these tubes to patch them together into a norm with the desired properties. In particular, for any $(a, b) \in \mathbb{R}^2$ define

$$B_{a,b} := \{(x, y) \in \mathbb{R}^2 \mid |\ell_{a,b}(x, y)| \leq 1\}.$$

Set $\zeta := e^{\frac{2\pi i}{3}}$ and

$$\begin{aligned}(a_1, b_1) &:= (\mathrm{Re}(\zeta), \mathrm{Im}(\zeta)),\\ (a_2, b_2) &:= (\mathrm{Re}(\zeta^2), \mathrm{Im}(\zeta^2)),\\ (a_3, b_3) &:= (\mathrm{Re}(\zeta^3), \mathrm{Im}(\zeta^3)) = (1, 0).\end{aligned}$$

Then the norm $\|\cdot\|_{\infty,E}$ given by the set $E := \{\ell_{a_1,b_1}, \ell_{a_2,b_2}, \ell_{a_3,b_3}\}$ has the desired property.

Indeed, we have $\langle E\rangle = (\mathbb{R}^2)^*$, so by (b) it is a norm and for any $v \in \mathbb{R}^2$ we have

$$\begin{aligned}\{v \in \mathbb{R}^2 \mid \|v\|_{\infty,E} \leq 1\} &= \{v \in \mathbb{R}^2 \mid \max_{1\leq k\leq 3} |\ell_{a_k,b_k}(v)| \leq 1\} \\ &= \bigcap_{1\leq k\leq 3} \{v \in \mathbb{R}^2 \mid |\ell_{a_k,b_k}(v)| \leq 1\} \\ &= \bigcap_{1\leq k\leq 3} B_{a_k,b_k}.\end{aligned}$$

Hence the set unit ball $\{v \in \mathbb{R}^2 \mid \|v\|_{\infty,E} \leq 1\}$ is a regular hexagon and looks as follows.