Linear Algebra II
Exercise Class

Eric Ceglie

25. March 2024

1 Computing the Jordan Normal Form

General Recipe.

(1) Compute the characteristic polynomial and determine the eigenvalues with their algebraic multiplicities.

(2) Compute the eigenspaces and determine the geometric multiplicities.

(3) Compute the "higher eigenspaces" $\ker((A-\lambda I)^k)$ until you obtain the generalized eigenspaces $\widetilde{E}_\lambda$. In this step, always make sure that the basis of $\ker((A-\lambda I)^{k-1})$ is included in the basis of $\ker((A-\lambda I)^k)$.

(4) Build the corresponding Jordan chains.

(5) Put everything into a transformation matrix and verify that it works.

We will illustrate how this recipe works by considering two examples.

Exercise 1.2. Compute the Jordan normal form J of the matrix

$$A := \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and find a corresponding transformation matrix $U \in \mathrm{GL}_4(\mathbb{C})$ with $U^{-1}AU = J$.

Solution. We apply the recipe as follows.

(1) Compute

$$\chi_A(x) = (x-2)(x-3)^3.$$

Hence A has the eigenvalues

$$\lambda_1 := 2, \quad \lambda_2 := 3$$

with algebraic multiplicities

$$a_{\lambda_1} = 1, \quad a_{\lambda_2} = 3.$$

(2) We have

$$E_{\lambda_1} = \ker(A - 2I_4) = \ker \begin{pmatrix} 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

and

$$E_{\lambda_2} = \ker(A - 3I_4) = \ker \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle.$$

Hence we also found the geometric multiplicities

$$g_{\lambda_1} = 1, \quad g_{\lambda_2} = 2.$$

Note that, in this case at least, this information suffices to conclude that the Jordan normal form must be given by

$$J := \begin{pmatrix} 2 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}.$$

We still need to find a corresponding transformation matrix.

(3) Recall that $\dim(\widetilde{E}_\lambda) = a_\lambda$. Hence for λ_1 we already found $\widetilde{E}_{\lambda_1} = E_{\lambda_1}$. For λ_2 compute

$$\ker(A - 3I_4)^2 = \ker \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \ker \begin{pmatrix} 1 & -2 & -2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\rangle.$$

Note that the blue vectors are the same as in step (2), here we only added a new linearly independent vector. Since $\dim(\ker(A - 3I_4)^2) = 3$, we also found $\widetilde{E}_{\lambda_2}$.

(4) Since $\dim(\widetilde{E}_{\lambda_1}) = 1$ we can just set

$$v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

as found in step (2). This is all we need to do for λ_1.

Now we turn to λ_2, which needs more work. We want to construct the following Jordan chains

$$\begin{array}{ccc} w_2 & & \in \ker(A - 3I_4)^2 \\ \big\downarrow{\scriptstyle (A-3I_4)\cdot} & & \\ w_1 & u_1 & \in \ker(A - 3I_4) \end{array}$$

forming a basis of $\widetilde{E}_{\lambda_2}$. This works as follows:

- Choose w_2 such that $w_2 \in \ker(A - 3I_4)^2 \setminus \ker(A - 3I_4)$, so the only natural choice is

  $$w_2 := \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

  found in step (3).
- Now set

  $$w_1 := (A - 3I_4)w_2 = \begin{pmatrix} 8 \\ 2 \\ 2 \\ 0 \end{pmatrix} \in \ker(A - 3I_4) = E_{\lambda_2}.$$

  This completes the first Jordan chain.
- Choose u_1 such that $u_1 \in \ker(A - 3I_4) = E_{\lambda_2}$ and $u_1 \notin \langle w_1 \rangle$. A natural choice would be

  $$u_1 := \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

  found in step (2).

(5) Set

$$U := (v_1, u_1, w_1, w_2) = \begin{pmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and verify that we indeed have

$$U^{-1}AU = J = \begin{pmatrix} 2 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}.$$

Note that you could equivalently just verify that $AU = UJ$ holds.

Try to figure out *why* this actually works. How does the order of the vectors v_1, u_1, w_1, w_2 in U influence the Jordan normal form? What happens if we choose $U' := (v_1, u_1, w_2, w_1)$? To answer this questions it might be useful to put this into the perspective of the fundamental concept of a transformation matrix (maybe recall the corresponding section in `https://n.ethz.ch/~eceglie/downloads/us_LA1/notes/LA1_Eric_Ceglie_061123.pdf`).

We consider another exercise which might explain better how to find the Jordan chains.

Exercise 1.3. Compute the Jordan normal form J of the matrix

$$A := \begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and find a corresponding transformation matrix $U \in \mathrm{GL}_4(\mathbb{C})$ with $U^{-1}AU = J$.

Solution.

(1) Compute

$$\chi_A(x) = (x-1)^5,$$

so we only have the eigenvalue $\lambda := 1$ with $a_\lambda = 5$.

(2) Compute

$$E_\lambda = \ker(A - I_5) = \ker \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle,$$

so we have $g_\lambda = 2$. Note that in this case this is not enough to fully determine the Jordan normal form.

(3) Compute

$$\ker(A - I_5)^2 = \ker \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^2 = \ker \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle,$$

where the blue vectors are the same as in step (2) and we only included two new linearly independent red vectors to obtain a basis. Note that we still have $\dim(\ker(A - I_5)^2) < 5 = a_\lambda$, so we continue our computation

$$\ker(A - I_5)^3 = \ker \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^3 = \ker \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle,$$

where again we only included a new linearly independent vector, the green one, to obtain a basis. Since now $\dim(\ker(A - I_5)^3) = 5 = a_\lambda$, this is now precisely the generalized eigenspace $\widetilde{E}_\lambda$.

(4) We want to construct the following Jordan chains

$$\begin{array}{cccc} v_3 & & & \in \ker(A - I_5)^3 \\ {\scriptstyle (A-I_5)\cdot}\downarrow & & & \\ v_2 & & w_2 & \in \ker(A - I_5)^2 \\ {\scriptstyle (A-I_5)\cdot}\downarrow & & \downarrow{\scriptstyle (A-I_5)\cdot} & \\ v_1 & & w_1 & \in \ker(A - I_5) \end{array}$$

forming a basis of $\widetilde{E}_\lambda$.

- To determine the first chain, start by choosing v_3 such that $v_3 \in \ker(A - I_5)^3 \setminus \ker(A - I_5)^2$, so the only natural choice in this case is

$$v_3 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- Set

$$v_2 := (A - I_5)v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \ker(A - I_5)^2$$

and

$$v_1 := (A - I_5)v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \in \ker(A - I_5) = E_\lambda.$$

This completes the first Jordan chain v_1, v_2, v_3.

- Now in order to guarantee that we get a new chain, choose w_2 such that

$$w_2 \in \ker(A - I_5)^2 \setminus \langle \ker(A - I_5) \cup \{v_2\} \rangle$$
$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \setminus \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle,$$

so both red vectors would work here, say

$$w_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Set

$$w_1 := (A - I_5)w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \ker(A - I_5) = E_\lambda.$$

This completes the second Jordan chain.

(5) Set

$$U := (v_1, v_2, v_3, w_1, w_2) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and verify that we indeed have

$$U^{-1}AU = J = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$