Linear Algebra II
Exercise Class

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1 The Determinant

Let K be a field.

1.1 Recap

Definition 1.1. A function $D : M_{n\times n}(K) \to K$ is called a *determinant map* if it satisfies

(1) $D(I_n) = 1$,

(2) D is n-linear and alternating as a function of the columns.

Theorem 1.2. *For all* $n \geq 1$ *there exists a unique determinant map* det *and it is characterized by*

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sng}(\sigma) \prod_{i=1}^{n} a_{\sigma i,i}$$

for $A \in M_{n\times n}(K)$.

1.2 Computing Determinants

Exercise 1.3. For any $\tau \in S_n$, consider the corresponding *permutation matrix*

$$P_\tau := (\delta_{i,\tau j})_{i,j=1,\ldots,n}.$$

Show that $\det(P_\tau) = \operatorname{sgn}(\tau)$ holds.

Proof. Using Theorem 1.2, we get

$$\begin{aligned}\det(P_\tau) &= \sum_{\sigma\in S_n} \operatorname{sng}(\sigma)\prod_{i=1}^{n}\delta_{\sigma i,\tau j}\\ &= \operatorname{sgn}(\tau) + \sum_{\sigma\in S_n\setminus\{\tau\}} \operatorname{sng}(\sigma)\underbrace{\prod_{i=1}^{n}\delta_{\sigma i,\tau j}}_{=0} = \operatorname{sgn}(\tau)\end{aligned}$$

since if $\sigma \in S_n \setminus \{\tau\}$ then there exists a $i \in \{1,\dots,n\}$ with $\sigma i \neq \tau i$. □

Exercise 1.4. Let $n \geq 1$ and consider the matrix $A_n := (a_{ij})_{i,j=1,\dots,n}$ given by

$$a_{ij} := \begin{cases} 0 & \text{if } i = j \\ i\cdot j & \text{if } i \neq j \end{cases}$$

for all $i, j = 1,\dots,n$. Compute $\det A_n$.
Solution. The given matrix looks as follows

$$A_n = \begin{pmatrix} 0 & 2 & 3 & \dots & n \\ 2 & 0 & 2\cdot 3 & \dots & 2\cdot n \\ 3 & 3\cdot 2 & 0 & \dots & 3\cdot n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n\cdot 2 & \dots & n(n-1) & 0 \end{pmatrix}.$$

We now apply some elementary transformations, leaving the determinant invariant. Subtracting the first row from the second row twice, three times from the third row and so on, we get

$$\begin{pmatrix} 0 & 2 & 3 & \dots & n \\ 2 & -4 & 0 & \dots & 0 \\ 3 & 0 & -9 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & \dots & 0 & -n^2 \end{pmatrix}.$$

Now proceed by adding the $\frac{1}{2}$ times the second row to the first row, $\frac{1}{3}$ times the third row to the first and so on, we get

$$\begin{pmatrix} n-1 & 0 & 0 & \dots & 0 \\ 2 & -4 & 0 & \dots & 0 \\ 3 & 0 & -9 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & \dots & 0 & -n^2 \end{pmatrix},$$

which is now an upper triangular matrix. Hence we obtain the determinant just by multiplying the diagonal entries, i.e. we have

$$\det A_n = (n-1)\prod_{k=2}^{n}(-k^2) = (-1)^{n-1}(n-1)(n!)^2.$$

The following theorem provides a A *very* useful tool for computing determinants.

Theorem 1.5 (Laplace expansion)**.** *Let $A \in M_{n\times n}(K)$ for $n \geq 1$ and for each $i, j = 1, \dots, n$ denote by $A_{ij} \in M_{(n-1)\times(n-1)}(K)$ the matrix obtained by deleting the i-th row and the j-th column from A. Then we have*

$$\forall 1 \leq i \leq n: \quad \det(A) = \sum_{j=1}^{n}(-1)^{i+j} a_{ij} \det(A_{ij})$$

and

$$\forall 1 \leq j \leq n: \quad \det(A) = \sum_{i=1}^{n}(-1)^{i+j} a_{ij} \det(A_{ij}).$$

Example 1.6. Let

$$A := \begin{pmatrix} 1 & 0 & 4 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 5 & 7 \end{pmatrix}.$$

Using Laplace expansion 1.5 along the first row, we get

$$\begin{aligned} \det A &= 1 \cdot \det \begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 5 & 7 \end{pmatrix} + 4 \cdot \det \begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \\ &= -2 \cdot 7 + 24 \cdot 7 = 22 \cdot 7 = 154. \end{aligned}$$

Exercise 1.7. Let $(a_k)_{k\geq 1} \subseteq K$ be a sequence and for $n \geq 1$ define the matrix

$$A_n := (a_{\min(i,j)})_{i,j=1,\dots,n}.$$

(a) Show that

$$\det(A_{n+1}) = \det(A_n) \cdot (a_{n+1} - a_n)$$

holds for all $n \geq 1$.

(b) Find an explicit formula for $\det(A_n)$ and prove it.

Solution.

(a) *Proof.* The given matrix looks as follows

$$A_{n+1} = \begin{pmatrix} a_1 & a_1 & a_1 & \dots & a_1 \\ a_1 & a_2 & a_2 & \dots & a_2 \\ a_1 & a_2 & a_3 & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_{n+1} \end{pmatrix}.$$

Observe that we may subtract the second to last row from the last row obtaining

$$\begin{pmatrix} a_1 & a_1 & a_1 & \dots & a_1 \\ a_1 & a_2 & a_2 & \dots & a_2 \\ a_1 & a_2 & a_3 & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n+1} - a_n \end{pmatrix},$$

which leaves the determinant invariant. The upper left $n \times n$-block is now precisely A_n and thus, by using Laplace expansion 1.5, we obtain

$$\det(A_{n+1}) = \det(A_n) \cdot (a_{n+1} - a_n)$$

concluding the proof. □

(b) *Claim.* We have $\det(A_n) = a_1 \cdot (a_2 - a_1) \cdots (a_n - a_{n-1})$.

Proof. We use (a) to do a proof by induction. We clearly have $\det(A_1) = a_1$. Now let $n \geq 1$ and assume that the formula already holds for A_n. Using (a) and the induction , we get

$$\det(A_{n+1}) = \det(A_n)(a_{n+1} - a_n) = a_1 \cdot (a_2 - a_1) \cdots (a_n - a_{n-1})$$

concluding the proof. □

1.3 The Adjugate of a Matrix

Definition 1.8. The *adjugate matrix* of a matrix $A \in M_{n\times n}(K)$ is given by

$$\operatorname{adj}(A) := ((-1)^{i+j} \cdot \det(A_{ij}))^T_{1\leq i,j\leq n}.$$

Theorem 1.9. *Let $A \in M_{n\times n}(K)$. Then we have*

$$\det(A)I_n = A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A.$$

Moreover, the matrix A is invertible if and only if $\det(A) \neq 0$ in K and in this case the formula

$$A^{-1} = \det(A)^{-1} \cdot \operatorname{adj}(A) \tag{1}$$

holds.

Example 1.10. Let

$$A := \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Observe that $\det A = 10$ and thus A is invertible and we may apply formula (1) to obtain its inverse. In particular, we have

$$\begin{aligned} A^{-1} &= \det(A)^{-1} \cdot \operatorname{adj}(A) \\ &= \frac{1}{10} \begin{pmatrix} 2 & -2 & 2 \\ -(-2) & 3 & -3 \\ 0 & -(-10) & 0 \end{pmatrix}^T \\ &= \frac{1}{10} \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}. \end{aligned}$$