Linear Algebra II
Exercise Class

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1 Inner Products

1.1 Recap

Definition 1.1. Let V be a $\mathbb{C}$-vector space. An inner product on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C},\ (v, w) \mapsto \langle v, w \rangle$$

such that

(1) $\forall w \in V$ the function $V \to \mathbb{C},\ v \mapsto \langle v, w \rangle$ is linear,

(2) $\forall v, w \in V:\ \langle v, w \rangle = \overline{\langle w, v \rangle}$,

(3) $\forall v \in V \setminus \{0\}:\ \langle v, v \rangle > 0$.

Proposition 1.2. *If $\langle \cdot, \cdot \rangle$ is an inner product on V then $\|v\| := \sqrt{\langle v, v \rangle}$ defines a norm on V.*

Theorem 1.3. *Every finite-dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$ over $\mathbb{C}$ has a orthonormal basis $b_1, \ldots, b_n$. In this case, for any $v \in V$ we have*

$$v = \sum_{k=1}^{n} \langle v, b_k \rangle b_k.$$

1.2 Exercises

Theorem 1.4 (Riesz-representatino theorem)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space of $\mathbb{C}$ and $\ell \in V^*$. Then there exists a unique $v_0 \in V$ such that*

$$\ell(v) = \langle v, v_0 \rangle \tag{1}$$

for all $v \in V$.

Exercise 1.5. Prove Riesz-representatino theorem 1.4.

Proof. Let $\ell \in V^*$ be arbitrary.
UNIQUENESS. Assume $v_0, w_0 \in V$ both satisfy (1), which means that we have

$$\langle u, v_0 \rangle = \langle u, w_0 \rangle$$

for all $u \in V$. Taking $u := v_0 - w_0$ and using linearity in the first component, we obtain $\|v_0 - w_0\|^2 = 0$ and thus $v_0 - w_0 = 0$ by the third axiom of being a norm.
EXISTENCE. By Theorem 1.3, there exists an orthonormal basis $b_1, \ldots, b_n$ of V. Set

$$v_0 := \sum_{k=1}^{n} \overline{\ell(b_k)} b_k$$

and observe that then for any $v \in V$ we have

$$\begin{aligned} \langle v, v_0 \rangle &= \left\langle v, \sum_{k=1}^{n} \overline{\ell(b_k)} b_k \right\rangle \\ &= \sum_{k=1}^{n} \ell(b_k) \langle v, b_k \rangle \\ &= \ell \left(\sum_{k=1}^{n} b_k \langle v, b_k \rangle \right) \\ &= \ell(v), \end{aligned}$$

where we applied the formula from Theorem 1.3 twice and linearity of ℓ. □

Exercise 1.6. Let $(V, \langle \cdot, \cdot \rangle)$ be a $\mathbb{C}$-vector space with an inner product and $T : V \to V$ a linear map.

(a) Assume that we have

$$\forall v, w \in V : \quad \langle Tv, w \rangle = \langle v, Tw \rangle. \tag{2}$$

Show that any eigenvalue of T is real.

(b) Assume that there exists another linear map $S : V \to V$ with $T \circ S = S \circ T = \mathrm{id}_V$ such that

$$\forall v, w \in V : \quad \langle Tv, w \rangle = \langle v, Sw \rangle. \tag{3}$$

Show that for any eigenvalue $\lambda \in \mathbb{C}$ of T we have $|\lambda| = 1$.

(c) Now assume that $V = \mathbb{C}^n$ for some $n \geq 1$ and that $\langle \cdot, \cdot \rangle$ is the standard euclidean inner product. For any matrix $A \in M_{n \times n}(\mathbb{C})$, translate the condition

$$\forall v, w \in V : \quad \langle Av, w \rangle = \langle v, Aw \rangle \tag{4}$$

into a condition for the matrix A, i.e. a condition which does not involve the inner product.

Solution.

(a) *Proof.* Assume that (2) holds and let $\lambda \in \mathbb{C}$ be an eigenvalue of T. Then by definition there exists some $v \in V \setminus \{0\}$ with $Tv = \lambda v$. Now observe that

$$\begin{aligned}\lambda\|v\|^2 &= \lambda\langle v, v\rangle \\ &= \langle \lambda v, v\rangle \\ &= \langle Tv, v\rangle \\ &= \langle v, Tv\rangle \\ &= \langle v, \lambda v\rangle \\ &= \overline{\lambda}\langle v, v\rangle = \overline{\lambda}\|v\|^2.\end{aligned}$$

Since $v \neq 0$ we have $\|v\|^2 > 0$ and thus we may divide by it to obtain

$$\lambda = \overline{\lambda},$$

which proves $\lambda \in \mathbb{R}$. □

(b) *Proof.* Assume that (3) holds and let $\lambda \in \mathbb{C}$ be an eigenvalue of T. Then by definition there exists some $v \in V \setminus \{0\}$ with $Tv = \lambda v$. Now observe that

$$\begin{aligned}\|v\|^2 &= \langle v, v\rangle \\ &= \langle v, STv\rangle \\ &= \langle Tv, Tv\rangle \\ &= \|Tv\|^2 \\ &= \|\lambda v\|^2 \\ &= |\lambda|^2\|v\|^2,\end{aligned}$$

where we used $S \circ T = \mathrm{id}_V$ and (3). Now again since $\|v\|^2 > 0$, we may divide by it to obtain

$$|\lambda|^2 = 1,$$

which implies $|\lambda| = 1$. □

(c) Let $A = (a_{ij})_{i,j=1,\dots,n} \in M_{n\times n}(\mathbb{C})$ be arbitrary and assume (4) holds. Observe that for any $i, j \in \{1, \dots, n\}$ we have

$$\langle Ae_i, e_j\rangle = \left\langle \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}, e_j \right\rangle = a_{ji}$$

and similarly we obtain

$$\langle e_i, Ae_j \rangle = \overline{a_{ij}}.$$

Applying (4) this implies

$$\forall i, j \in \{1, \dots, n\} : \quad a_{ij} = \overline{a_{ji}},$$

which is equivalent to

$$A = \overline{A^T}.$$

Exercise 1.7. Let V be an arbitrary $\mathbb{C}$-vector space. Show that there exists an inner product on V.

Hint. Choose a basis and use the fact that any vector can be uniquely represented as a finite linear combination of basis vectors to define an inner product.

Proof. Let $\mathcal{B} = (b_i)_{i \in I}$ be a basis of V. Then for any two vectors $v, w \in V$ there exists unique values $(\alpha_i)_{i \in I}, (\beta_i)_{i \in I} \subseteq \mathbb{C}$ such that all but finitely many are equal to zero and

$$v = \sum_{i \in I} \alpha_i b_i, \quad w = \sum_{i \in I} \beta_i b_i.$$

As these representations are unique, we can use them to define

$$\langle v, w \rangle := \sum_{i \in I} \alpha_i \overline{\beta_i},$$

which is again a finite sum as all but finitely many coefficients are equal to zero. We now show that this indeed defines an inner product.
Let $v, w, u \in V$ and $\lambda \in \mathbb{C}$ be arbitrary. Write them uniquely as

$$v = \sum_{i \in I} \alpha_i b_i, \quad w = \sum_{i \in I} \beta_i b_i, \quad u = \sum_{i \in I} \gamma_i b_i.$$

(1) We have

$$\begin{aligned} \langle v + \lambda w, u \rangle &= \sum_{i \in I} (\alpha_i + \lambda \beta_i) \overline{\gamma_i} \\ &= \sum_{i \in I} \alpha_i \overline{\gamma_i} + \lambda \sum_{i \in I} \beta_i \overline{\gamma_i} \\ &= \langle v, u \rangle + \lambda \langle w, u \rangle, \end{aligned}$$

which proves linearity in the first component.

(2) We have

$$\begin{aligned}\langle v, w\rangle &= \sum_{i\in I} \alpha_i \overline{\beta_i} \\ &= \sum_{i\in I} \overline{\beta_i \overline{\alpha_i}} \\ &= \overline{\sum_{i\in I} \beta_i \overline{\alpha_i}} \\ &= \overline{\langle w, v\rangle}.\end{aligned}$$

(3) Assume that $v \neq 0$ so there exists some $k \in I$ with $\alpha_k \neq 0$. Then we have

$$\begin{aligned}\langle v, v\rangle &= \sum_{i\in I} \alpha_i \overline{\alpha_i} \\ &= \sum_{i\in I} |\alpha_i|^2 \\ &\geq |\alpha_k|^2 > 0.\end{aligned}$$

This proves that $\langle \cdot, \cdot \rangle$ defines an inner product on V. □