Lecture 10: Estimation

1 State Estimation

In the previous chapters, we assumed that all the state variables of a given systen system were available at each time. In real systems, however, this is not the case: one knows just the output $y(t)$ and the input $u(t)$. Hence, one has to figure out how to get the actual state $x(t)$. The idea is to use an **observer** to get an estimate of $x(t)$, also called $\hat{x}(t)$. A whole course about estimation if offered at IDSC in the master by Prof. D'Andrea: *Recursive Estimation*.

1.1 Preliminary Definitions

Definition 1. A system is said to be *observable* if for any time $T > 0$ it is possible to determine the state of the system $x(T) \in \mathbb{R}^n$ through the measurements of $u(t)$ and $y(t)$, with $t \in [0, T]$.

A pair (A, C) is observable if and only if $\text{rank}(\mathcal{O}) = n = \dim(A)$, where

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{1.1}$$

is the observability matrix.

Definition 2. A pair (A, C) is detectable if all the unobservable modes are stable.

Definition 3. A linear system is **reachable** if for any $x_0, x_\text{f} \in \mathbb{R}^n$, there exists a $T > 0$ and $u(t) : [0, T] \to \mathbb{R}$ such that the corresponding solution satisfies $x(0) = x_0$ and $x(T) = x_\text{f} \in \mathbb{R}^n$.

Reachability test: A system (A, B) is reachable if and only if $\text{rank}(\mathcal{R}) = n$, where $x \in \mathbb{R}^n$ and

$$\mathcal{R} = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}. \tag{1.2}$$

Remark. Note that for continuous linear time invariant systems, controllability is the same as reachability. In general, reachability implies controllability, but not the converse.

1.2 Problem Definition

Given a linear, time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.3}$$

at each time instant t construct an estimate of the state $\hat{x}(t)$ by only measuring the system's inputs and outputs, such that

$$\lim_{t \to \infty} (x(t) - \hat{x}(t)) = 0. \tag{1.4}$$

1.3 The Luenberger Observer

In order to do this, one creates a numerical copy of the system (an observer)

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t)\\ \hat{y}(t) &= C\hat{x}(t) + Du(t),\end{aligned} \tag{1.5}$$

and one observes the dynamics of the estimation error

$$\hat{e} = x(t) - \hat{x}(t). \tag{1.6}$$

It holds

$$\begin{aligned}\dot{\hat{e}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t)\\ &= A\hat{e}(t).\end{aligned} \tag{1.7}$$

If matrix A has all the eigenvalues in the left half-plane, the error $\hat{e}(t)$ will converge to zero, resulting in a correct state estimation. But is this what we want? Essentially, our error is converging to zero because the states of the two systems are designed to converge to zero. In particular, we are not using the output as an information. How can we solve the problem even for unstable systems? For the following, consider the structure reported in Figure 1. Let's add feedback from the measured output by considering the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu + L(y(t) - \hat{y}(t)). \tag{1.8}$$

It holds

$$\begin{aligned}\dot{\hat{e}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ \\ &= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) - L(y(t) - \hat{y}(t))\\ &= Ax(t) - A\hat{x}(t) - L(Cx(t) - C\hat{x}(t))\\ &= (A - LC)\hat{e}(t).\end{aligned} \tag{1.9}$$

With this new equation, one can choose a matrix L such that the matrix $A - LC$ has eigenvalues with negative real parts and hence such that the error $\hat{e}(t)$ will converge to 0. This observer is known as the Luenberger observer.

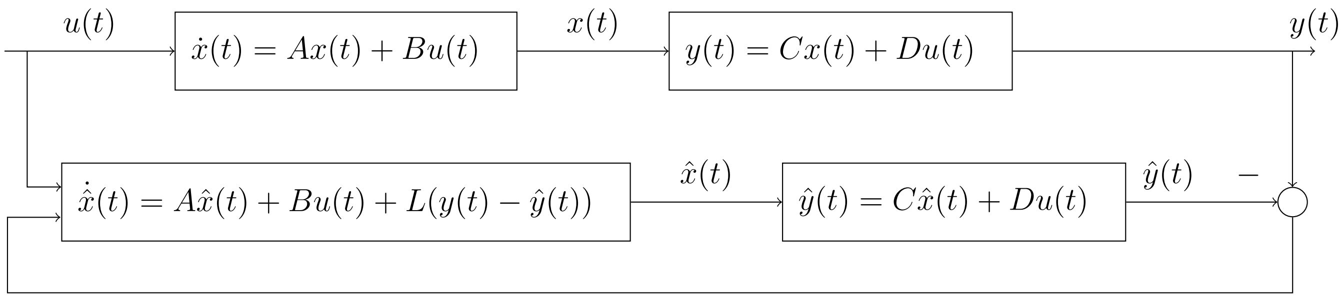

Figure 1: Luenberger Observer

1.3.1 Duality of Estimation and Control

The structure of Equation 1.9 is similar to the one of state feedback we introduce in the previous chapter. Considering the state feedback dynamic equation

$$\dot{x}(t) = (A - BK)x(t), \tag{1.10}$$

one can find in Equation 1.9 some analogies. In particular, it holds

$$\dot{\hat{e}}(t) = (\underbrace{A^\intercal}_{\tilde{A}} - \underbrace{C^\intercal}_{\tilde{B}} \underbrace{L^\intercal}_{\tilde{K}})^\intercal \hat{e}(t). \tag{1.11}$$

Since the problem has the same form, one can use the same methodology to solve it. One recalls that pole placement is allowed if and only if the system is reachable, i.e. if $\text{rank}(\mathcal{R}) = n = \dim(A)$, where $\mathcal{R}$ is the reachability matrix defined in Equation 1.2 By using the analogies, one can define similarly

$$\begin{aligned}\tilde{\mathcal{R}} &= \begin{pmatrix}\tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B}\end{pmatrix} \\ &= \begin{pmatrix}C^\intercal & A^\intercal C^\intercal & \dots & (A^\intercal)^{n-1}C^\intercal\end{pmatrix} \\ &= \begin{pmatrix}C \\ CA \\ \vdots \\ CA^{n-1}\end{pmatrix}^\intercal \\ &= \mathcal{O}^\intercal,\end{aligned} \tag{1.12}$$

where $\mathcal{O}$ is the observability matrix defined in Equation 1.1. Using the well known rule

$$\text{rank}(\mathcal{O}) = \text{rank}(\mathcal{O}^\intercal), \tag{1.13}$$

one can impose $\text{rank}(\mathcal{O})$ to be n in order for observer pole placement to be feasible. Starting from the Ackermann formula for state feedback

$$K = \begin{pmatrix}0 & \dots & 1\end{pmatrix}\mathcal{R}^{-1}p^*_{\text{cl}}(A), \tag{1.14}$$

one can write

$$\begin{aligned}\tilde{K} &= L^\intercal \\ &= \begin{pmatrix}0 & \dots & 0 & 1\end{pmatrix}\tilde{\mathcal{R}}^{-1}p^*_{\text{cl}}(A) \\ \Rightarrow L &= p^*_{\text{cl}}(A)\mathcal{O}^{-1}\begin{pmatrix}0 & \dots & 0 & 1\end{pmatrix}^\intercal.\end{aligned} \tag{1.15}$$

1.3.2 Putting Things Together

Considering Figure 2, one can define the augmented state

$$\tilde{x}(t) = \begin{pmatrix}x(t) \\ \hat{e}(t)\end{pmatrix}. \tag{1.16}$$

The dynamics read

$$\begin{aligned}\dot{\tilde{x}}(t) &= \begin{pmatrix}\dot{x}(t) \\ \dot{\hat{e}}(t)\end{pmatrix} \\ &= \underbrace{\begin{pmatrix}A - BK & BK \\ 0 & A - LC\end{pmatrix}}_{A_{\text{cl}}}\begin{pmatrix}x(t) \\ \hat{e}(t)\end{pmatrix}\end{aligned} \tag{1.17}$$

Since A_{cl} is upper triangular, it holds

$$\sigma(A_{\text{cl}}) = \sigma(A - BK) \cup \sigma(A - LC). \tag{1.18}$$

This is known as the separation principle. Intuitively, this means that control and estimation do not interact with each other, hence can designed independently.

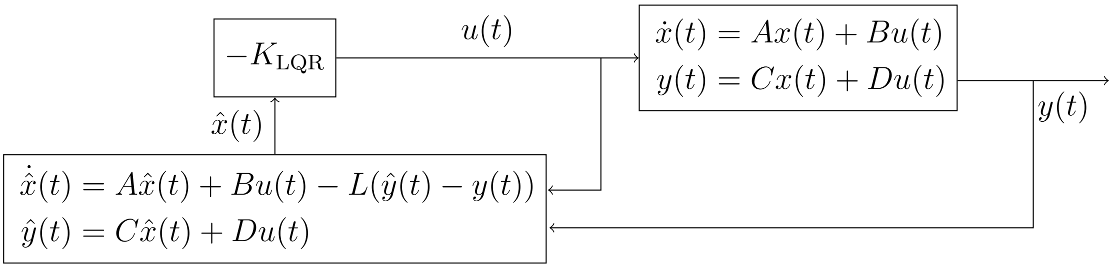

Figure 2: Observer Problem: Closed loop system.

In general one follows this procedure:

1. Design the controller first. Find K to place the poles of $A - BK$ where you desire in the LHP.
2. Design the observer. Find L to place the poles of $A - LC$ in the LHP. As a rule of thumb, make the observer 10 times faster than the controller.

1.4 Linear Quadratic Gaussian (LQG) Control

LQR relies on the assumption that the states are known. How can one integrate the defined estimation procedure in the LQR framework? Is the optimality defined for the LQR method affected by this? In the following, we recall the LQR problem definition and its solution.

1.4.1 LQR Problem Definition

With LQR one wants to find a stabilizing input $u_{\text{LQR}}(t),\ t \in [0, \infty]$ such that

$$u_{\text{LQR}}(t) = \underset{u}{\operatorname{argmin}} \int_0^\infty u(t)^\intercal Ru(t) + x(t)^\intercal Qx(t) + 2x(t)^\intercal Nu(t)\mathrm{d}t \tag{1.19}$$

satisfying

- the dynamics
$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \tag{1.20}$$
and
$$z(t) = Ex(t) + Fu(t), \tag{1.21}$$
with $u(t) \in \mathbb{R}^{m\times 1}$, $z \in \mathbb{R}^{k\times 1}$ and $x \in \mathbb{R}^{n\times 1}$.
- $Q > 0$, $R > 0$, $Q = Q^\intercal$, $R = R^\intercal$, with
$$\begin{aligned} R &= F^\intercal \bar{Q} F + \rho \bar{R}, \quad \rho \in \mathbb{R}^+ \\ Q &= E^\intercal \bar{Q} E \\ N &= E^\intercal \bar{Q} F. \end{aligned} \tag{1.22}$$

1.4.2 LQR Problem Solution

If

1. The system (A, B) is stabilizable and
2. the pair $(\tilde{A}, \tilde{Q}) = (A - BR^{-1}N^\intercal, Q - NR^{-1}N^\intercal)$ is detectable,

then

$$u_{\text{LQR}}(t) = -\underbrace{R^{-1}(N + PB)^\intercal}_{K_{\text{LQR}}} x(t), \tag{1.23}$$

where P is the real, symmetric, positive definite solution of the algebraic Riccati equation (ARE)

$$\underbrace{(A - BR^{-1}N^\intercal)^\intercal}_{\bar{A}^\intercal} P + P \underbrace{(A - BR^{-1}N^\intercal)}_{\bar{A}} + P \underbrace{(-BR^{-1}B^\intercal)}_{\bar{R}} P + \underbrace{(Q - NR^{-1}N^\intercal)}_{\bar{Q}} = 0 \tag{1.24}$$

1.4.3 Simplified Case

It turns out that choosing $N = 0$ results in nice robustness properties. By writing P_∞ instead of P (we are solving the finit horizon), one can simplify the ARE as

$$A^\intercal P_\infty + P_\infty A - P_\infty BR^{-1}B^\intercal P_\infty + Q = 0. \tag{1.25}$$

1.4.4 Stady-state Kalman Filter

The observer problem, i.e. find L in

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \tag{1.26}$$

such that $(A - LC)$ is stable, shows duality with the control problem, i.e.

$$C^\intercal \to B, \quad A^\intercal \to A, \quad L^\intercal \to K. \tag{1.27}$$

Thank to this duality, one can solve the estimation problem by solving the control one. The algebraic Riccati equation for estimation is

$$AP_\infty + P_\infty A^\intercal - P_\infty C^\intercal R^{-1} C P_\infty + Q = 0. \tag{1.28}$$

The matrix L can be found with

$$L^\intercal = R^{-1}CP_\infty \Rightarrow L = P_\infty C^\intercal R^{-1}. \tag{1.29}$$

The duality exists also for the technical conditions for the ARE:

$$\begin{aligned} (A^\intercal, C^\intercal) \text{ stabilizable } &\leftrightarrow (A, C) \text{ detectable .} \\ (A^\intercal, Q) \text{ detectable } &\leftrightarrow (A, Q) \text{ stabilizable .} \end{aligned} \tag{1.30}$$

Deterministic Interpretation

It is worth mentioning that the Kalman filter's theory is not that short. Among others, stochastic interpretation, recursive formulatiom, finite horizon and discrete time implementation represent important topics to be discussed. However, for the aim of this course, we use the deterministic interpretation of Kalman filters. Considering the effect of disturbances/noises on the plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t),\ x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) + n(t), \end{aligned} \tag{1.31}$$

where $w(t)$ represents the process noise and $n(t)$ the measurement noise. The Kalman Filter can be interpreted deterministically as minimizing an uncertainty measure

$$\|x_0\|_2^2 + \int_0^T \|w\|_2^2 + \|n\|_2^2 \mathrm{d}t, \tag{1.32}$$

i.e. estimating the last energy/most likely initial condition, disturbance and measurement noise that justify the measurements.

1.4.5 Summary

The linear quadratic gaussian (LQG) regulator is the union of a LQR controller and a Kalman Filter. One can see such a closed loop system in Figure 3. The closed loop is

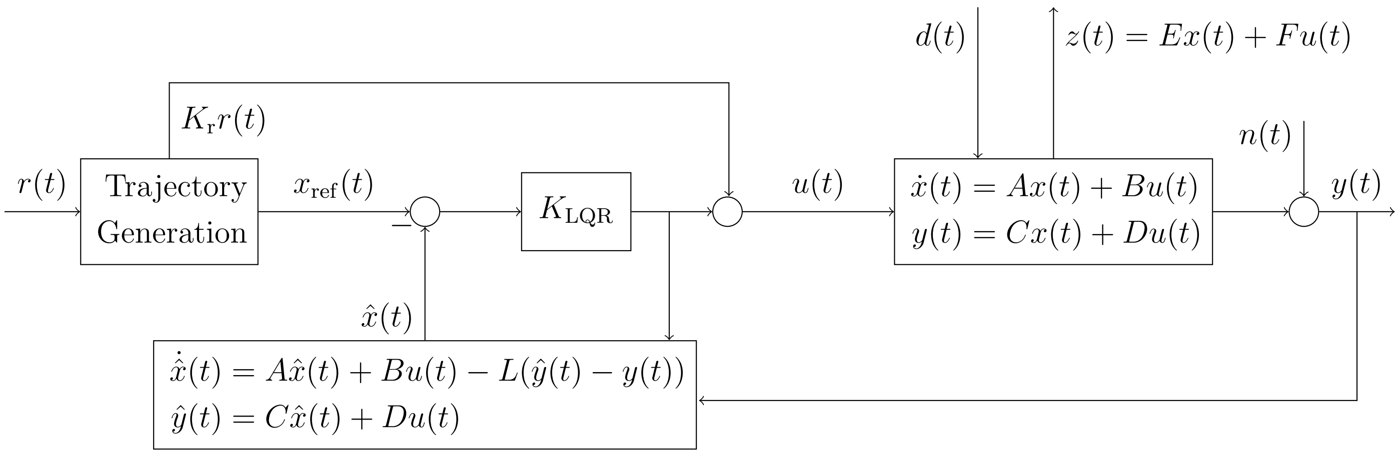

Figure 3: LQG Problem: Closed loop system.

stable if and only if K is a stabilizing state feedback gain and L is a stabilizing estimation gain.

1.5 Examples

Example 1. The dynamics of a system are given as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u(t)\\ y(t) &= x_1(t).\end{aligned} \tag{1.33}$$

You want to design a state observer. The observer should use the measurements for $y(t)$ and $u(t)$ in order to estimate the state variables $\hat{x}(t) \sim x(t)$.

(a) Which dimension should the observer matrix L have?

(b) Compute the observer matrix L for $R = 1$ and $Q = BB^\intercal$.

(c) You have already computed a state feedback matrix $K = \begin{pmatrix}1 & 1\end{pmatrix}$ for the system above. What is the complete transfer function of the controller $C(s)$?

Solution.

(a) Since $C \in \mathbb{R}^{1\times 2}$ matrix and $L \cdot C$ has the same dimensions of $A \in \mathbb{R}^{2\times 2}$, L is a 2×1 matrix, i.e.

$$L = \begin{pmatrix}L_1\\ L_2\end{pmatrix}. \tag{1.34}$$

(b) First of all, let's read from 1.33 the system matrices:

$$\begin{aligned}A &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\\ B &= \begin{pmatrix}0\\ 1\end{pmatrix}\\ C &= \begin{pmatrix}1 & 0\end{pmatrix}\\ D &= 0.\end{aligned} \tag{1.35}$$

Plugging these matrices into the algebraic Riccati equation and using the unknown matrix

$$\Psi = \begin{pmatrix}\psi_1 & \psi_2\\ \psi_2 & \psi_3\end{pmatrix}, \tag{1.36}$$

one gets:

$$\begin{aligned}\frac{1}{R}\cdot\Psi\cdot C^T\cdot C\cdot\Psi - \Psi\cdot A^T - A\cdot\Psi - B\cdot B^T &= 0\\ \Psi\cdot\begin{pmatrix}1\\ 0\end{pmatrix}\cdot\begin{pmatrix}1 & 0\end{pmatrix}\cdot\Psi - \Psi\cdot\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot\Psi - \begin{pmatrix}0\\ 1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\end{pmatrix} &= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}\psi_1^2 & \psi_1\cdot\psi_2\\ \psi_1\cdot\psi_2 & \psi_2^2\end{pmatrix} - \begin{pmatrix}2\psi_2 & \psi_3\\ \psi_3 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} &= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.\end{aligned} \tag{1.37}$$

The matrix Ψ is symmetric and positive definite and with these informations we can compute its elements:

- From the last term of the equation one gets

$$\psi_2^2 = 1 \Rightarrow \psi_2 = \pm 1. \tag{1.38}$$

- By plugging this into the first equation one gets $\psi_1 = \pm\sqrt{2}$. Because the positive definite condition, one gets $\psi_1 = \sqrt{2}$, $\psi_2 = 1$.
- Because of the form of C we don't care about ψ_3.

From these calculations it follows

$$\begin{aligned} L^T &= \frac{1}{R} \cdot C \cdot \Psi \\ &= \frac{1}{1} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 1 \\ 1 & * \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2} & 1 \end{pmatrix}, \end{aligned} \tag{1.39}$$

and so

$$L = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}. \tag{1.40}$$

(c) By looking at Figure 4, one can write the transfer function of the feedback controller as

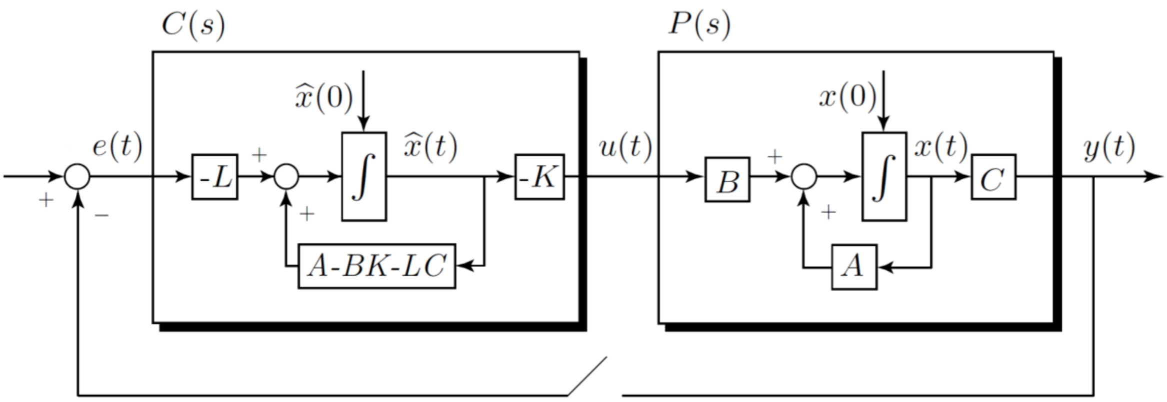

Figure 4: Structure of LQG controller.

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$$\hat{C}(s) = K \cdot (s \cdot \mathbb{I} - (A - B \cdot K - L \cdot C))^{-1} \cdot L. \tag{1.41}$$

By plugging in the found matrices one gets

$$\begin{aligned} (A - B \cdot K - L \cdot C) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \end{pmatrix} - \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \sqrt{2} & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2} & 1 \\ -2 & -1 \end{pmatrix}. \end{aligned} \tag{1.42}$$

It follows

$$
\begin{aligned}
(s\cdot\mathbb{I}-(A-B\cdot K-L\cdot C))^{-1} &= \begin{pmatrix} s+\sqrt{2} & -1 \\ 2 & s+1 \end{pmatrix}^{-1} \\
&= \frac{1}{(s+\sqrt{2})\cdot(s+1)+2}\cdot\begin{pmatrix} s+1 & 1 \\ -2 & s+\sqrt{2} \end{pmatrix}.
\end{aligned} \tag{1.43}
$$

By plugging this into the formula one gets

$$
\begin{aligned}
\hat{C}(s) &= K\cdot(s\cdot\mathbb{I}-(A-B\cdot K-L\cdot C))^{-1}\cdot L \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix}\cdot\frac{1}{(s+\sqrt{2})\cdot(s+1)+2}\cdot\begin{pmatrix} s+1 & 1 \\ -2 & s+\sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\
&= \frac{1}{(s+\sqrt{2})\cdot(s+1)+2}\cdot\begin{pmatrix} s-1 & s+1+\sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\
&= \frac{1}{(s+\sqrt{2})\cdot(s+1)+2}\cdot(\sqrt{2}s-\sqrt{2}+s+1+\sqrt{2}) \\
&= \frac{(\sqrt{2}+1)s+1}{(s+\sqrt{2})\cdot(s+1)+2}.
\end{aligned} \tag{1.44}
$$

Example 2. You are working for your semester thesis at a project which includes a water reservoir. Your task is to determine the disturbance $d(t)$ that acts on the reservoir. Figure 5 shows the situation. The only state of the system is the water volume $x(t) = V(t)$. The volume flows in the reservoir $V_{\text{in}}^*(t)$ are the known system input $u(t)$ and the unknown disturbance $d(t) > 0$. The volume flow of the system is assumed to be only dependend on the water volume, i.e.

$$V_{\text{out}}^*(t) = -\beta \cdot x(t). \tag{1.45}$$

The system output $y(t)$ is the water level $h(t)$. The model of this reservoir reads

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= -\beta \cdot x(t) + u(t) + d(t), \\ y(t) &= \frac{1}{\alpha} \cdot x(t), \ \alpha > 0, \ \beta > 0. \end{aligned} \tag{1.46}$$

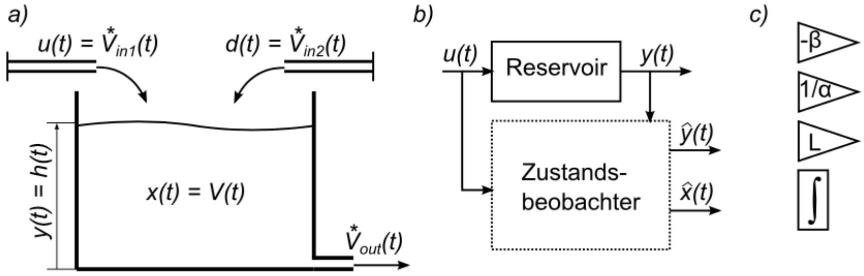

Figure 5: a) Drawing of the reservoir; b) Inputs and Outputs of the observer; c) Blocks for signal flow diagram.

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The goal is to determine $d(t)$. Your supervisor has already tried to solve the model equations for $d(t)$: he couldn't determine the change in volume $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ with enough precision. Hence, you want to solve this problem with a state observer.

(a) Draw the signal flow diagram of such a state observer. Use the blocks of Figure 5c).

(b) The state feedback matrix L is in this case some scalar value. Which value can L be, in order to get an asymptotically stable state observer?

(c) Introduce a new signal $\hat{d}(t)$ in the state observer. This should approximate the real disturbance $d(t)$.

(d) Find the state space description of the observer with inputs $u(t)$ and $y(t)$ and output $\hat{d}(t)$.

Solution.

(a) The signal flow diagram can be seen in Figure 6.

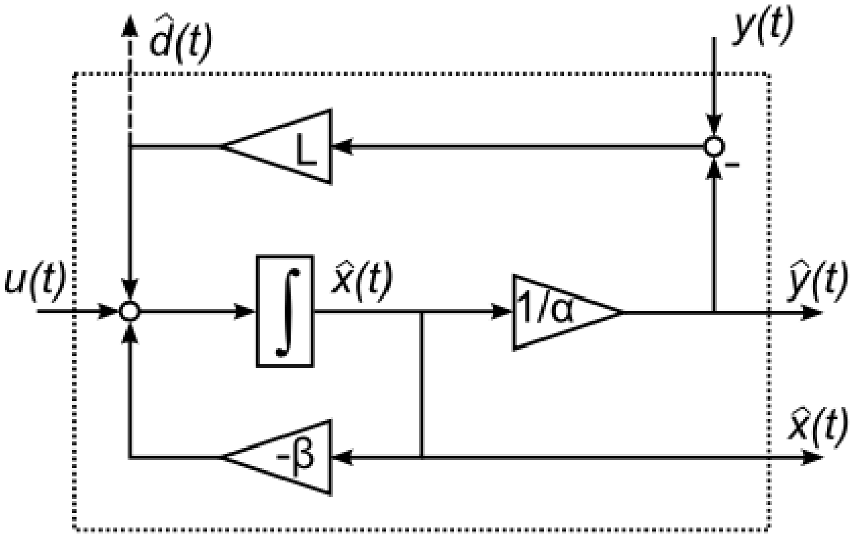

Figure 6: Signal flow diagram of the state observer.

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(b) The stability of the ovserver depends on the eigenvalues of $A - L \cdot C$. In this case, since $A - L \cdot C$ is a scalar,

$$A - L \cdot C < 0$$

should hold. This leads to

$$L > \frac{A}{C}. \tag{1.47}$$

With the given informations it follows

$$L > -\frac{\beta}{\alpha}. \tag{1.48}$$

(c) The dashed line in Figure 6 represents the new output $\hat{d}(t)$. The integrator in Figure 6 has now 3 inputs. The arrow from downwards from the reservoir is $V^*_{\text{out}}(t)$, the arrow from left is the input flow $u(t) = V^*_{\text{in1}}(t)$. If we simulate the system without the dashed arrow, there is a deviation between the measured $y(t)$ and the simulated $\hat{y}(t)$. This results from the extra inflow $d(t) = V^*_{\text{in}}(t)$, which is not considered in the simulation.

(d) The new state-space description reads

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \left(-\beta - \frac{L}{\alpha}\right) \cdot \hat{x}(t) + (1) \cdot u(t) + (L) \cdot y(t) \tag{1.49}$$

$$\hat{d}(t) = \left(-\frac{L}{\alpha}\right) \cdot \hat{x}(t) + (0) \cdot u(t) + (L) \cdot y(t). \tag{1.50}$$

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[1] Essentials of Robust Control, Kemin Zhou.

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[3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.