

Lecture 11: \mathcal{H}_∞ Control

1 \mathcal{H}_∞ Control

The big disadvantage of LQR/LQG is that one cannot directly impose frequency domain specifications to the control loop. A solution to this problem is given by the \mathcal{H}_∞ control formulation.

1.1 Problem Formulation

In \mathcal{H}_∞ control we consider the closed-loop system representation reported in Figure 1. By referring to Figure 1, one can define

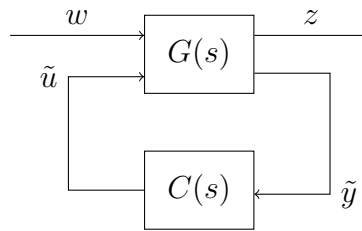


Figure 1: General system for \mathcal{H}_∞ control.

- $G(s)$ is called the *extended system* and is real, rational and proper.
- $C(s)$ is the controller and is real, rational and proper.
- $w(t) \in \mathbb{C}^{m_1 \times 1}$ is called *exogenous input*, and contains at least the reference signal $r(t)$ and possibly other exogenous signals, such as a noise model $n(t)$.
- $z(t) \in \mathbb{C}^{p_1 \times 1}$ is called the *performance output* and is a virtual output signal only used for design.
- $\tilde{u}(t) \in \mathbb{C}^{m_2 \times 1}$ is the control input, computed by the controller $C(s)$.
- $\tilde{y}(t) \in \mathbb{C}^{p_2 \times 1}$ is the measured output, available to the controller $C(s)$.

Remark. As a side note, *exogenous* means: caused or produced by factors external to a model.

With \mathcal{H}_∞ control we are interested in finding the controller $C(s)$ that stabilizes internally and externally the closed loop system and minimizes

$$\begin{aligned}
 \|T_{zw}(s)\|_\infty &= \sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2} \\
 &= \sup_{w \neq 0} \bar{\sigma}(T_{zw}(j\omega)) \\
 &:= \gamma_{\min},
 \end{aligned} \tag{1.1}$$

where $T_{zw}(s)$ is the transfer function which relates signals $z(t)$ and $w(t)$. Intuitively, this is equivalent to

- Minimize the energy ($\|\cdot\|_2$ norm) gain of the closed-loop system.
- Have a chance to incorporate constraints on regulated variables in frequency space, e.g. the tracking error $E(s)$.

One can hence state the aim of \mathcal{H}_∞ control to be:

Optimal \mathcal{H}_∞ Control: Find all admissible controllers $C(s)$ such that $\|T_{zw}(j\omega)\|_\infty$ is minimized.

Differently to what we observed in \mathcal{H}_2 control, the optimal \mathcal{H}_∞ controllers are not *unique*. Moreover, the process of finding an optimal controller is complicated, numerically and theoretically. This said, in practice is often not necessary to design an optimal controller: often it is sufficient to find controllers which are close to optimality, but *easier* to compute, i.e. *suboptimal controllers*.

Suboptimal \mathcal{H}_∞ Control: Given $\gamma > 0$, find all admissible controllers $C(s)$ such that $\|T_{zw}(j\omega)\|_\infty < \gamma$.

1.2 Mixed Sensitivity Approach

1.2.1 Transfer Functions Recap

By considering the standard MIMO control system structure with 0 disturbance depicted in Figure 2 and defining the signals of interest to be $E(s)$, $U(s)$ and $Y(s)$, one can derive them as

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - N(s) - P(s)C(s)E(s) \\ \Rightarrow E(s) &= (\mathbb{I} + P(s)C(s))^{-1} (R(s) - N(s)) \\ &= S(s)(R(s) - N(s)) \end{aligned} \tag{1.2}$$

$$\begin{aligned} U(s) &= C(s)E(s) \\ &= C(s)S(s)(R(s) - N(s)), \end{aligned} \tag{1.3}$$

$$\begin{aligned} Y(s) &= N(s) + P(s)C(s)E(s) \\ &= N(s) + P(s)C(s)S(s)(R(s) - N(s)) \\ &= N(s) + T(s)(R(s) - N(s)) \\ &= (\mathbb{I} - T(s))N(s) + T(s)R(s) \\ &= S(s)N(s) + T(s)R(s), \end{aligned} \tag{1.4}$$

which can be rewritten in matrix form as

$$\begin{pmatrix} E(s) \\ U(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} S(s) & -S(s) \\ C(s)S(s) & -C(s)S(s) \\ T(s) & S(s) \end{pmatrix} \begin{pmatrix} R(s) \\ N(s) \end{pmatrix}. \tag{1.5}$$

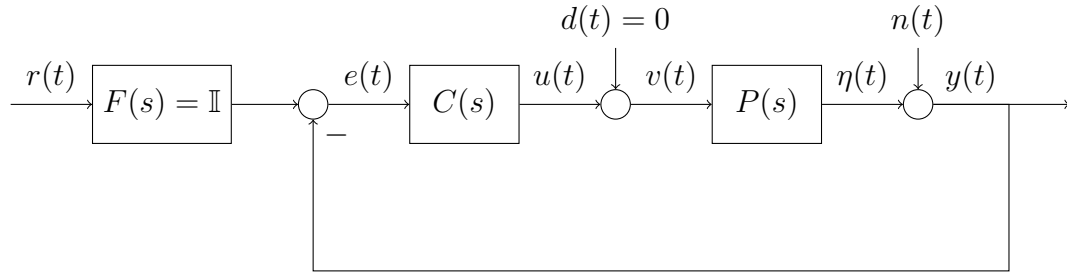


Figure 2: Standard feedback control system structure.

1.2.2 How to ensure Robustness?

As previously mentioned, \mathcal{H}_∞ control approach allows to introduce specifications in the frequency domain. In particular, we always focused ourselves in the analysis of the useful system transfer functions $S(s)$ (sensitivity function) and $T(s)$ (complementary sensitivity function). As a general reminder, recall that a small sensitivity $S(s)$ corresponds to disturbance rejection and a small complementary sensitivity function $T(s)$ corresponds to noise attenuation and robustness on modeling errors. We now recall the bound we defined in previous classes

$$\|W_1(j\omega)S(j\omega)\| + \|W_2(j\omega)T(j\omega)\| < 1, \quad (1.6)$$

where $W_1(s)$ and $W_2(s)$ are the robust weighting functions for the sensitivity and the complementary sensitivity, respectively. In order to ensure robustness, we would like to minimize the left term in Equation 1.6. Since there exist no controller which is able to solve this problem directly, one can write the two conditions separately, i.e.

$$\begin{aligned} \|W_1(s)S(s)\|_\infty &< 1 \text{ nominal performance,} \\ \|W_2(s)T(s)\|_\infty &< 1 \text{ robust stability.} \end{aligned} \quad (1.7)$$

1.2.3 How to use this in \mathcal{H}_∞ Control?

Once the weighting functions are designed, one needs to augment the original plant in order to let the approach meet the \mathcal{H}_∞ problem definition. In particular, the general form of such an augmentation with $\tilde{u}(t) = u(t)$, $\tilde{y}(t) = e(t)$ and $w(t) = r(t)$ can be seen in Figure 3. One can note that the signals resulting from the weighting are three and are $z_e(t)$, $z_u(t)$, $z_y(t)$. The error $e(t)$ is fed through the weighting function $W_e(s)$ ($W_1(s)$ in our previous considerations). Since the transfer function from the reference signal $r(t)$ to the error $e(t)$ is known to be the sensitivity function $S(s)$ (refer to Equation 1.5), one can write

$$Z_e(s) = W_e(s)S(s)R(s). \quad (1.8)$$

The measured output $y(t)$ is fed through the weighting function $W_y(s)$ ($W_2(s)$ in our previous considerations). Since the transfer function from the reference $r(t)$ to the output $y(t)$ is known to be the complementary sensitivity function $T(s)$ (refer to Equation 1.5), one can write

$$Z_y(s) = W_y(s)T(s)R(s). \quad (1.9)$$

The clever reader will notice that a third weighting function is present. The input $u(t)$ is fed through the weighting function $W_u(s)$. Since the transfer function from the reference

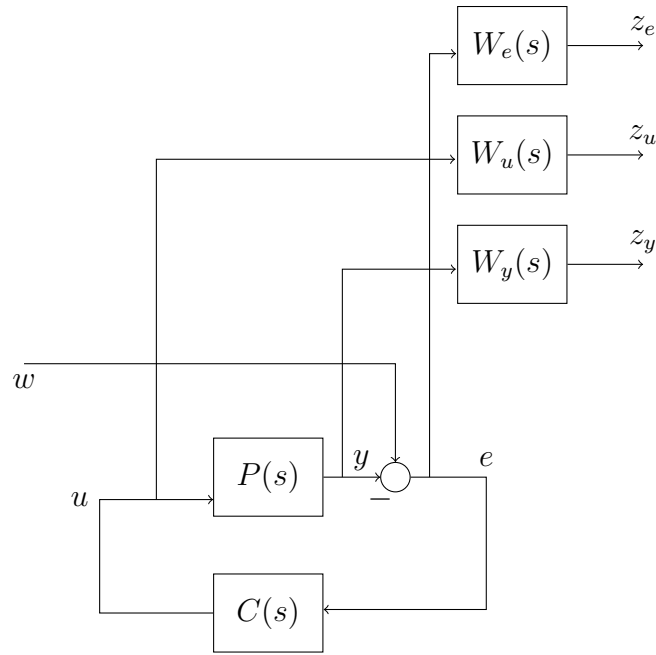


Figure 3: General extended system structure.

$r(t)$ to the input $u(t)$ is known to be $C(s)S(s)$ (refer to Equation 1.5), one can write

$$Z_u(s) = W_u(s)C(s)S(s)R(s). \quad (1.10)$$

Summarizing, one can write

$$\underbrace{\begin{pmatrix} Z_e(s) \\ Z_u(s) \\ Z_y(s) \end{pmatrix}}_{Z(s)} = \underbrace{\begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix}}_{T_{zw}(s)} \underbrace{R(s)}_{W(s)}. \quad (1.11)$$

1.3 Finding $T_{zw}(s)$

1.3.1 General Form

By looking at Figure 1, one can write the system into standard form, which reads

$$\begin{aligned} \begin{pmatrix} Z(s) \\ \tilde{Y}(s) \end{pmatrix} &= G(s) \begin{pmatrix} W(s) \\ \tilde{U}(s) \end{pmatrix} \\ &= \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} W(s) \\ \tilde{U}(s) \end{pmatrix}. \end{aligned} \quad (1.12)$$

Furthermore, it holds

$$\begin{aligned} \tilde{U}(s) &= C(s)\tilde{Y}(s) \\ &= C(s) \left(G_{21}(s)W(s) + G_{22}(s)\tilde{U}(s) \right), \end{aligned} \quad (1.13)$$

from which it follows

$$\tilde{U}(s) = (\mathbb{I} - C(s)G_{22}(s))^{-1} C(s)G_{21}(s)W(s). \quad (1.14)$$

Combining Equation 1.12 and Equation 1.14 results in

$$\begin{aligned}
 Z(s) &= G_{11}(s)W(s) + G_{12}(s)\tilde{U}(s) \\
 &= G_{11}(s)W(s) + G_{12}(s) (\mathbb{I} - C(s)G_{22}(s))^{-1} C(s)G_{21}(s)W(s) \\
 &= \underbrace{(G_{11}(s) + G_{12}(s) (\mathbb{I} - C(s)G_{22}(s))^{-1} C(s)G_{21}(s))}_{T_{zw}(s)} W(s).
 \end{aligned} \tag{1.15}$$

The infinity norm of $T_{zw}(s)$ is per definition

$$\|T_{zw}(j\omega)\|_\infty = \max_{\omega} \left(\max_i \sigma_i(T_{zw}(j\omega)) \right), \tag{1.16}$$

which corresponds to the maximum magnitude of its frequency response. Minimizing the infinity means in practice, minimizing this maximum singular value, i.e. minimizing the worst-case amplification from $w(t)$ to $z(t)$ at any frequency.

1.3.2 Applying Mixed Sensitivity Approach

If the plant is augmented using the mixed sensitivity approach, it holds

$$T_{zw}(s) = \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix}. \tag{1.17}$$

Let's define $\hat{S}(s)$, $\hat{T}(s)$ and $\hat{R}(s)$ to be acceptable upper bounds for the sensitivity $S(s)$, the complementary sensitivity $T(s)$ and the transfer function $r \rightarrow u$, $C(s)S(s)$. By setting

$$\begin{aligned}
 W_e(s) &= \hat{S}(s)^{-1} \\
 W_u(s) &= \hat{R}(s)^{-1} \\
 W_y(s) &= \hat{T}(s)^{-1},
 \end{aligned} \tag{1.18}$$

one can write the control problem as:

Suboptimal \mathcal{H}_∞ Control: Find $C(s)$ such that for sufficiently small $\gamma \in \mathbb{R}_+$ it holds.

$$\left\| \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix} \right\|_\infty \leq \gamma. \tag{1.19}$$

Remark. Note that

$$\bar{\sigma}(T_{zw}(s)) = \bar{\sigma} \rightarrow \underline{\sigma}(T_{zw}(s)^{-1}) = \frac{1}{\bar{\sigma}}. \tag{1.20}$$

One can then in general define the generalized optimization problem related to this control problem to be

$$\min_{\|T_{zw}(s)\|_\infty \leq \gamma} \gamma. \tag{1.21}$$

One can have essentially two cases:

- If the solution of the optimization problem results into $\gamma^* \leq 1$, then the imposed specifications are fulfilled and satisfied by $C(s)$.
- If the solution of the optimization problem results into $\gamma^* > 1$, the conditions are not satisfied and one should use relaxed weights.

What is done in practice is an iterative procedure, which solves the optimization problem by restricting the weighting functions.

1.4 Implementation

By fixing a γ^* , one can solve the optimization problem. The augmented plant $G(s)$ has to be represented in state space form, i.e. we need to express the weighting functions' dynamics. In general, one can always write

$$\begin{aligned} P(s) &= C(s\mathbb{I} - A)^{-1}B + D, \\ W_e(s) &= C_e(s\mathbb{I} - A_e)^{-1}B_e + D_e, \\ W_u(s) &= C_u(s\mathbb{I} - A_u)^{-1}B_u + D_u, \\ W_y(s) &= C_y(s\mathbb{I} - A_y)^{-1}B_y + D_y. \end{aligned} \tag{1.22}$$

1.4.1 State Space Representation

The extended system can be re-written as a dynamic system with $A_{\text{ext}}, B_{\text{ext}}, C_{\text{ext}}, D_{\text{ext}}$ as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{x}_e(t) \\ \dot{x}_u(t) \\ \dot{x}_y(t) \end{pmatrix} &= A_{\text{ext}} \begin{pmatrix} x(t) \\ x_e(t) \\ x_u(t) \\ x_y(t) \end{pmatrix} + B_{\text{ext}} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} \\ \begin{pmatrix} z_e(t) \\ z_u(t) \\ z_y(t) \\ e(t) \end{pmatrix} &= C_{\text{ext}} \begin{pmatrix} x(t) \\ x_e(t) \\ x_u(t) \\ x_y(t) \end{pmatrix} + D_{\text{ext}} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}. \end{aligned} \tag{1.23}$$

By using Figure 3, one can write

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u_e(t) \\ &= A_e x_e(t) + B_e (r(t) - y(t)) \\ &= A_e x_e(t) + B_e (w(t) - Cx(t) - Du(t)) \\ &= A_e x_e(t) + B_e w(t) - B_e Cx(t) - B_e Du(t), \\ z_e(t) &= y_e(t) \\ &= C_e x_e(t) + D_e u_e(t) \\ &= C_e x_e(t) + D_e (w(t) - Cx(t) - Du(t)) \\ &= C_e x_e(t) - D_e Cx(t) + D_e w(t) - D_e Du(t), \end{aligned} \tag{1.24}$$

and

$$\begin{aligned} \dot{x}_y(t) &= A_y x_y(t) + B_y u_y(t) \\ &= A_y x_y(t) + B_y y(t) \\ &= A_y x_y(t) + B_y (Cx(t) + Du(t)) \\ &= A_y x_y(t) + B_y Cx(t) + B_y Du(t), \\ z_y(t) &= y_y(t) \\ &= C_y x_y(t) + D_y u_y(t) \\ &= C_y x_y(t) + D_y (Cx(t) + Du(t)) \\ &= C_y x_y(t) + D_y Cx(t) + D_y Du(t), \end{aligned} \tag{1.25}$$

and

$$\begin{aligned}
 \dot{x}_u(t) &= A_u x_u(t) + B_u u_u(t) \\
 &= A_u x_u(t) + B_u u(t), \\
 z_u(t) &= y_u(t) \\
 &= C_u x_u(t) + D_u u_u(t) \\
 &= C_u x_u(t) + D_u u(t).
 \end{aligned} \tag{1.26}$$

Combining these results with Equation 1.23, one gets

$$\begin{aligned}
 A_{\text{ext}} &= \begin{pmatrix} A & 0 & 0 & 0 \\ -B_e C & A_e & 0 & 0 \\ 0 & 0 & A_u & 0 \\ B_y C & 0 & 0 & A_y \end{pmatrix}, \quad B_{\text{ext}} = \left(\begin{array}{c|c} 0 & B \\ B_e & -B_e D \\ 0 & B_u \\ 0 & B_y D \end{array} \right) = (B_{\text{ext},w} \quad B_{\text{ext},u}), \\
 C_{\text{ext}} &= \begin{pmatrix} -D_e C & C_e & 0 & 0 \\ 0 & 0 & C_u & 0 \\ D_y C & 0 & 0 & C_y \\ -C & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{\text{ext},z} \\ C_{\text{ext},y} \end{pmatrix}, \quad D_{\text{ext}} = \left(\begin{array}{c|c} D_e & -D_e D \\ 0 & D_u \\ 0 & D_y D \\ \mathbb{I} & -D \end{array} \right) = \begin{pmatrix} D_{\text{ext},zw} & D_{\text{ext},zu} \\ D_{\text{ext},yw} & D_{\text{ext},yu} \end{pmatrix}.
 \end{aligned} \tag{1.27}$$

Compactly, one can write

$$\left(\begin{array}{c|cc} A_{\text{ext}} & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right) = \left(\begin{array}{c|cc} A_{\text{ext}} & B_{\text{ext},w} & B_{\text{ext},u} \\ \hline C_{\text{ext},z} & D_{\text{ext},zw} & D_{\text{ext},zu} \\ C_{\text{ext},y} & D_{\text{ext},yw} & D_{\text{ext},yu} \end{array} \right) \tag{1.28}$$

1.4.2 \mathcal{H}_∞ Solution

Once that one has the extended plant $G(s)$ and the state space description of the system, one can solve the optimization problem.

Simplified Case

Assuming

- $C_{\text{ext},z}^\top D_{\text{ext},zu} = 0$,
- $B_{\text{ext},w} D_{\text{ext},yw}^\top = 0$, i.e. process noise and sensor noise are uncorrelated,
- $D_{\text{ext},zu}^\top D_{\text{ext},zu} = \mathbb{I}$,
- $D_{\text{ext},yw}^\top D_{\text{ext},yw} = \mathbb{I}$,

find a controller $C(s)$ such that $\|T_{zw}\|_\infty < \gamma$ for $\gamma > 0$. It turns out that by simplifying the problem, the solution has similarities with the one of LQG (state feedback). The procedure to solve this problem is:

Kochrezept \mathcal{H}_∞ Control

A controller $C(s)$ which satisfies the objective exists if and only if the conditions contained in the different steps are fulfilled.

I) Fix a large value for γ .

II) Find the quadratic, real matrix $X_\infty \geq 0$ which solves the algebraic Riccati equation

$$A_{\text{ext}}^\top X_\infty + X_\infty A + C_{\text{ext},z}^\top C_{\text{ext},z} + X_\infty \left(\frac{1}{\gamma^2} B_{\text{ext},w} B_{\text{ext},w}^\top - B_{\text{ext},u} B_{\text{ext},u}^\top \right) X_\infty \quad (1.29)$$

and such that it is stabilizing, i.e.

$$\text{Re} \left(\lambda_i \left(A_{\text{ext}} + \left(\frac{1}{\gamma^2} B_{\text{ext},w} B_{\text{ext},w}^\top - B_{\text{ext},u} B_{\text{ext},u}^\top \right) X_\infty \right) \right) < 0 \quad \forall i, \quad (1.30)$$

where $\lambda_i(\cdot)$ denotes the i -th eigenvalue.

III) Find the quadratic, real matrix $Y_\infty \geq 0$ which solves the algebraic Riccati equation

$$A_{\text{ext}} Y_\infty + Y_\infty A_{\text{ext}}^\top + B_{\text{ext},w} B_{\text{ext},w}^\top + Y_\infty \left(\frac{1}{\gamma^2} C_{\text{ext},z}^\top C_{\text{ext},z} - C_{\text{ext},y}^\top C_{\text{ext},y} \right) Y_\infty = 0, \quad (1.31)$$

and it is stabilizing, i.e.

$$\text{Re} \left(\lambda_i \left(A_{\text{ext}} + Y_\infty \left(\frac{1}{\gamma^2} C_{\text{ext},z}^\top C_{\text{ext},z} - C_{\text{ext},y}^\top C_{\text{ext},y} \right) \right) \right) < 0 \quad \forall i, \quad (1.32)$$

where $\lambda_i(\cdot)$ denotes the i -th eigenvalue.

IV) It must holds

$$\gamma \mathbb{I} - Y_\infty X_\infty > 0, \quad (1.33)$$

or, equivalently,

$$\max_i |\lambda_i(X_\infty Y_\infty)| = \rho(X_\infty Y_\infty) \leq \gamma^2, \quad (1.34)$$

where ρ denotes the spectral radius.

V) Reduce γ until no solution is found.

VI) If the resulting minimal $\gamma^* > 1$, the feasibility conditions we introduced in the previous chapter are no more valid. In order to make the problem feasible, one need to relax the weights $W_e(s), W_u(s), W_y(s)$. If the resulting minimal $\gamma^* \leq 1$, the result is acceptable. One can use the matrices X_∞ and Y_∞ to calculate the \mathcal{H}_∞ control dynamics. Considering the extended state

$$\hat{x}(t) = \begin{pmatrix} x(t) \\ x_e(t) \\ x_u(t) \\ x_y(t) \end{pmatrix}, \quad (1.35)$$

one can write the controller dynamics to be

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= A_\infty\hat{x}(t) - ZLy(t) \\ u(t) &= F\hat{x}(t),\end{aligned}\tag{1.36}$$

where

$$\begin{aligned}A_\infty &= A_{\text{ext}} + \left(\frac{1}{\gamma^2} B_{\text{ext},w} B_{\text{ext},w}^\top \right) X_\infty + B_{\text{ext},u} F + Z L C_{\text{ext},y} \\ F &= -B_{\text{ext},u}^\top X_\infty \\ L &= -Y_\infty C_{\text{ext},y}^\top \\ Z &= \left(\mathbb{I} - \frac{1}{\gamma^2} Y_\infty X_\infty \right)^{-1}.\end{aligned}\tag{1.37}$$

Remark. In order to solve this kind of problems, a popular strategy is bisection. Let γ^* be the optimal solution. By maintaining lower and upper bounds $\gamma_- < \gamma^* < \gamma_+$ one uses the following procedure:

1. Initialize $\gamma_- = 0$ and $\gamma_+ = \alpha$, where α is the \mathcal{H}_∞ norm of the \mathcal{H}_2 optimal design (LQG). Let K_+ be the optimal LQG controller.

2. Let

$$\gamma \leftarrow \frac{\gamma_- + \gamma_+}{2}.\tag{1.38}$$

Check if a controller exists such that $\|T_{zw}\|_\infty < \gamma$. If yes, set $\gamma_+ = \gamma$ and K_+ to the controller just designed. If not, set $\gamma_- \leftarrow \gamma$.

3. Repeat from step 2. until

$$\gamma_+ - \gamma_- < \varepsilon,\tag{1.39}$$

where ε is a user-defined threshold.

4. Return K_+ .

1.4.3 Feasibility Conditions

Conditions for the feasibility of the problem are

- (a) The controllability of the extended plant $G(s)$ must be verified: if there are non controllable states, one needs to make sure that these states remain bounded. The pair (A_{ext}, B_2) must be **stabilizable**.
- (b) The extended plant $G(s)$ must be fully observable: if there are not observable states, one needs to make sure that these are stable. The pair $(C_{\text{ext},y}, A_{\text{ext}})$ must be detectable.
- (c) The four matrices

$$D_{\text{ext},zu}, \quad D_{\text{ext},yw}, \quad \begin{pmatrix} A_{\text{ext}} - \gamma\omega\mathbb{I} & B_{\text{ext},u} \\ C_{\text{ext},z} & D_{\text{ext},zu} \end{pmatrix}, \quad \begin{pmatrix} A_{\text{ext}} - \gamma\omega\mathbb{I} & B_{\text{ext},w} \\ C_{\text{ext},y} & D_{\text{ext},yw} \end{pmatrix}\tag{1.40}$$

must have full rank $\forall \omega$.

- (d) $\bar{\sigma}(D_{\text{ext},zw}) < \gamma$.

Example 1. Given an extended system of a SISO plant with performance output

$$\begin{pmatrix} Z_e(s) \\ Z_u(s) \\ Z_y(s) \end{pmatrix} = \underbrace{\begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix}}_{=T_{zw}} R(s) \quad (1.41)$$

where

- $W_e(s)$, $W_u(s)$ and $W_y(s)$ are weights for the corresponding sensitivities,
- $S(s)$ is the sensitivity and $T(s)$ is the complementary sensitivity,
- $R(s)$ is the reference signal to the feedback loop.

The state space representation of the extended system is given as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2\tilde{u}(t) \\ z(t) &= C_1x(t) + D_{11}w(t) + D_{12}\tilde{u}(t) \\ \tilde{y}(t) &= C_2x(t) + D_{21}w(t) + D_{22}\tilde{u}(t) \end{aligned} \quad (1.42)$$

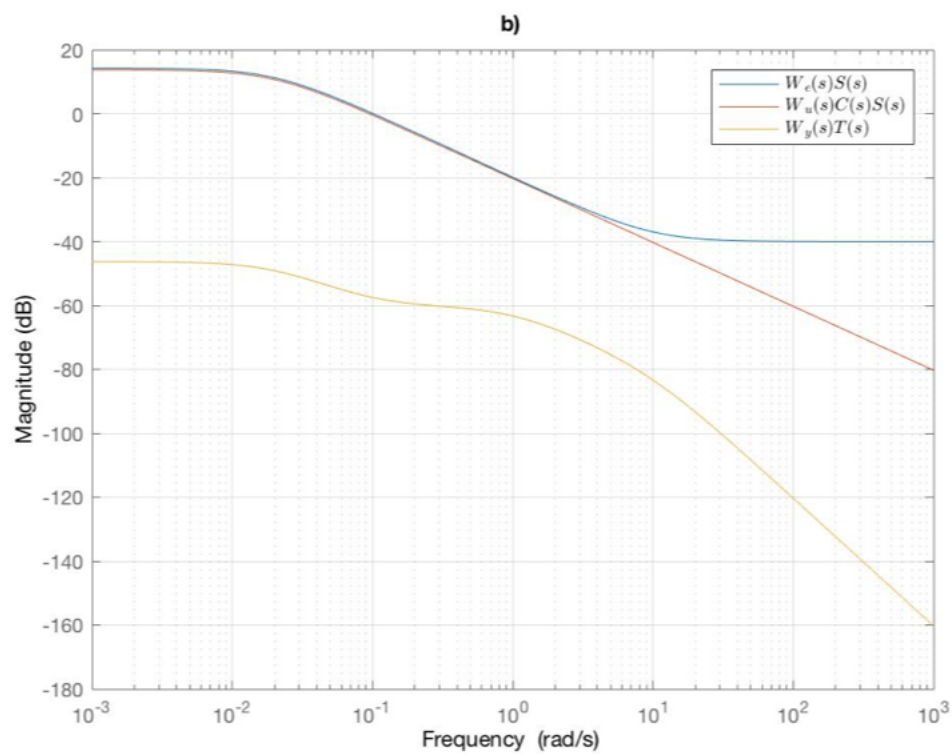
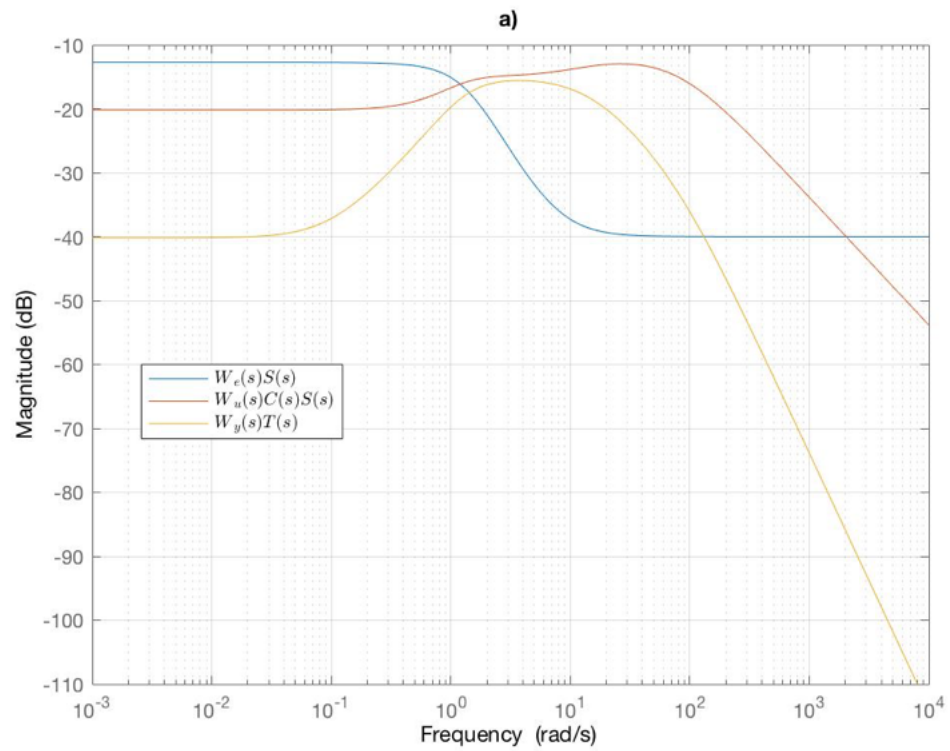
Note: The system matrix A and state vector $x(t)$ correspond to the extended system state and not only to the plant's one.

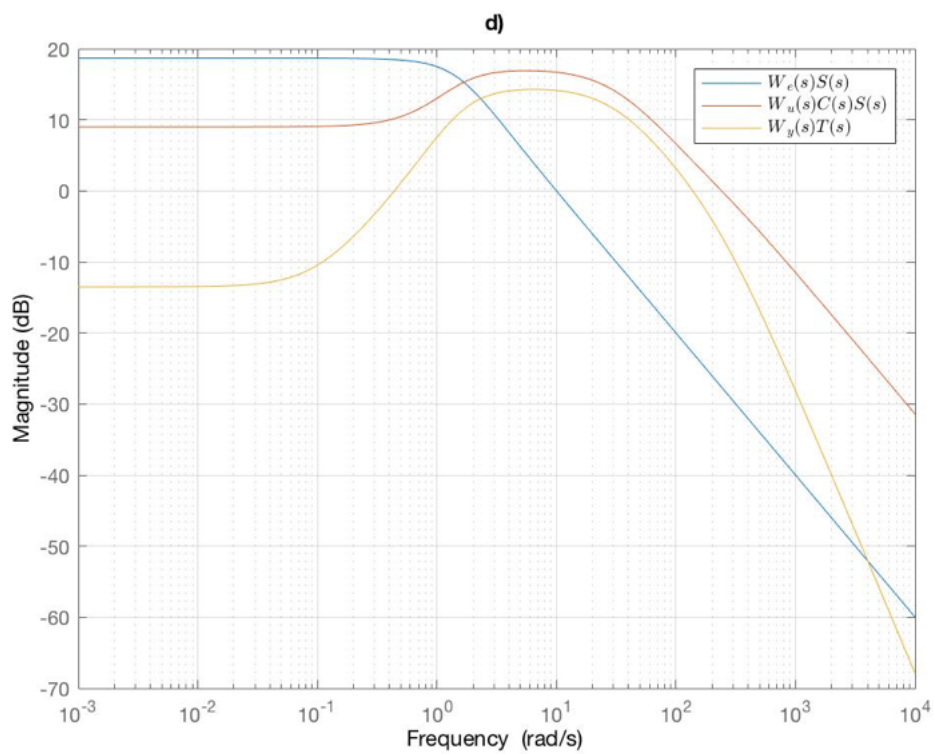
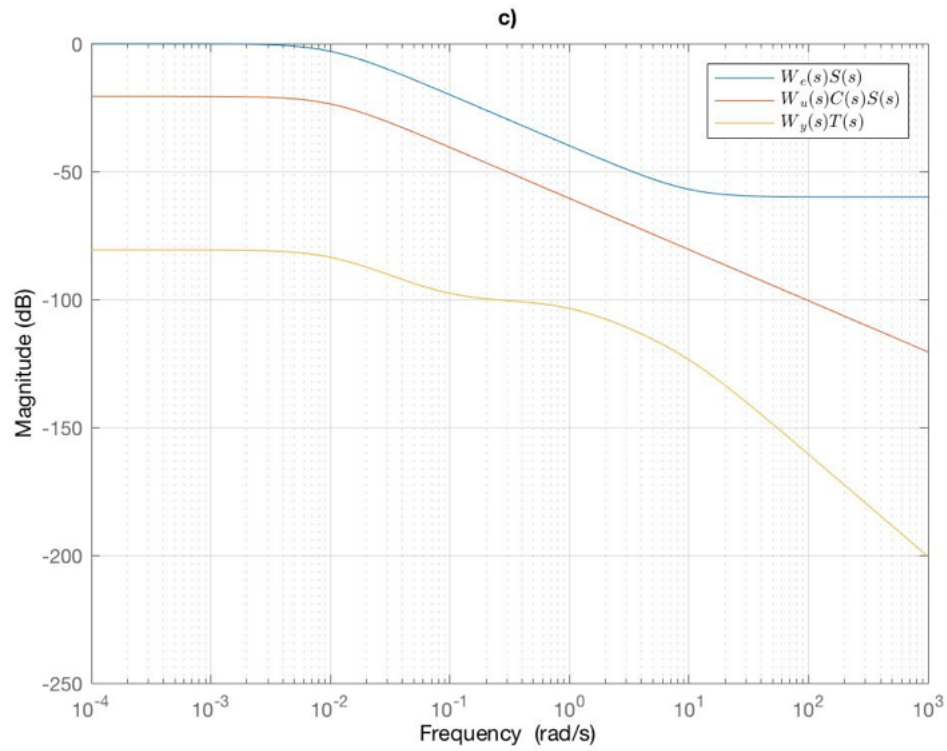
Assume that an \mathcal{H}_∞ -controller $C(s)$ was found so that

$$\left\| \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix} \right\|_\infty \leq 1 \quad (1.43)$$

holds.

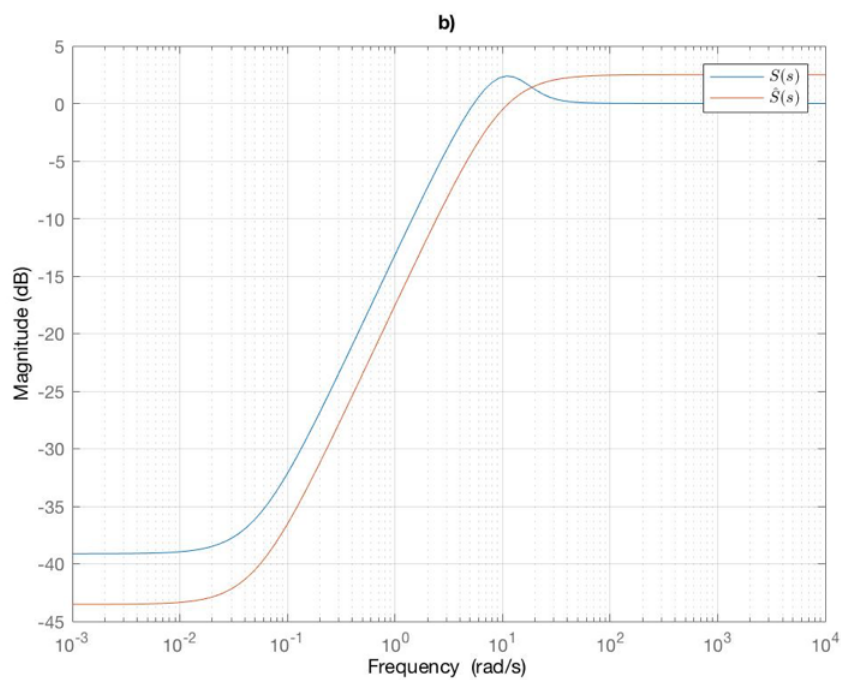
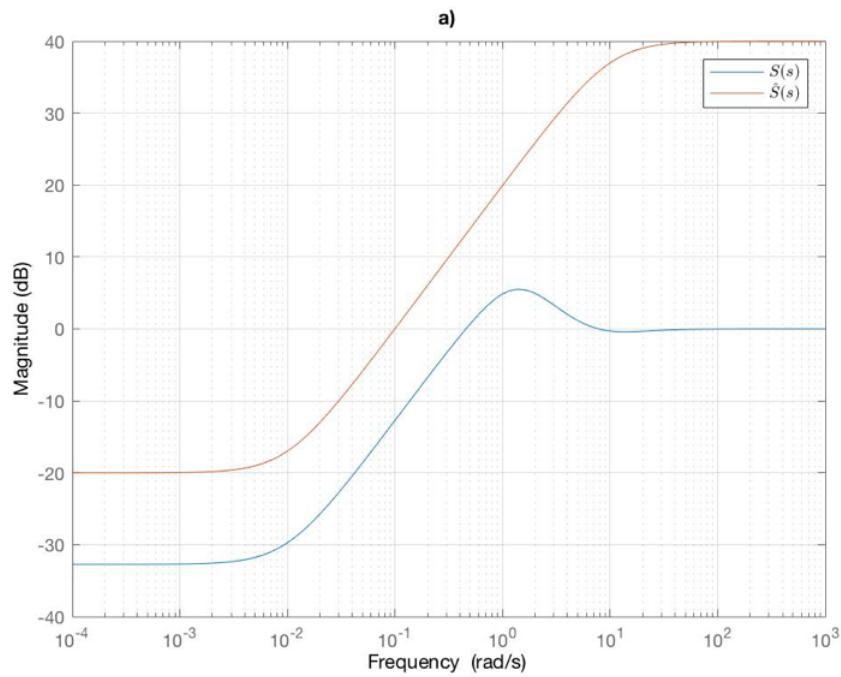
1. Which of the following magnitude plots are possible for the given extended system?





- ☐ a)
- ☐ b)
- ☐ c)
- ☐ d)

2. Let $W_e(s) = \hat{S}(s)^{-1}$ where $\hat{S}(s)$ is a designer defined upper boundary for $S(s)$. Which magnitude plot of $S(s)$ and $\hat{S}(s)$ could correspond to the the given system?



☐ a)

☐ b)

Given that all matrices in (1.42) are the same as the one derived in the lecture. One can show that the eigenvalues of A are the eigenvalues of all separately considered subsystems. Assume for the next two subtasks that the not considered conditions (summarized on slide 18, lecture 11) for well-posedness of the \mathcal{H}_∞ -problem hold.

3. Given stabilizability to the extended system. There exists a solution to the problem if all eigenvalues of the weight system matrices have negative real part.

☐ True.

☐ False.

4. Assume for the given extended system that $\bar{\sigma}(D_{11}) = 1.15$. There exists a solution.

☐ True.

☐ False.

Solution.

1. ☒ a)

☐ b)

☒ c)

☐ d)

From the lecture we know that if

$$\left\| \begin{pmatrix} W_e(s)S(s) \\ W_u(s)C(s)S(s) \\ W_y(s)T(s) \end{pmatrix} \right\|_\infty \leq 1 \quad (1.44)$$

holds, then each individual inequality holds as well. Therefore the Bode magnitude plots of $W_e(s)S(s)$, $W_u(s)C(s)S(s)$ and $W_y(s)T(s)$ must not exceed the 0dB-line. We see that only the plots of **a)** and **c)** satisfy this condition.

2. ☒ a)

☐ b)

Since $\left\| \hat{S}(s)^{-1}S(s) \right\|_\infty \leq 1$, the magnitude plot of $S(s)$ has to be always below the one of $\hat{S}(s)$. This means, that the \mathcal{H}_∞ -controller is exact the solution of the minimization problem which leads to a $S(s)$ satisfying the above condition.

3. ☒ True.

☐ False.

From the lecture we know that a sufficient condition for well-posedness of the problem is $[A, B_2]$ stabilizable and $[A, C_2]$ detectable. Since $C_2 = \begin{pmatrix} -C_s & 0 & 0 & 0 \end{pmatrix}$, where C_s corresponds to the plant's LTI-representation, only the states of the plant can be observed. However, the states of the plant are not influenced by the remaining one. In other words, there is no possibility to observe the weight's states. Therefore the poles of the weights has to be stable in order to have well-posedness.

4. ☐ True.
☒ False.

From the lecture we know that a sufficient condition for well-posedness of the problem is $\bar{\sigma}(D_{11}) = \gamma$, where in our case $\gamma = 1$. Therefore the problem is not well-posed.

References

- [1] Essentials of Robust Control, Kemin Zhou.
- [2] Karl Johan Amstroem, Richard M. Murray *Feedback Systems for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2009.
- [3] Sigurd Skogestad, *Multivariate Feedback Control*. John Wiley and Sons, New York, 2001.