Lecture 12: Nonlinear Control

1 Elements of Nonlinear Control

1.1 Equilibrium Point and Linearization

A nonlinear system can be written as

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)), \end{aligned} \tag{1.1}$$

where $f(\cdot)$ and $g(\cdot)$ are nonlinear functions. Recall that $(x_\mathrm{e}, u_\mathrm{e})$ represents an **equilibrium point** if and only if

$$\begin{aligned} 0 &= f(x_\mathrm{e}, u_\mathrm{e}, t) \\ y_\mathrm{e} &= g(x_\mathrm{e}, u_\mathrm{e}, t). \end{aligned} \tag{1.2}$$

As the analysis of the nonlinear system is often difficult, we previously considered such a system in a neighbourhood of its equilibrium points. Mathematically, this translates into considering the Taylor expansion of the functions $f(\cdot)$ and $g(\cdot)$ around the equilibrium points of the system and neglecting high order terms. Let $\delta x = x - x_\mathrm{e}$ and $\delta u = u - u_\mathrm{e}$. It holds then

$$\begin{aligned} \delta\dot{x} &= f(x_\mathrm{e} + \delta x, u_\mathrm{e} + \delta u, t) \\ &= \frac{\partial f}{\partial x}\bigg|_{x_\mathrm{e}, u_\mathrm{e}} \delta x + \frac{\partial f}{\partial u}\bigg|_{x_\mathrm{e}, u_\mathrm{e}} \delta u + \text{ high order terms} \\ &= A\delta x + B\delta u + \text{ high order terms.} \end{aligned} \tag{1.3}$$

By proceeding analogously for $g(\cdot)$ and neglecting high order terms, one gets

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u, \end{aligned} \tag{1.4}$$

where $C = \frac{\partial g}{\partial x}\Big|_{x_\mathrm{e}, u_\mathrm{e}}$ and $D = \frac{\partial g}{\partial u}\Big|_{x_\mathrm{e}, u_\mathrm{e}}$.

Remark.

- Note that in general, matrices A, B, C, D are time-varying. However, if $f(\cdot), g(\cdot)$ do not depend explicitly on time t, the linearized model will be time-invariant.
- $\delta x, \delta u, \delta y$ describe a deviation from the equilibrium point. The linearized dynamics are given by

$$\begin{aligned} x &= x_\mathrm{e} + \delta x \\ y &= y_\mathrm{e} + \delta y \\ u &= u_\mathrm{e} + \delta u. \end{aligned} \tag{1.5}$$

1.2 Nominal Stability

During the course Control Systems I, you learned about different stability concepts. Moreover, you have learned the differences between internal and external stability: let's recall them here. Consider a generic nonlinear system defined by the dynamics

$$\dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n. \tag{1.6}$$

1.2.1 Internal/Lyapunov Stability

Internal stability, also called Lyapunov stability, characterises the stability of the trajectories of a dynamic system subject to a perturbation near the to equilibrium. Let now $\hat{x} \in \mathbb{R}^n$ be an equilibrium of system (1.6).

Definition 1. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *Lyapunov* stable if

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad \text{s.t. } \|x(0) - \hat{x}\| < \delta \Rightarrow \|x(t) - \hat{x}\| < \varepsilon. \tag{1.7}$$

In words, an equilibrium point is said to be *Lyapunov* stable if for any bounded initial condition and zero input, the state remains bounded.

Definition 2. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *asymptotically stable* in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and attractive, i.e. if

$$\lim_{t\to\infty} (x(t) - \hat{x}) = 0, \quad \forall x(0) \in \Omega. \tag{1.8}$$

In words, an equilibrium is said to be asymptotically stable if, for any bounded initial condition and zero input, the state converges to the equilibrium.

Definition 3. An equilibrium $\hat{x} \in \mathbb{R}^n$ is said to be *unstable* if it is not stable.

Remark. Note that stability is a property of the equilibrium and not of the system in general.

1.2.2 External/BIBO Stability

External stability, also called BIBO stability (Bounded Input-Bounded Output), characterises the stability of a dynamic system which for bounded inputs gives back bounded outputs.

Definition 4. A signal $s(t)$ is said to be bounded, if there exists a finite value $B > 0$ such that the signal magnitude never exceeds B, that is

$$|s(t)| \leq B \quad \forall t \in \mathbb{R}. \tag{1.9}$$

Definition 5. A system is said to be *BIBO-stable* if

$$\|u(t)\| \leq \varepsilon \quad \forall t \geq 0, \text{ and } x(0) = 0 \Rightarrow \|y(t)\| < \delta \quad \forall t \geq 0, \quad \varepsilon, \delta \in \mathbb{R}. \tag{1.10}$$

In words, for any bounded input, the output remains bounded.

1.2.3 Stability for LTI Systems

Above, we focused on general nonlinear system. However, in Control Systems I you learned that the output $y(t)$ for a LTI system of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{1.11}$$

can be written as

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}(\tau) + Du(t). \tag{1.12}$$

The transfer function relating input to output is a rational function

$$P(s) = C(s\mathbb{I} - A)^{-1}B + D = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d. \tag{1.13}$$

Furthermore, it holds:

- The zeros of the numerator of Equation (1.13) are the *zeros* of the system, i.e. the values s_i which fulfill
$$P(s_i) = 0. \tag{1.14}$$
- The zeros of the denominator of Equation (1.13) are the *poles* of the system, i.e. the values s_i which fulfill $\det(s_i\mathbb{I} - A) = 0$, or, in other words, the eigenvalues of A.

One can show, that the following Theorem holds:

Theorem 1. The equilibrium $\hat{x} = 0$ of a linear time invariant system is stable if and only if the following two conditions are met:

1. For all $\lambda \in \sigma(A)$, $\quad \mathrm{Re}(\lambda) \leq 0$.
2. The algebraic and geometric multiplicity of all $\lambda \in \sigma(A)$ such that $\mathrm{Re}(\lambda) = 0$ are equal.

Remark. For linear systems, the stability of an equilibrium point does not depend on the point itself. For nonlinear systems, it does.

1.3 Local Stability

Let $x = x_\mathrm{e}$ be an equilibrium for the autonomous nonlinear system

$$\dot{x}(t) = f(x(t)), \tag{1.15}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a continuously differentiable function and $\mathcal{D}$ is a neighborhood of x_e. Let

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=x_\mathrm{e}}. \tag{1.16}$$

Then:

1. x_e is aymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues of A.
2. x_e is unstable if $\mathrm{Re}(\lambda_i) > 0$ for one or more of the eigenvalues of A.

It holds:

- In linear systems, local stability $\Leftrightarrow$ global stability.
- In nonlinear systems, this is not true.

1.3.1 Region of Attraction

Definition 6. A function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be Lipschitz on Ω if for $K \geq 0$ it holds

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \leq K, \quad \forall x, y \in \Omega. \tag{1.17}$$

Definition 7. Let x_e be an asymptotically stable equilibrium point of the system $\dot{x}(t) = f(x(t))$, where $f(\cdot)$ is a locally Lipschitz function defined over a domain $\mathcal{D} \subset \mathbb{R}^n$ and x_e is contained in $\mathcal{D}$. The **region of attraction** (also known as region of asymptotic stability, domain of attraction) is the set of all points $x_0 \in \mathcal{D}$ such that the solution of

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{1.18}$$

is defined for all $t \geq 0$ and converges to x_e as $t \to \infty$. Note that x_e is said to be **globally asymptotically stable** if the region of attraction is the whole space $\mathbb{R}^n$.

1.4 Lyapunov Stability

1.4.1 Lyapunov Principle - General Systems

1. The Lyapunov Principle is valid for all finite-order systems: as long as the linearized system has no eigenvalues on the imaginary axis.
2. The local stability properties of an arbitrary-order nonlinear system are fully understood once the eigenvalues of the linearization are known.
3. Particularly, if the linearization of a nonlinear system around an isolated equilibrium point x_e is asymptotically stable (or unstable), then this equilibrium is an asymptotically stable (or unstable) equilibrium of the nonlinear system as well. We can but not say that this holds also for the concept of *stable system* ($\mathrm{Re}(\lambda) = 0$).

If we are interested in non-local results or in the case of stable systems, we should use the Lyapunov's direct method.
A scalar function $\alpha(p)$ with $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function if $\alpha(0) = 0$ and $\alpha(p) \geq \alpha(q) \forall p > q$. A function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a candidate global Lyapunov function if

- The function is strictly positive, i.e., $V(x,t) > 0 \ \forall x \neq 0, \forall t$ and $V(0) = 0$ and
- there are two nondecreasing functions α and β which satisfy the inequalities

$$\beta(||x||) \leq V(x,t) \leq \alpha(||x||) \tag{1.19}$$

Remark. If these conditions are not met, only local assumptions can be made.

Theorem 2. The system

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \neq 0, \tag{1.20}$$

is globally/locally stable in the sense of Lyapunov if there is a global/local Lyapunov function candidate $V(x,t)$ for which the following inequality holds $\forall x(t) \neq 0$ and $\forall t$:

$$\dot{V}(x(t), t) = \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(x(t), t) \leq 0 \tag{1.21}$$

Theorem 3. The same system is globally/locally asymptotically stable if there is a global/local Lyapunov function candidate $V(x,t)$ such that $-\dot{V}(x(t),t)$ satisfies all conditions of a global/local Lyapunov function candidate.

Remark. In general it is difficult to find suitable functions. A good way to approach the problem is to use physical laws (Lyapunov functions can be seen as generalized energy functions).

For linear systems one can find the Lyapunov function

$$V(x(t)) = x(t)^{\intercal} \cdot P \cdot x(t), \quad P = P^{\intercal} > 0, \tag{1.22}$$

where P is the solution of the Lyapunov equation

$$PA + A^{\intercal}P = -Q. \tag{1.23}$$

For arbitrary $Q = Q^{\intercal} > 0$, a solution to this equation exists if and only if A is a Hurwitz matrix.

Remark. Lyapunov theorems provide sufficient but not necessary conditions!

Example 1. Consider the nonlinear system described by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^2 \\ \dot{x}_2 &= x_1^2 x_2 + 2x_2^3 - 6x_2. \end{aligned} \tag{1.24}$$

a) Linearize the system around the equilibrium $x_{1,e} = x_{2,e} = 0$ and find matrix A.

b) Can you say something about the stability of the nonlinear system?

c) Evaluate the stability of the nonlinear system using the Lyapunov function $V = \frac{1}{2}(x_1^2 + x_2^2)$ and find the region of attraction about the equilibrium point.

Solution.

1. The linearization matrix A reads

$$\begin{aligned} A &= \begin{pmatrix} \frac{\partial}{\partial x_1}(x_1 x_2^2) & \frac{\partial}{\partial x_2}(x_1 x_2^2) \\ \frac{\partial}{\partial x_1}(x_1^2 x_2 + 2x_2^3 - 6x_2) & \frac{\partial}{\partial x_2}(x_1^2 x_2 + 2x_2^3 - 6x_2) \end{pmatrix} \Bigg|_{(0,0)} \\ &= \begin{pmatrix} x_2^2 & 2x_1 x_2 \\ 2x_1 x_2 & x_1^2 + 6x_2^2 - 6 \end{pmatrix} \Bigg|_{(0,0)} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}. \end{aligned} \tag{1.25}$$

2. The eigenvalues of matrix A are $\lambda_1 = 0$ and $\lambda_2 = -6$. Using the Lyapunov principle, we cannot evaluate the stability of the nonlinear system, since the linearized one is just stable around the equilibrium.

3. The derivative of the Lyapunov function reads

$$\begin{aligned} \dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1 \cdot (x_1 x_2^2) + x_2 \cdot (x_1^2 x_2 + 2x_2^3 - 6x_2) \\ &= x_1^2 x_2^2 + x_2^2 x_1^2 + 2x_2^4 - 6x_2^2 \\ &= 2x_2^2 \cdot (x_1^2 + x_2^2 - 3). \end{aligned} \tag{1.26}$$

In order for $\dot{V}$ to be negative definite, it must hold $x_1^2 + x_2^2 < 3$.

1.5 Gain Scheduling

As for most systems stability is guaranteed in some neighborhood of the equilibrium point, we are limited when we design a stabilizing controller. A first method to overcome this problem could be to stabilize the system around each equilibrium point and to design a local controller to get stability. The procedure can be defined as

I) Given the nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1.27}$$

choose n equilibrium points $(x_{\text{e,i}}, u_{\text{e,i}})$, $i = 1, \dots, n$.

II) For each of these equilibria, linearize the system and design a local control law

$$u_{\text{l}}(x(t)) = u_{\text{l,e}} - K(x(t) - x_{\text{l,e}}) \tag{1.28}$$

for the linearization.

III) The global control law consists of:

- Choosing the correct control law, as a function of the state: $i = \sigma(x)$.
- Use that control law: $u(x) = u_{\sigma(x)}(x)$.

1.6 Feedback Linearization

1.6.1 Input-State Feedback Linearization

Input-state feedback linearization is the ability to use feedback to convert a nonlinear state equation into a linear state equation by canceling nonlinearities. This requires the nonlinear state equation to have the structure

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right], \\ y(t) &= h(x(t)). \end{aligned} \tag{1.29}$$

where

- The pair (A, B) is controllable.
- $$\begin{aligned} \alpha &: \mathbb{R}^n \to \mathbb{R}^p \\ \beta &: \mathbb{R}^n \to \mathbb{R}^{p\times p} \end{aligned} \tag{1.30}$$

  are defined on the domain $\mathcal{D}_x \subset \mathbb{R}^n$, which contains the origin.
- The matrix $\beta(x(t))$ is assumed to be invertible $\forall x \in \mathcal{D}_x$.

If the system is in the form presented in Equation 1.29, one can linearize it using the feedback law

$$u(t) = \alpha(x(t)) + \beta(x(t))v(t). \tag{1.31}$$

Remark. The form presented in Equation 1.31 has a specific meaning. In fact, it holds

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right] \\ &= Ax(t) + B\beta^{-1}(x(t))\left[\alpha(x(t)) + \beta(x(t))v(t) - \alpha(x(t))\right] \\ &= Ax(t) + Bv(t), \end{aligned} \tag{1.32}$$

where $v(t)$ can be chosen with respect to the design constraints. This allows to linearize the dynamics of the system.

1.6.2 Input-State Linearizability

Let $z(x(t)) = T(x(t))$ be a change of variables (also known as bijection). If both T and T^{-1} are continuously differentiable, we call it a **diffeomorphism**. A nonlinear system

$$\dot{x}(t) = f(x(t)) + \Gamma(x(t))u(t), \tag{1.33}$$

where

$$f : \mathcal{D}_x \to \mathbb{R}^n \tag{1.34}$$

and

$$\Gamma : \mathcal{D}_x \to \mathbb{R}^{p \times p} \tag{1.35}$$

are sufficiently smooth on a domain $\mathcal{D}_x \subset \mathbb{R}^n$, is said to be **input-state linearizable** if there exists a diffeomorphism

$$T : \mathcal{D}_x \subset \mathbb{R}^n \tag{1.36}$$

such that

$$\mathcal{D}_z = T(\mathcal{D}_x) \tag{1.37}$$

contains the origin and the change of variables $z(x(t)) = T(x(t))$ transforms the system into the form

$$\dot{z}(x(t)) = Az(x(t)) + B\beta^{-1}(x(t)) \left[u(t) - \alpha(x(t))\right], \tag{1.38}$$

with (A, B) controllable and $\beta(x(t))$ invertible for all $x \in \mathcal{D}_x$.

Conditions for Linearizability - General Case

But when is this the case? In general, holds

$$\begin{aligned} \dot{z}(x(t)) &= \frac{\partial T}{\partial x}\dot{x}(t) \\ &= \frac{\partial T}{\partial x}\left[f(x(t)) + \Gamma(x(t))u(t)\right]. \end{aligned} \tag{1.39}$$

On the other hand, one can also write

$$\begin{aligned} \dot{z}(t) &= Az(t) + B\beta^{-1}(x(t)) \left[u(t) - \alpha(x(t))\right] \\ &= AT(x(t)) + B\beta^{-1}(x(t)) \left[u(t) - \alpha(x(t))\right]. \end{aligned} \tag{1.40}$$

Using Equation 1.39 and Equation 1.40, one can write the general equality which must hold for all $x(t)$ and $u(t)$ in the domain of interest:

$$\frac{\partial T}{\partial x}\left[f(x(t)) + \Gamma(x(t))u(t)\right] = AT(x(t)) + B\beta^{-1}(x(t)) \left[u(t) - \alpha(x(t))\right]. \tag{1.41}$$

If one sets $u(t) = 0$, one can split the equation into two:

$$\begin{aligned} \frac{\partial T}{\partial x}f(x(t)) &= AT(x(t)) - B\beta^{-1}(x(t))\alpha(x(t)) \\ \frac{\partial T}{\partial x}\Gamma(x(t)) &= B\beta^{-1}(x(t)). \end{aligned} \tag{1.42}$$

Each correct transformation $T(\cdot)$ must satisfy the partial differential equations given in Equation 1.42.
Having $T(x(t))$ which fulfills these partial differential equations is a necessary and sufficient conditions that a transformation from the form in Equation 1.33 to the form in Equation 1.38 exists.

Conditions for Linearizability - Single Input

With a single input ($p = 1$), one can define a linear transformation $\xi(x(t)) = Mz(x(t))$ with M invertible and write

$$\dot{\xi} = MAM^{-1}\xi + MB\beta^{-1}(x(t))\left[u(t) - \alpha(x(t))\right]. \tag{1.43}$$

We choose M such that the controller canonical form can be written as

$$\left(\begin{array}{c|c} A_c + B_c\gamma^\intercal & B_c \\ \hline C_c & D_c \end{array}\right) = \left(\begin{array}{cccccc|c} 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & \dots & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 0 \\ -\gamma_0 & -\gamma_1 & \dots & \dots & --\gamma_{n-2} & -\gamma_{n-1} & 1 \\ \hline c_0 & \dots & c_m & 0 & \dots & 0 & 0 \end{array}\right). \tag{1.44}$$

This means

$$MAM^{-1} = A_c + B_c\gamma^\intercal \tag{1.45}$$

and

$$MB = B_c. \tag{1.46}$$

The term

$$B_c\gamma^\intercal\xi = B_c\gamma^\intercal MT(x(t)) \tag{1.47}$$

is included into the nonlinearity

$$B_c\beta^{-1}(x(t))\alpha(x(t)), \tag{1.48}$$

which allows to reformulate the partial differential equations as

$$A_cT(x(t)) - B_c\beta^{-1}(x(t))\alpha(x(t)) = \begin{pmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_{n-1}(x) \\ T_n(x) \end{pmatrix} \tag{1.49}$$

and

$$B_c\beta^{-1}(x(t)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\beta(x(t))} \end{pmatrix}. \tag{1.50}$$

Finally, one can write

$$\begin{aligned} \frac{\partial T_1}{\partial x(t)} f(x(t)) &= T_2(x(t)) \\ \frac{\partial T_2}{\partial x(t)} f(x(t)) &= T_3(x(t)) \\ &\vdots \\ \frac{\partial T_{n-1}}{\partial x(t)} f(x(t)) &= T_n(x(t)) \\ \frac{\partial T_n}{\partial x} f(x(t)) &= -\frac{\alpha(x(t))}{\beta(x(t))} \end{aligned} \tag{1.51}$$

and

$$\begin{aligned}\frac{\partial T_1}{\partial x}\gamma(x(t)) &= 0\\ \frac{\partial T_2}{\partial x}\gamma(x(t)) &= 0\\ &\vdots\\ \frac{\partial T_{n-1}}{\partial x}\gamma(x(t)) &= 0\\ \frac{\partial T_n}{\partial x}\gamma(x(t)) &= \frac{1}{\beta(x(t))}\end{aligned} \tag{1.52}$$

1.7 Examples

Example 2. 1. Consider the continuous-time system

$$\dot{x}(t) = 0.5x(t), \quad x(t) \in \mathbb{R}, \tag{1.53}$$

and the test function

$$V(x) = 2x. \tag{1.54}$$

Which of the following statements is true?

- ☐ $V(x)$ is a Lyapunov function for this system and therefore the system is asymptotically stable.
- ☐ $V(x)$ is not a Lyapunov function for this system and therefore the system is not stable.
- ☐ $V(x)$ is not a Lyapunov function for this system. Furthrermore, given this information, we cannot conclude anything about the stability of the system.

2. Which of the following functions are positive definite

- ☐ $V(x) = x_1(t)^2 + x_2(t)^2$.
- ☐ $V(x) = x_1(t)^2$.
- ☐ $V(x) = (x_1(t) + x_2(t))^2$.
- ☐ $V(x) = -x_1(t)^2 - (3x_1(t) + 2x_2(t))^2$.
- ☐ $V(x) = x_1(t)x_2(t) + x_2(t)^2$.
- ☐ $V(x) = x_1(t)^2 + \frac{2x_2(t)^2}{1+x_2(t)^2}$.

3. You are given the nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t)^2\\ \dot{x}_2(t) &= x_1(t)^2x_2(t) + 2x_2(t)^3 - 6x_2(t).\end{aligned} \tag{1.55}$$

Evaluate the stability of the origin using the Lyapunov function

$$\frac{1}{2}(x_1^2(t) + x_2(t)^2). \tag{1.56}$$

- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 3\}$.

- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{3}\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 2\}$.
- ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{2}\}$.
- ☐ None of the above.

Solution.

1.

- ☐ $V(x)$ is a Lyapunov function for this system and therefore the system is asymptotically stable.
- ☐ $V(x)$ is not a Lyapunov function for this system and therefore the system is not stable.
- ☑ $V(x)$ is not a Lyapunov function for this system. Furthermore, given this information, we cannot conclude anything about the stability of the system.

Solution: The test function is not a Lyapunov function. One can verify this by observing that:

- $V(x)$ is not a positive definite function or
-

$$\begin{aligned}\dot{V}(x) &= \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ &= \frac{\partial V}{\partial x}\dot{x}(t) \\ &= 2 \cdot 0.5 \cdot x \\ &= x\end{aligned} \tag{1.57}$$

is not a negative definite function.

Since $V(x)$ is not a Lyapunov function, we cannot conclude anything about the stability of the system. Moreover, we know that the system is unstable only from the positive eigenvalue $\lambda_1 = 0.5$, and not from $V(x)$.

2.

- ☑ $V_1(x(t)) = x_1(t)^2 + x_2(t)^2$.
- ☐ $V_2(x(t)) = x_1(t)^2$.
- ☐ $V_3(x(t)) = (x_1(t) + x_2(t))^2$.
- ☐ $V_4(x(t)) = -x_1(t)^2 - (3x_1(t) + 2x_2(t))^2$.
- ☐ $V_5(x(t)) = x_1(t)x_2(t) + x_2(t)^2$.

Solution:

- $V_1(x(t)) > 0\ \forall x(t) \neq 0$ and $V_1(x(t)) = 0$ if $x = 0$.
- $V_2(x(t)) > 0\ \forall x(t) \neq 0$ and $V_2(x(t)) = 0$ if $x_1 = 0$. This still holds for any $x_2(t) \neq 0$, which makes $V_2(x(t))$ positive semi-definite.
- $V_3(x(t)) \geq 0\ \forall x(t)$, but can be 0 as soon as $x_1(t) = -x_2(t)$.
- $V_4(x(t)) < 0\ \forall x(t) \neq 0$.
- As soon as $x_1(t)x_2(t) < x_2(t)^2$, $V_5(x(t)) < 0$.

3. ☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 3\}$.

☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{3}\}$.

☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq 2\}$.

☐ The largest region of attraction of the system is $\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 \leq \sqrt{2}\}$.

☑ None of the above.

Solution: It holds

$$\begin{aligned}\dot{V}(x_1(t), x_2(t)) &= \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} \\ &= x_1(t)\left(x_1(t)x_2(t)^2\right) + x_2(t)\left(x_1(t)^2 x_2(t) + 2x_2(t)^3 - 6x_2(t)\right) \\ &= 2x_1(t)^2 x_2(t)^2 + 2x_2(t)^4 - 6x_2(t)^2 \\ &= 2x_2(t)^2\left(x_1(t)^2 + x_2(t)^2 - 3\right).\end{aligned} \tag{1.58}$$

In order to find the region of attraction for which the system is asymptotically stable, $\dot{V}(x)$ must be negative definite. This is the case if

$$\{x(t) \in \mathbb{R}^2 | x_1(t)^2 + x_2(t)^2 < 3\}. \tag{1.59}$$

This ensures that the region of attraction for the origin is at least the one presented in Equation 1.59. However, the choice of another Lyapunov function could result in a larger region of attraction. This explains why none of the first four answers is correct.

Example 3. You are given the system

$$\dot{x}(t) = f(x(t)) + gu(t), \tag{1.60}$$

with

$$f(x(t)) = \begin{pmatrix} x_2(t) \\ -a\sin(x_1(t)) - b(x_1(t) - x_3(t)) \\ x_4(t) \\ c(x_1(t) - x_3(t)) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} \tag{1.61}$$

where a, b, c and d are positive constants. We want to find a diffeomorphismus such that $T_1(x(t))$ fulfills:

$$\frac{\partial T_i}{\partial x} g = 0, \quad i = 1, 2, 3; \ \frac{\partial T_4}{\partial x} g \neq 0. \tag{1.62}$$

The system has clearly an equilibrium point at $x = 0$. From the first condition

$$\frac{\partial T_1}{\partial x} g = 0, \tag{1.63}$$

one knows that

$$\frac{\partial T_1}{\partial x_4} g = 0. \tag{1.64}$$

This means that one must choose $T_1(x(t))$ independent of $x_4(t)$. Using this, one can write

$$T_2(x(t)) = \frac{\partial T_1}{\partial x_1} x_2(t) + \frac{\partial T_1}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right) + \frac{\partial T_1}{\partial x_3} x_4(t). \tag{1.65}$$

From the second condition

$$\frac{\partial T_2}{\partial x} g = 0, \tag{1.66}$$

one knows that

$$\frac{\partial T_1}{\partial x_4} = 0. \tag{1.67}$$

This implies

$$\frac{\partial T_2}{\partial x_4} = 0 \Rightarrow \frac{\partial T_1}{\partial x_3} = 0. \tag{1.68}$$

$T_1(x(t))$ needs to be independent of $x_3(t)$ and hence

$$T_2(x(t)) = \frac{\partial T_1}{\partial x_1} x_2(t) + \frac{\partial T_1}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right), \tag{1.69}$$

and

$$T_3(x(t)) = \frac{\partial T_2}{\partial x_1} x_2(t) + \frac{\partial T_2}{\partial x_2} \left(-a\sin(x_1(t)) - b(x_1(t) - x_3(t)\right) + \frac{\partial T_2}{\partial x_3} x_4(t). \tag{1.70}$$

From the third condition

$$\frac{\partial T_3}{\partial x} g = 0, \tag{1.71}$$

one knows that

$$\frac{\partial T_3}{\partial x_4} = 0. \tag{1.72}$$

This implies

$$\frac{\partial T_3}{\partial x_4} = 0 \Rightarrow \frac{\partial T_2}{\partial x_3} = 0 \Rightarrow \frac{\partial T_1}{\partial x_2} = 0. \tag{1.73}$$

$T_1(x(t))$ needs to be independent of $x_2(t)$ and hence

$$T_4(x(t)) = \frac{\partial T_3}{\partial x_1} x_2(t) + \frac{\partial T_3}{\partial x_2} \left(-a \sin(x_1(t)) - b(x_1(t) - x_3(t)\right) + \frac{\partial T_3}{\partial x_3} x_4(t). \tag{1.74}$$

The last condition

$$\frac{\partial T_4}{\partial x} g \neq 0 \tag{1.75}$$

is satisfied if

$$\frac{\partial T_3}{\partial x_3} \neq 0 \Rightarrow \frac{\partial T_2}{\partial x_2} \neq 0 \Rightarrow \frac{\partial T_1}{\partial x_1} \neq 0. \tag{1.76}$$

With $T_1(x(t)) = x_1(t)$, one can write

$$\begin{aligned}
z_1(x(t)) &= T_1(x(t)) = x_1(t) \\
z_2(x(t)) &= T_2(x(t)) = x_2(t) \\
z_3(x(t)) &= T_3(x(t)) = -a \sin(x_1(t)) - b(x_1(t) - x_3(t)) \\
z_4(x(t)) &= T_4(x(t)) = -a x_2(t) \cos(x_1(t)) - b(x_2(t) - x_4(t)).
\end{aligned} \tag{1.77}$$

Example 4. Your SpaghETH startup, which cooks pasta on the polyterrasse everyday, is growing every week more and although no particular production issues occur you are concerned about ecology. Since each tank of pasta you cook needs water and a correct salt seasoning for it to taste that delicious, you need a lot of salt and water, which are often wasted. For this reason, you open a research branch in your startup which decides to design a duct-hydraulic system to counteract the waste of water and salt. The idea is to use a two water tank system, which helps you seasoning the water and changing it, without substituting the whole pot. The dynamics of the system are given by

$$\begin{aligned} \dot{x}_1(t) &= 1 + u(t) - \sqrt{1 + x_1(t)} \\ \dot{x}_2(t) &= \sqrt{1 + x_1(t)} - \sqrt{1 + x_2(t)} \\ y &= x_2(t). \end{aligned} \tag{1.78}$$

a) Linearize the nonlinear system around the equilibrium

$$\begin{pmatrix} x_{1,\text{eq}}(t) & x_{2,\text{eq}}(t) & u_{\text{eq}}(t) \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \end{pmatrix}. \tag{1.79}$$

b) Determine the coordinate transformation such that the system can be written in the form

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= \alpha(z) + \beta(z)u(t) \\ y(t) &= z_1(t). \end{aligned} \tag{1.80}$$

c) Find a feedback control law by exactly linearizing the system.

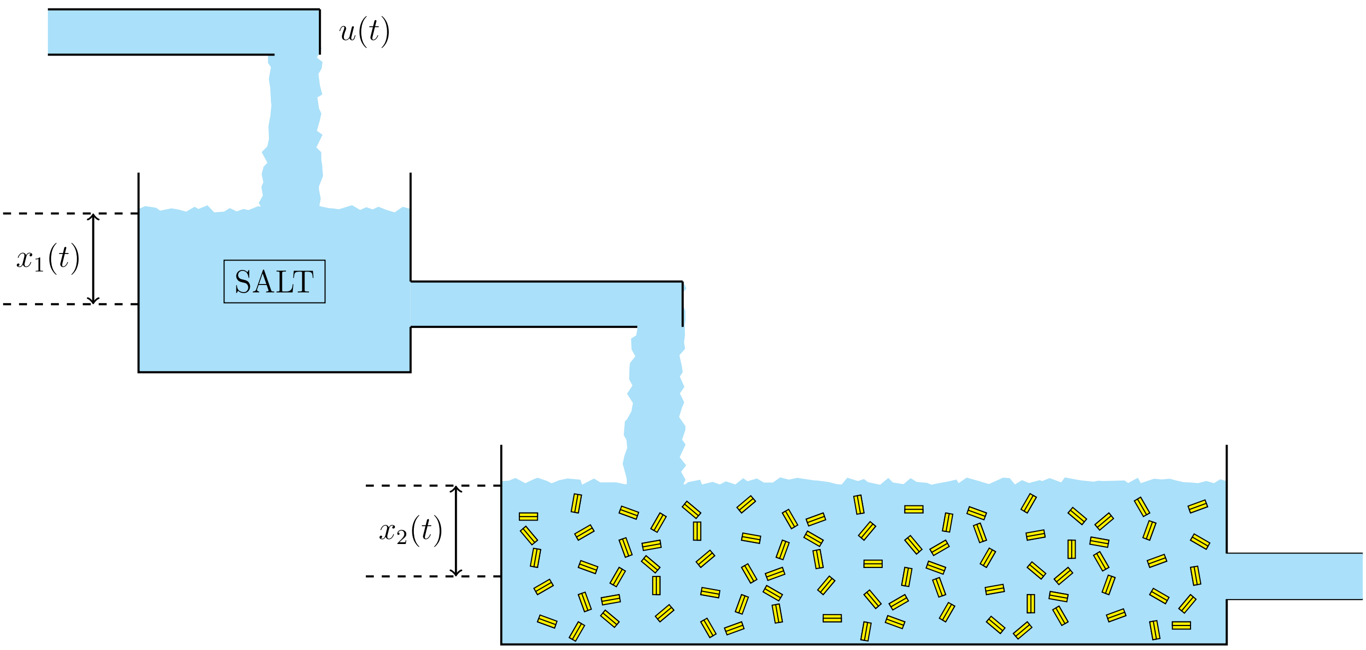

Figure 1: Sketch of the system.

Solution.

a) It holds

$$\begin{aligned}
A &= \begin{pmatrix} -\frac{1}{2\sqrt{1+x_1(t)}} & 0 \\ \frac{1}{2\sqrt{1+x_1(t)}} & -\frac{1}{2\sqrt{1+x_2(t)}} \end{pmatrix} \Bigg|_{x_{1,\text{eq}}(t)=x_{2,\text{eq}}(t)=3} \\
&= \begin{pmatrix} -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}, \\
B &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
C &= \begin{pmatrix} 0 & 1 \end{pmatrix}, \\
D &= 0.
\end{aligned} \tag{1.81}$$

b) By choosing the states

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}, \tag{1.82}$$

one gets

$$\begin{aligned}
\dot{z}_1(t) &= z_2(t) \\
\dot{z}_2(t) &= \frac{\partial}{\partial t}\dot{y}(t) \\
&= \frac{\partial}{\partial t}\dot{x}_2(t) \\
&= \frac{1}{2\sqrt{1+x_1(t)}}\dot{x}_1(t) - \frac{1}{2\sqrt{1+x_2(t)}}\dot{x}_2(t) \\
&= \frac{1}{2\sqrt{1+x_1(t)}}\left(1+u(t)-\sqrt{1+x_1(t)}\right) - \frac{1}{2\sqrt{1+x_2(t)}}\left(\sqrt{1+x_1(t)}-\sqrt{1+x_2(t)}\right) \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{1+x_1(t)}} - \frac{\sqrt{1+x_1(t)}}{\sqrt{1+x_2(t)}}\right) + \frac{u(t)}{2\sqrt{1+x_1(t)}}.
\end{aligned} \tag{1.83}$$

Furthermore, we know

$$\begin{aligned}
z_1(t) &= y(t) = x_2(t) \\
z_2(t) &= \dot{y}(t) = \dot{x}_2(t) \\
&= \sqrt{1+x_1(t)} - \sqrt{1+x_2(t)},
\end{aligned} \tag{1.84}$$

from which it follows

$$\begin{aligned}
x_2(t) &= z_1(t) \\
\sqrt{1+x_1(t)} &= z_2(t) + \sqrt{1+z_1(t)}
\end{aligned} \tag{1.85}$$

Plugging Equation 1.85 into Equation results in

$$\begin{aligned}
\dot{z}_1(t) &= z_2(t) \\
\dot{z}_2(t) &= \frac{1}{2}\left(\frac{1}{z_2(t)+\sqrt{1+z_1(t)}} - \frac{z_2(t)+\sqrt{1+z_1(t)}}{\sqrt{1+z_1(t)}}\right) + \frac{1}{2}\frac{u(t)}{z_2(t)+\sqrt{1+z_1(t)}} \\
&= \alpha(z(t)) + \beta(z(t))u(t).
\end{aligned} \tag{1.86}$$

c) With the form obtained in Equation 1.86, one can write

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(t), \tag{1.87}$$

where

$$\begin{aligned} u(t) &= \frac{1}{\beta(z(t))} \left(v(t) - \alpha(z(t))\right) \\ &= 2(z_2(t) + \sqrt{1 + z_1(t)}) \left(v(t) - \frac{1}{2} \left(\frac{1}{z_2(t) + \sqrt{1 + z_1(t)}} - \frac{z_2(t) + \sqrt{1 + z_1(t)}}{\sqrt{1 + z_1(t)}} \right) \right). \end{aligned} \tag{1.88}$$

References

[1] Hassan K. Khalil, *Nonlinear Systems*. Michigan State University.